

A Partial Order Approach to Decentralized Control

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Abstract—In this paper we employ the theory of partially ordered sets to model and analyze a class of decentralized control problems. We show that posets provide a natural way of modeling problems where communication constraints between subsystems have a hierarchical structure. We show that such problems have appealing algebraic properties that can be exploited to parameterize the set of stabilizing controllers. While much of the paper is devoted to problems where the plant and controller have identical communication constraints, we also generalize our theory to case where they may have different communication constraints.

I. INTRODUCTION

Traditional control theory deals with the problem of synthesizing centralized controllers, i.e., controllers that assume that all output measurements are available for processing. Many important control problems however, are large-scale, complex, and decentralized. For such problems, implementing centralized controllers is often not feasible, and the study of decentralized decision-making becomes important. Examples of such large-scale systems include flocks of aerial vehicles and the power distribution grid.

It is well-known that in general decentralized control is a hard problem, and significant research efforts have been directed towards its many different aspects; see for instance the classical survey [2] for many of the earlier results. More recently, Blondel and Tsitsiklis [3] have shown that in certain instances, decentralized control problems are in fact computationally intractable. On the other hand, Voulgaris [14], [15] presented several cases where decentralized control is in fact tractable. We are able to unify all these examples under an appealing theoretical framework. Rotkowitz and Lall [11] have presented a criterion known as quadratic invariance that characterizes a class of problems in decentralized control that have the property that problems become convex in the Youla parameter. Our results are related to this property and we show the connection to their work in our paper.

In this paper we consider linear time invariant systems with communication constraints, both within the plant and the controller. In most of the paper we are interested in the setting where the system is composed of several interacting subsystems. This subsystem approach enables us to partition the overall transfer function into several local transfer functions. The communication constraints among the subsystems manifest themselves via sparsity constraints in the plant and controller transfer functions. In this paper we consider a

specific class of such communication constraints that arise naturally in many decentralized decision-making problems.

The fundamental object of interest in this paper is the notion of a *partially ordered set* or *poset*. We argue that posets provide the right language and technical tools to talk about a more general notion of *causality* (also referred to as hierarchical control in the literature) among subsystems. We show that many interesting examples of decentralized control that have been shown to be tractable in the literature are in fact specific instances of this poset paradigm.

To any given poset (which is a combinatorial object) we can associate its *incidence algebra* [10], an algebraic object. By exploiting the algebraic structure, we show that several appealing properties follow for this class of problems. These include a nice parameterization of all controllers satisfying the sparsity constraints, convexity guarantees, quadratic invariance, and some deeper theoretical insight relating feedback invariance to certain lattices of invariant subspaces. In most of this paper we consider problems where the communication constraints within the controller are required to mirror the communication constraints within the plant. In the last section we provide a brief outline of a poset-based approach to deal with problems where constraints within the controller may be different from the plant constraints. We show that the natural generalization to this setting involves considering two posets (on the inputs and outputs respectively) along with a *Galois connection*.

The main contributions of this paper are the following:

- 1) We introduce the notion of a partially ordered set (poset) as a means of modeling causality-like communication constraints between subsystems in a decentralized control setting.
- 2) We exploit algebraic properties of the problem to show that the set of controllers that satisfy the sparsity constraints can be parameterized explicitly.
- 3) Using the notion of Galois connections, we generalize the poset-based technique from the setting where plant and controller have same communication constraints to the setting where they may have different constraints.

Posets are well-studied objects in combinatorics. The associated notions of incidence algebras and Galois connections were first studied by Rota [10] in a combinatorics setting. Since then, order-theoretic concepts have been used in engineering and computer science; we mention a few specific works below. Cousot and Cousot used these ideas to develop tools for formal verification of computer programs in their seminal paper [5]. In control theory too, these ideas have

been used by some authors in the past, albeit in somewhat different settings. Ho and Chu used posets to study team theory problems [7]. They were interested in sequential decision making problems where agents must make decisions at different time steps. They study the form of optimal decision-makers when the problems have poset structure. Mullans and Elliot [9] use posets to generalize the notion of time and causality, and study evolution of systems on locally finite posets. Del Vecchio and Murray [13] have used ideas from lattice and order theory to construct estimators for discrete states in hybrid systems.

It is possible to generalize and state some of our results in a more coordinate-independent setting using the language of structural matrix algebras. This generalization proves to be insightful because it enables one to describe the incidence algebra (a coordinate-dependent object) in terms of an associated distributive lattice of invariant subspaces. Due to space reasons, we are not able to include this material in this paper, we refer the interested reader to [12].

The rest of this paper is organized as follows. In Section II we introduce the order-theoretic and control-theoretic preliminaries that will be used throughout the paper. In Section III we present some examples of communication structures that can be modeled via posets. In Section IV we show how the algebraic properties of systems on posets may be exploited to construct a parameterization of all stabilizing controllers. In Section V, we generalize our results to the case where plant and controller do not necessarily have the same communication constraints. In Section VI we conclude our paper and mention future directions of research.

II. PRELIMINARIES

A. Order-theoretic Preliminaries

Definition 1: A partially ordered set (or *poset*) consists of a set \mathcal{P} along with a binary relation \preceq which is reflexive, antisymmetric and transitive.

In this paper we will be dealing with *finite* posets, i.e., where the set \mathcal{P} is of finite cardinality.

Example 1: An example of a poset with three elements (i.e., $\mathcal{P} = \{a, b, c\}$) with order relations $a \preceq b$ and $a \preceq c$ is shown in Figure 1.

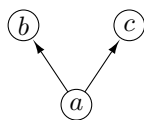


Fig. 1. A poset on the set $\{1, 2, 3\}$.

Definition 2: Let \mathcal{P} be a poset. Let \mathbb{Q} be a ring. The set of all functions

$$f : \mathcal{P} \times \mathcal{P} \rightarrow \mathbb{Q}$$

with the property that $f(x, y) = 0$ if $x \not\preceq y$ is called the *incidence algebra* of \mathcal{P} over \mathbb{Q} . It is denoted by $I_{\mathcal{P}}(\mathbb{Q})$. If the ring is clear from the context, we will simply denote this by $I_{\mathcal{P}}$ (we will usually work over the field of rational proper transfer functions).

Since the poset \mathcal{P} is finite, the set of functions in the incidence algebra may be thought of as matrices with a specific sparsity pattern given by the order relations of the poset.

Definition 3: Let \mathcal{P} be a poset. The function $\zeta(\mathcal{P}) \in I_{\mathcal{P}}(\mathbb{Q})$ defined by

$$\zeta(\mathcal{P})(x, y) = \begin{cases} 0, & \text{if } x \not\preceq y \\ 1, & \text{otherwise} \end{cases}$$

is called the *zeta-function* of \mathcal{P} .

Clearly, the zeta-function of the poset is a member of the incidence algebra.

Example 2: The matrix representation of the zeta function for the poset from Example 1 is as follows:

$$\zeta_{\mathcal{P}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The incidence algebra is the set of all matrices in $\mathbb{Q}^{3 \times 3}$ which have the same sparsity pattern as its zeta function.

Given two functions $f, g \in I_{\mathcal{P}}(\mathbb{Q})$, their sum $f + g$ and scalar multiplication cf are defined as usual. The product $h = f \cdot g$ is defined as follows:

$$h(x, y) = \sum_{z \in \mathcal{P}} f(x, z)g(z, y). \quad (1)$$

As mentioned above, we will frequently think of the functions in the incidence algebra of a poset as square matrices (of appropriate dimensions) inheriting a sparsity pattern dictated by the poset. The above definition of function multiplication is made so that it is consistent with standard matrix multiplication.

Theorem 1: Let \mathcal{P} be a poset. Under the usual definition of addition, and multiplication as defined in (1) the incidence algebra is an associative algebra (i.e. it is closed under addition, scalar multiplication and function multiplication).

Proof: Closure under addition and scalar multiplication is obvious. Let $f, g \in I_{\mathcal{P}}$, and consider elements x, y such that $x \not\preceq y$. If $x \not\preceq y$, there cannot exist a z such that $x \preceq z \preceq y$. Hence, in the above sum, either $f(x, z) = 0$ or $g(z, y) = 0$ for every z , and thus $h(x, y) = 0$. ■

A standard corollary of this theorem is the following.

Corollary 1: Suppose $A \in I_{\mathcal{P}}$ is invertible. Then $A^{-1} \in I_{\mathcal{P}}$.

B. Control-theoretic Preliminaries

In paper we are interested in discrete-time linear systems defined by difference equations as follows:

$$\begin{aligned} x[k+1] &= Ax[k] + B_1w[k] + B_2u[k] \\ z[k] &= C_1x[k] + D_{11}w[k] + D_{12}u[k] \\ y[k] &= C_2x[k] + D_{21}w[k] + D_{22}u[k]. \end{aligned}$$

In this description, $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^{n_u}$ is the control input, $y \in \mathbb{R}^{n_y}$ is the plant output, $w \in \mathbb{R}^{n_w}$ is the exogenous input, $z \in \mathbb{R}^{n_z}$ is the system output. Rather than working with state-space models, we will throughout be interested in

working with their frequency domain representation via the z -transform.

It can be shown [6] that if the above system is stabilizable and detectable (we assume this throughout the paper) then the system (denoted by P) can be described in frequency domain by the following map:

$$P(z) = \begin{bmatrix} P_{11}(z) & P_{12}(z) \\ P_{21}(z) & P_{22}(z) \end{bmatrix},$$

where $P(z) \in \mathbb{C}^{(n_w+n_u) \times (n_z+n_y)}$ is the overall system transfer function. Through the rest of this paper, we abbreviate notation and define $P_{22} = G$. We also assume G to be strictly proper, so that the stabilization of P is equivalent to the stabilization of G (see [6]). Furthermore, in Section III we assume that system G has an equal number of inputs and outputs (i.e. $n_u = n_y$). We will think of G being composed of several subsystems, each subsystem having one input and one output. We denote $\mathcal{R}_p^{m \times n}$ to be the set of rational-proper transfer matrices of dimension $m \times n$. We denote the set of stable proper transfer matrices by \mathcal{RH}_∞ . The entries of P can be shown to be rational proper transfer functions, i.e. $P \in \mathcal{R}_p^{(n_w+n_u) \times (n_z+n_y)}$. Given a controller $K \in \mathcal{R}_p^{n_y \times n_u}$, the closed-loop system has transfer function:

$$f(P, K) = P_{11} + P_{12}K(I - GK)^{-1}P_{21}.$$

We are interested in optimal controller-synthesis problems of the form:

$$\begin{aligned} & \text{minimize} && \| f(P, K) \| \\ & \text{subject to} && K \text{ stabilizes } P \\ & && K \in S, \end{aligned} \quad (2)$$

where S is some subspace of the space of controllers. In this paper, $\| \cdot \|$ is any norm on $\mathcal{R}_p^{n_z \times n_w}$, chosen to appropriately capture the performance of the closed-loop system. In this paper S will represent sparsity constraints on the controller K . It may be noted that for general P and S there is no known technique for solving problem (2).

Problem (2) as presented is a nonconvex problem in K . If the subspace constraint $K \in S$ were not present, then several techniques exist for solving the problem [6]. One approach towards a solution to the problem is to write an explicit parameterization of all stabilizing controllers for the problem. It is desirable to have the closed-loop transfer function be an affine function in the parameter, so that the problem becomes convex. We present one such parameterization here [4].

Let \mathcal{H}_{stab} denote the set of stable closed loop transfer matrices achieved by some controller that internally stabilizes the plant, i.e.,

$$\mathcal{H}_{stab} = \left\{ H \mid H = P_{11} + P_{12}K(I - GK)^{-1}P_{21}, \right. \\ \left. K \text{ stabilizes } P \right\}.$$

Let $R = K(I - GK)^{-1}$. Then it is well-known [4] that \mathcal{H}_{stab} can be parameterized in terms of R via

$$\mathcal{H}_{stab} = \{ P_{11} + P_{12}RP_{21} \mid RG \in \mathcal{RH}_\infty, R \in \mathcal{RH}_\infty, \\ I + GR \in \mathcal{RH}_\infty, (I + GR)G \in \mathcal{RH}_\infty \}. \quad (3)$$

Furthermore, given R , the controller K may be recovered by

$$K := h_G(R) = (I + RG)^{-1}R. \quad (4)$$

Lemma 1: Assume $G \in I_{\mathcal{P}}$. Then $R \in I_{\mathcal{P}}$ if and only if $K \in I_{\mathcal{P}}$.

Proof: Follows from the definition of R , (4) and Corollary 1. ■

Lastly, if G is stable, then \mathcal{H}_{stab} has a simple parameterization:

$$\mathcal{H}_{stab} = \{ P_{11} + P_{12}RP_{21} \mid R \in \mathcal{RH}_\infty \}. \quad (5)$$

III. EXAMPLES OF COMMUNICATION STRUCTURES ARISING FROM POSETS

In this section we study some examples of posets. Some communication structures that have been studied in the past (and arise in practice) can be described using posets [14], [15]. The intuition behind modeling communication among subsystems via posets is as follows. We say that subsystems i and j satisfy $i \preceq j$ if all the information that is available to subsystem j is also available to subsystem i . It means that subsystem i is more *information-rich*. We will formalize this notion in the next section.

A. Independent subsystems

For the trivial poset on the set $\{1, 2, \dots, n\}$ where there are no partial order relations between any of the elements (i.e. all the elements are independent of each other) corresponds to the case where the subsystems exchange no communication whatsoever (all subsystems have access to only their own information, thus K and G are diagonal). It is readily seen that this is just the case where one is required to stabilize n independent plants using independent controllers, a problem that reduces to a classical control problem. At the other extreme is the case where the poset is *totally ordered*. This is the case of nested control [14], which we study below.

B. Nested systems

This is a class of systems where the transfer functions have a block-triangular structure. Nested systems have been analyzed by Voulgaris [15], [14]. Such structures arise in cases where there are several subsystems, with each subsystem contained within the subsequent subsystem so that the arrangement forms a nest. There is one-way communication among the subsystems (say from the inside to the outside, or vice-versa).

For simplicity, consider a system with just two subsystems P_1 and P_2 . The internal subsystem is P_1 which can communicate information to the outer subsystem P_2 (but not vice-versa). The task at hand is to design a controller that obeys this same nested-communication architecture. The following is the set of plant outputs, control inputs, exogenous outputs and exogenous inputs respectively:

$$y := \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad u := \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad z := \begin{bmatrix} z_1 \\ z_{12} \\ z_2 \end{bmatrix} \quad w := \begin{bmatrix} w_1 \\ w_{12} \\ w_2 \end{bmatrix}$$

The sparsity pattern generated by the communication constraints for the controller and the plant are as follows:

$$G = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix} \quad K = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix}.$$

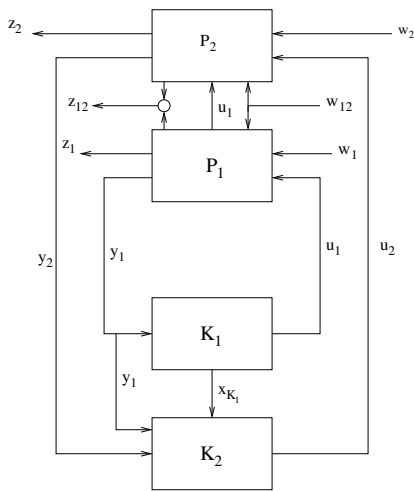


Fig. 2. A system with nested communication constraints.

Figure 2 depicts such a nested system in a block diagram. It is easy to see that G and K are matrices in the incidence algebra generated by the poset over $\{1, 2\}$ with $2 \preceq 1$. This is consistent with the intuition that since subsystem 2 has access to information from subsystem 1 to make decisions (K_{21} is not required to be 0), subsystem 2 is more information-rich. Voulgaris [14] showed that for such nested systems, the optimal control problem can be reduced to a convex problem in the Youla parameter. In the next section we will see that this result follows as a special case of a more general result that is true for all poset-systems.

C. Other examples

The example cited in the above subsection shows that nested systems are just special cases of those arising from posets. Clearly, many other communication structures can be modeled as posets. Some such examples include multi-chains, lattices and closures of directed acyclic graphs. A few such examples are shown in Fig. 3.

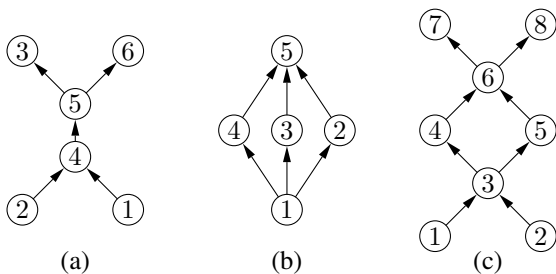


Fig. 3. Examples of other poset communication structures: (a) A multichain (b) A lattice (c) A directed acyclic graph.

IV. SYSTEMS WITH SAME PLANT AND CONTROLLER COMMUNICATION CONSTRAINTS

In this section we consider systems that are composed by interconnecting several subsystems. Each subsystem is assumed to be linear and time invariant. We consider an input-output framework where each subsystem is given as a transfer function matrix G . If $G \in \mathcal{R}_p^{n \times n}$, the system is

composed of n subsystems. Subsystem i consists of input i and output i (the transfer function between which is G_{ii}). In addition, input i can also affect another subsystem (say subsystem j) in which case $G_{ij}(z) \neq 0$ for almost all z . As in [11], we would like to consider communication constraints between the subsystems being modeled as sparsity constraints on the matrix $G(z)$. To this end we define some notation.

Suppose we have a collection of subsystems that are interconnected in a way that is consistent with the partial order structure of a poset $\mathcal{P} = (\{1, \dots, n\}, \preceq)$. The partial order represents the communication structure in the plant as follows:

Definition 4: The plant $G \in \mathcal{R}_p^{n \times n}$ is said to be *communication-constrained* by poset \mathcal{P} if whenever $j \not\preceq i$, subsystem i does not communicate information to subsystem j (i.e., $G_{ji}(z) = 0$).

This definition formalizes the notion of information-richness that we mentioned in the previous section, i.e., that $j \preceq i$ implies that j has access to more information that i since $G_{ji} \neq 0$.

In this section we are interested in the case where the controller K must mirror the communication constraints of the plant, i.e. if $i \not\preceq j$ then $K_{ij}(z) = 0$ (i.e K must also lie in the incidence algebra of \mathcal{P}). We denote the set of all stabilizing controllers that lie in the incidence algebra by $C_{stab}(\mathcal{P})$. Let the set of all closed loop transfer functions that are stabilized by $K \in C_{stab}(\mathcal{P})$ be denoted by $\mathcal{H}_{stab}(\mathcal{P})$. We are interested in parameterizing $C_{stab}(\mathcal{P})$ and $\mathcal{H}_{stab}(\mathcal{P})$. Our approach will be as follows. First we will construct an explicit stabilizing controller in $C_{stab}(\mathcal{P})$. Using this controller in the feedback loop, we reduce the problem to the case where the plant is stable and then use equation (5) to parameterize the set of all closed loop maps. Before we do so, we must make an important remark and a related assumption.

Remark Suppose we have a plant $G \in I_{\mathcal{P}}(\mathcal{R}_p)$ with G_{ij} unstable for some $i \neq j$. The task of internally stabilizing the plant G with a controller $K \in I_{\mathcal{P}}(\mathcal{R}_p)$ is *impossible*. This is because G_{ij} does not have a feedback path. To illustrate this consider an example with two subsystems forming a nest (i.e. the block-triangular case we saw Section III-B). Suppose

$$G = \begin{bmatrix} 0 & 0 \\ G_{21} & 0 \end{bmatrix}, \quad K = \begin{bmatrix} K_{11} & 0 \\ K_{21} & K_{22} \end{bmatrix},$$

where G_{21} is unstable and K is some stabilizing controller in the incidence algebra. By Theorem 1 and Corollary 1 it is easy to see that $R = K(I - GK)^{-1}$ is also in the incidence algebra (i.e., it is lower triangular). However, if this were the case, one can readily check that

$$(I + GR)G = \begin{bmatrix} 0 & 0 \\ G_{21} & 0 \end{bmatrix}.$$

It is impossible for this to be stable, thus the controller cannot be internally stabilizing, yielding a contradiction.

In addition, for the problem to make sense we also need

that the G_{ii} be internally stabilizable.

Assumptions

- (a) For $i \neq j$, $G_{ij}(z)$ is stable
- (b) The G_{ii} are stabilizable.

The argument in the remark above shows that if we are interested in plants with poset constraints, and if we want to stabilize the plant with a controller that also satisfies the same poset constraints, then Assumption (a) is necessary. This assumption is reasonable because it encompasses the case where subsystems pass static signals among each other (i.e. $G_{ij}(z)$ are constants). In fact, there may even be dynamics between subsystems as long as these dynamics are stable.

A. Explicit construction of a stabilizing controller

Armed with Assumption (a), we can now explicitly construct a stabilizing controller as follows. Suppose we are given $G \in I_{\mathcal{P}}(\mathcal{R}_p)$. Without loss of generality, we assume that G is lower triangular. (It is always possible to put it in triangular form by constructing a linear extension of the poset i.e. extending the partial order to a total order which is consistent [1, Prop 1.4].) We choose an R which is diagonal and such that $R_{ii}, R_{ii}G_{ii}, (I + G_{ii}R_{ii}), (I + G_{ii}R_{ii})G_{ii}$ are stable. We can always choose such an R because the existence of such R_{ii} are equivalent to the G_{ii} being internally stabilizable. Moreover, since R is diagonal, it is also in the incidence algebra. The diagonal entries of $RG, R, I + GR$, and $(I + GR)G$ are stable by construction. It can be easily verified that the off-diagonal entries of these matrices are stable because they are sums and products of stable entries (recall that by assumption (a) the G_{ij} are stable for $i \neq j$). Hence all four of the transfer functions are stable, and by (3) we have a stable closed loop. We choose the corresponding stabilizing controller to be simply $K_{nom} = (I + RG)^{-1}R$.

B. Parametrization of all stabilizing controllers in the incidence algebra

Note that since R is diagonal (and hence trivially in the incidence algebra), by Lemma 1, K_{nom} is in the incidence algebra and also stabilizing. We use K_{nom} in the closed loop to stabilize the plant, so that the problem is reduced to the case of a stable plant. Now we treat the system with K_{nom} in the closed loop as the “new plant”, which is already stable. Let

$$\tilde{P}(z) = \begin{bmatrix} \tilde{P}_{11}(z) & \tilde{P}_{12}(z) \\ \tilde{P}_{21}(z) & \tilde{G}(z) \end{bmatrix},$$

where \tilde{P} is the closed loop map obtained via K_{nom} . Since this is stable, by (5) we have that the set of all stable closed loop maps achievable by a controller in $I_{\mathcal{P}}$ is

$$\mathcal{H}_{stab}(\mathcal{P}) = \left\{ \tilde{P}_{11} + \tilde{P}_{12}R\tilde{P}_{21} \mid R \in \mathcal{RH}_{\infty} \cap I_{\mathcal{P}} \right\}.$$

Finally, the set of all stabilizing controllers in the incidence algebra is

$$\mathcal{C}_{stab}(\mathcal{P}) = \left\{ (I + R\tilde{G})^{-1}R \mid R \in \mathcal{RH}_{\infty} \cap I_{\mathcal{P}} \right\}.$$

Using this parameterization, one can reduce the optimal control problem (2) to the convex problem:

$$\begin{aligned} & \text{minimize} && \| \tilde{P}_{11} + \tilde{P}_{12}R\tilde{P}_{21} \| \\ & \text{subject to} && R \in \mathcal{RH}_{\infty} \cap I_{\mathcal{P}}. \end{aligned}$$

Remark Rotkowitz and Lall present a property known as *quadratic invariance* in their paper [11]. A plant P and a subspace (of controllers) S is defined to be quadratically invariant if for every $K \in S$, $KGK \in S$. Note that our problem is quadratically invariant. We have $G \in I_{\mathcal{P}}(\mathcal{R}_p^{n \times n})$ and the subspace constraint is also $K \in I_{\mathcal{P}}(\mathcal{R}_p^{n \times n})$. By Theorem 1, $I_{\mathcal{P}}(\mathcal{R}_p^{n \times n})$ is an algebra of matrices, hence $KGK \in I_{\mathcal{P}}(\mathcal{R}_p^{n \times n})$. One of the advantages of our approach is that the problem has more structure, and hence (unlike their work) we do not need to assume the existence of a nominal stable controller (we explicitly construct a controller, and it need not be stable for our parameterization to work). Also our parameterization is free, i.e. it does not have any constraints apart from $R \in \mathcal{RH}_{\infty} \cap I_{\mathcal{P}}$.

V. SYSTEMS WITH DIFFERENT PLANT AND CONTROLLER COMMUNICATION CONSTRAINTS

A. Computational Tractability

In this section we examine a more general setting where the controller is not necessarily required to mirror the communication constraints of the plant. In the previous section, we viewed the system as a collection of interconnected subsystems. We move away from this view here. However, we still wish to study such systems from a partial-order point of view. Instead of having a partial order on subsystems, we now have a partial order on the set of inputs and a (possibly different) partial order on the set of outputs. The communication constraints between the inputs and outputs in the plant and the controller are given by a pair of maps between the two posets. We show that if the pair of maps have the special property of being a *Galois connection* [10],[8] the controller synthesis problem (subject to the communication constraints) is amenable to convex optimization.

Definition 5: Let \mathcal{P} and \mathcal{Q} be finite posets. A pair of maps (ϕ, ψ) where $\phi : \mathcal{P} \rightarrow \mathcal{Q}$ and $\psi : \mathcal{Q} \rightarrow \mathcal{P}$ is said to form a *Galois connection* if it satisfies the following property:

$$q \preceq \phi(p) \Leftrightarrow \psi(q) \preceq p \text{ for all } p \in \mathcal{P} \text{ and } q \in \mathcal{Q}.$$

Let $G(z) \in \mathcal{R}_p^{n_y \times n_u}$ be a system with n_u inputs and n_y outputs. Let $\mathcal{P} = (\{1, \dots, n_u\}, \preceq_{\mathcal{P}})$ and $\mathcal{Q} = (\{1, \dots, n_y\}, \preceq_{\mathcal{Q}})$ be posets on the index sets of the inputs and outputs respectively (we will often omit the subscript on the inequality symbol when it is clear from the context).

Definition 6:

- (1) We say that the plant is *communication-constrained* by ϕ if whenever $q \not\preceq \phi(p)$ input p cannot communicate with output q (i.e. $G_{qp} = 0$ for almost all z).
- (2) We say that the controller is *communication-constrained* by ψ if whenever $p \not\preceq \psi(q)$ output q cannot communicate with input p (i.e. $K_{pq} = 0$ for almost all z).

The set of all controllers that are communication-constrained by ψ is a subspace. This subspace is denoted by $S(\psi)$. Similarly, the set of all plants that are constrained by ϕ are denoted by $S(\phi)$.

Remark These definitions generalize the notion of an incidence algebra to the case when we have two posets. For instance if the two posets are the same (i.e. $\mathcal{P} = \mathcal{Q}$) and we choose the Galois connection to be the identity map between them, then it can be easily verified that $S(\phi) = S(\psi) = I_{\mathcal{P}}$.

Theorem 2: Let \mathcal{P} and \mathcal{Q} be posets on the index sets of the inputs and outputs respectively. Let $G \in \mathcal{R}_{\mathcal{P}}^{n_y \times n_u} \in S(\phi)$ and $K \in \mathcal{R}_{\mathcal{Q}}^{n_u \times n_y} \in S(\psi)$. Then $KGK \in S(\psi)$ and $GKG \in S(\phi)$.

Proof: We prove $KGK \in S(\psi)$ (the proof for $GKG \in S(\phi)$ is similar). Suppose $G \in S(\phi)$ and $K \in S(\psi)$. Assume $i \not\leq \psi(j)$, and consider $(KGK)_{ij} = \sum_k \sum_l K_{ik} G_{kl} K_{lj}$. If $i \not\leq \psi(k)$ then $K_{ik} = 0$. Similarly if $l \not\leq \psi(j)$ then $K_{lj} = 0$ and if $k \not\leq \phi(l)$ then $G_{kl} = 0$. Hence, nonzero terms in the above summation may only occur for indices that satisfy $i \leq \psi(k)$, $k \leq \phi(l)$, and $l \leq \psi(j)$. Using Definition 5, since (ϕ, ψ) form a Galois connection $k \leq \phi(l) \Leftrightarrow \psi(k) \leq l$. Hence, $i \leq \psi(k) \leq l \leq \psi(j)$. However, these three conditions imply that $i \leq \psi(j)$ (contradicting our initial assumption) and thus $(KGK)_{ij} = 0$. ■

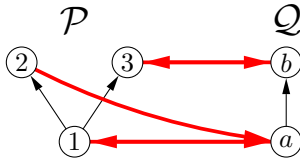


Fig. 4. Posets \mathcal{P} and \mathcal{Q} with a pair of maps that form a Galois connection.

Remark A consequence of Theorem 2 is that problems with such communication constraints are quadratically invariant [11], and can hence be recast as convex problems in the Youla parameter.

B. Example

Consider a system with two inputs ($\mathcal{P} = \{1, 2, 3\}$) and two outputs ($\mathcal{Q} = \{a, b\}$). The input and output spaces are endowed with a partial order structure and a pair of maps (ϕ, ψ) that form a Galois connection (see Fig. 4).

$$\begin{aligned} \phi(1) = a & \quad \phi(2) = a & \quad \phi(3) = b \\ \psi(a) = 1 & \quad \psi(b) = 3. \end{aligned}$$

This order-theoretic structure results in communication constraints on the plant G and controller K as shown in (6) below. In the plant, output a can be influenced by inputs 1, 2, and 3, and output b can only be influenced by input 1. On the controller side, input 1 can be affected by outputs a and b , input 3 by output b , while input 2 can be affected by none of the outputs. The corresponding G and K would be

sparse with the following sparsity structure:

$$G = \begin{bmatrix} * & * & * \\ 0 & 0 & * \end{bmatrix} \quad K = \begin{bmatrix} * & * \\ 0 & 0 \\ 0 & * \end{bmatrix}. \quad (6)$$

It can be easily verified that the problem with these sparsity constraints is quadratically invariant. The quadratic invariance of the problem depends only on the interconnection between the inputs and outputs in the controller and plant. The emphasis here is that when such constraints are modeled using order-theoretic considerations as explained above, quadratic invariance and the attendant convexity guarantees follow.

VI. CONCLUSION

We presented a new framework for modeling and analyzing a class of decentralized control problems based on posets. We showed that problems of this type have several appealing properties: (a) They unify many examples studied in the literature under a common theoretical framework, (b) the set of stabilizing controller has a nice parametrization (c) one can also generalize the theory to the setting where plant and controller may have different constraints.

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