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Null Controllability of Discrete-time Linear Systems with Input and State Constraints

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Abstract— **This paper presents necessary and sufficient conditions for null controllability of discrete-time linear systems subject to both input and state constraints. The classical results for linear systems without constraints by Kalman and Hautus and for linear systems with only input constraints by Evans, Nguyen and Sontag can be obtained from our main result as particular cases.**

Index Terms— **Linear systems, state constraints, controllability.**

I. Introduction

The notion of controllability has played a central role throughout the history of modern control theory. For linear systems Kalman [8] and Hautus [5] studied this property in the sixties and early seventies and came up with complete characterizations in the well-known algebraic conditions. Also in the case that input constraints are present for the linear system, with a constraint set being a closed convex cone, the controllability property has been characterized by Brammer [2] for continuous-time systems and by Evans and Murthy for the discrete-time single input case in [4], while Nguyen [9] and Evans [3] provided necessary and sufficient conditions in the general case. In case the input constraint set is a bounded set that contains the origin in its interior, the problem of null controllability of input constrained (continuous-time and discrete-time) linear systems is solved by Sontag in [10].

So, although the unconstrained and input-constrained null controllability problems are solved completely (provided certain conditions are imposed on the constraint sets), the case where state constraints are active on the linear system is largely overlooked in the literature. Only the continuous-time case was recently considered by the authors in [7], where a full algebraic characterization was given for outputs (combined states and inputs) taking values in a *convex cone* under a right-invertibility condition. These conditions include the original results by Kalman, Hautus and Brammer as special cases.

In this paper we will study the problem of null controllability for discrete-time linear systems with constraints on input and state variables, even allowing more general constraint sets as in the continuous-time case. These constraints will be formulated as constraints on a suitably defined output variable of the system that can take values in either a convex cone or a bounded set. We will present necessary and sufficient conditions for the null controllability using that the underlying linear system satisfies a right-invertibility condition on its transfer matrix. In other words, for the class of "right-invertible" linear systems we fully characterize null controllability of linear systems involving both state and input constraints or combinations of them. The original results of Kalman, Hautus, Nguyen, Evans, Murthy and Sontag (for the constraint sets considered here) are recovered as particular cases of these conditions, as in their cases the right-invertible condition is trivially satisfied.

II. Notation

The spaces $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{N}$ denote the set of real numbers, complex numbers and nonnegative integers, respectively. For a complex number $\lambda \in \mathbb{C}$, $|\lambda|$ denotes its modulus. For a matrix $A \in \mathbb{C}^{n\times m}$, we write A^T for its transpose and A^* for its complex conjugate transpose. Moreover, for a matrix $A \in \mathbb{R}^{n\times m}$, its kernel $\ker A$ is defined as $\{x \in \mathbb{R}^m \mid Ax = 0\}$ and its image $\operatorname{im} A$ by $\{Ax \mid x \in \mathbb{R}^m\}$. For two subspaces $\mathcal{X}_1$ and $\mathcal{X}_2$ of $\mathbb{R}^n$, we write $\mathcal{X}_1 \oplus \mathcal{X}_2 = \mathbb{R}^n$, when their intersection $\mathcal{X}_1 \cap \mathcal{X}_2$ is equal to $\{0\}$ and the sum $\mathcal{X}_1 + \mathcal{X}_2 := \{x_1 + x_2 \mid x_1 \in \mathcal{X}_1,\ x_2 \in \mathcal{X}_2\}$ is equal to $\mathbb{R}^n$. For a subspace $\mathcal{X}$ in $\mathbb{R}^n$, we denote its orthogonal complement $\mathcal{X}^\perp$ by $\{z \in \mathbb{R}^n \mid z^T x = 0 \text{ for all } x \in \mathcal{X}\}$. For a set $\mathcal{Y} \subseteq \mathbb{R}^n$, we define its dual cone $\mathcal{Y}^*$ as $\{w \in \mathbb{R}^p \mid w^T y \geqslant 0 \text{ for all } y \in \mathcal{Y}\}$. For two subsets $\mathcal{X}_1$ and $\mathcal{X}_2$ of $\mathbb{R}^n$, we denote their set difference $\{x \in \mathcal{X}_1 \mid x \notin \mathcal{X}_2\}$ by $\mathcal{X}_1 \setminus \mathcal{X}_2$. For two vectors $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$, $\operatorname{col}(x_1, x_2)$ will denote the vector in $\mathbb{R}^{n_1+n_2}$ obtained by stacking x_1 over x_2. The space of all sequences $\{u_k\}_{k\in\mathbb{N}}$ with $u_k \in \mathbb{R}^m$ is denoted by $\mathcal{S}^m$.

III. Problem definition

Consider the linear system

$$x[k+1] = Ax[k] + Bu[k] \tag{1a}$$
$$y[k] = Cx[k] + Du[k], \tag{1b}$$

where $x[k] \in \mathbb{R}^n$ is the state at time $k \in \mathbb{N}$, $u[k] \in \mathbb{R}^m$ is the input, $y[k] \in \mathbb{R}^p$ is the output, and all matrices are of appropriate sizes. For a given initial state x_0 and input sequence $u \in \mathcal{S}^m$, there exists a unique solution to (1) with $x[0] = x_0$, which is denoted by $x^{x_0,u}$. The corresponding

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output will be denoted by $y^{x_0,u}$. Sometimes we will write $\Sigma(A,B,C,D)$ to refer to the linear system (1).

Together with (1), we consider the output constraints

$$y[k] \in \mathcal{Y}, \tag{2}$$

where $\mathcal{Y}$ is a subset of $\mathbb{R}^p$. The inclusion (2) can express input constraints, state constraints or their combinations.

We say that a state $x_0 \in \mathbb{R}^n$ is *feasible as initial state* for (1)-(2) if there exists an input $u \in \mathcal{S}^m$ such that $y^{x_0,u}[k] \in \mathcal{Y}$ for all $k \in \mathbb{N}$. The set of all such initial states is denoted by $\mathcal{X}_0$. To indicate the need to restrict the set of feasible initial states we study the following example.

Example III.1 The "discrete-time double integrator"

$$\begin{aligned} x_1[k+1] &= x_1[k] + x_2[k] && \text{(3a)}\\ x_2[k+1] &= x_2[k] + u[k] && \text{(3b)}\\ y[k] &= x_1[k], && \text{(3c)} \end{aligned}$$

is given with the "position" constraint $y[k] \geqslant 0$. Hence, $\mathcal{Y} = [0,\infty)$. Clearly, one has

$$\mathcal{X}_0 = \{\bar{x} \in \mathbb{R}^2 \mid \bar{x}_1 \geqslant 0 \text{ and } \bar{x}_1 + \bar{x}_2 \geqslant 0\}. \tag{4}$$

Indeed, observe that $y[0] = x_1[0]$ and $y[1] = x_1[0] + x_2[0]$ cannot be influenced by the control input u. Hence, for a state $\bar{x}$ to be feasible as an initial state, one needs at least that $y[0] \in \mathcal{Y}$ and $y[1] \in \mathcal{Y}$. This is also sufficient as $y[k+2] = x_1[k] + 2x_2[k] + u[k]$ can be given any desirable value by proper selection of $u[k]$ for all $k \geqslant 0$.

In case we would use the "velocity" constraint $y[k] = x_2[k] \geqslant 0$ instead, we obtain that

$$\mathcal{X}_0 = \{\bar{x} \in \mathbb{R}^2 \mid \bar{x}_2 \geqslant 0\}, \tag{5}$$

which only requires one inequality constraint as $y[0] = x_2[0]$ and $y[k+1] = x_2[k] + u[k]$.

We say that a linear system of the form (1) is *null controllable under the constraints* (2) if for each feasible initial state $x_0 \in \mathcal{X}_0$ there exist an input $u \in \mathcal{S}^m$ and a positive number $T \in \mathbb{N}$ such that $x^{x_0,u}[T] = 0$ and $y^{x_0,u}[k] \in \mathcal{Y}$ for all $k = 0, 1, \ldots, T$.

IV. Classical controllability results

Two particular cases of our framework are among the classical results of systems theory.

A. Linear systems.

First we consider the unconstrained case, i.e. $\mathcal{Y} = \mathbb{R}^p$. Clearly, one gets $\mathcal{X}_0 = \mathbb{R}^n$. In this case, the solution to the null controllability problem is an easy consequence of the classical controllability conditions of Kalman and Hautus [5], [6]. Indeed, in the continuous-time case without constraints ($\mathcal{Y} = \mathbb{R}^p$) it is well known that the concepts of reachability (steering the origin to any state), controllability (steering any state to any other state) and null controllability (steering any state to the origin) are equivalent concepts (see e.g. [11]). However, in the discrete-time unconstrained case, controllability and reachability are equivalent, but null controllability is a weaker concept (e.g. the system $x[k+1] = 0$ is null controllable, but not controllable/reachable), which is characterized as follows.

Theorem IV.1 *Consider the linear system* (1) *and the constraints* (2) *with* $\mathcal{Y} = \mathbb{R}^p$*. Then, it is null controllable if, and only if,*

$$\left.\begin{array}{ll} \lambda \in \mathbb{C} \setminus \{0\}, & z \in \mathbb{C}^n, \\ z^*A = \lambda z^*, & z^*B = 0, \end{array}\right\} \Rightarrow \quad z = 0. \tag{6}$$

B. Linear systems with input constraints.

Let $C = 0$ and $D = I$. Note that the problem reduces now to establishing null controllability for the system

$$x[k+1] = Ax[k] + Bu[k]$$

with input constraints $u[k] = y[k] \in \mathcal{Y}$ for all $k \in \mathbb{N}$. Clearly, one gets $\mathcal{X}_0 = \mathbb{R}^n$. We consider two situations for the set $\mathcal{Y}$:

Assumption IV.2 The set $\mathcal{Y}$ is a solid closed polyhedral cone, i.e. there exists a matrix $Y \in \mathbb{R}^{q\times p}$ such that $\mathcal{Y} = \{y \in \mathbb{R}^p \mid Yy \geqslant 0\}$ and $\mathcal{Y}$ has a non-empty interior.

Assumption IV.3 The set $\mathcal{Y}$ is a *bounded* set (i.e. there exists an $M \in \mathbb{R}$ such that $\|y\| \leqslant M$ for all $y \in \mathcal{Y}$) that contains zero in its interior.

Under Assumption IV.2, the answer to the null controllability question can be based upon the controllability result by Evans [3] or by Nguyen [9], as formulated in the following theorem.

Theorem IV.4 *Consider the linear system* (1) *and the constraints* (2) *with* $C = 0$ *and* $D = I$ *and* $\mathcal{Y}$ *a solid closed polyhedral cone as in Assumption IV.2. Then, it is null controllable if, and only if, the following implications hold:*

$$\left.\begin{array}{ll} \lambda \in \mathbb{C} \setminus \{0\}, & z \in \mathbb{C}^n, \\ z^*A = \lambda z^*, & z^*B = 0, \end{array}\right\} \Rightarrow \quad z = 0 \tag{7a}$$

$$\left.\begin{array}{ll} \lambda \in (0,\infty), & z \in \mathbb{R}^n \\ z^TA = \lambda z^T, & B^Tz \in \mathcal{Y}^* \end{array}\right\} \Rightarrow \quad z = 0. \tag{7b}$$

In case the input constraint set $\mathcal{Y}$ is a *bounded* set, Sontag [10] provides the solution to the null controllability problem.

Theorem IV.5 *Consider the linear system* (1) *and the constraints* (2) *with* $C = 0$ *and* $D = I$ *and* $\mathcal{Y}$ *a bounded set that contains zero in its interior as in Assumption IV.3. Then, it is null controllable if, and only if, the following implications hold:*

$$\left.\begin{array}{ll} \lambda \in \mathbb{C} \setminus \{0\}, & z \in \mathbb{C}^n \\ z^*A = \lambda z^*, & z^*B = 0 \end{array}\right\} \Rightarrow \quad z = 0. \tag{8a}$$

$$\left.\begin{array}{ll} \lambda \in \mathbb{C}, \ |\lambda| > 1, \\ z \in \mathbb{C}^n, \ z^*A = \lambda z^*, \end{array}\right\} \Rightarrow \quad z = 0. \tag{8b}$$

Note that conditions (7a) and (8a) are equivalent to null controllability without constraints as provided in Theorem IV.1.

The main contribution of the paper is to give necessary and sufficient conditions for controllability in the presence of both input and state constraints.

V. Linear systems with input/state constraints

We will use the following assumption in the paper.

Assumption V.1 The transfer matrix $D+C(zI-A)^{-1}B$ is right invertible as a rational matrix.

To make it easier to deal with the constraints (2), we will transform (1) into a canonical form that is based on [1]. We will briefly recall some of the notions from [1] and [11] and we refer to the Section VIII for some more particular facts.

A. Preliminaries in geometric control theory

Consider the linear system (1). We define the *controllable subspace* and *unobservable subspace* as $\langle A \mid \operatorname{im} B\rangle := \operatorname{im} B + A\operatorname{im} B + \cdots + A^{n-1}\operatorname{im} B$ and $\langle \ker C \mid A\rangle := \ker C \cap A^{-1}\ker C \cap \cdots \cap A^{1-n}\ker C$, respectively.

We say that a subspace $\mathcal{V}$ is *output-nulling controlled invariant* if for some matrix K the inclusions

$$(A-BK)\mathcal{V} \subseteq \mathcal{V} \text{ and } \mathcal{V} \subseteq \ker(C-DK) \tag{9}$$

hold. As the set of such subspaces is non-empty and closed under subspace addition, it has a maximal element $\mathcal{V}^*$. The notation $\mathcal{K}(\mathcal{V})$ stands for the set $\{K \mid (A-BK)\mathcal{V} \subseteq \mathcal{V} \text{ and } \mathcal{V} \subseteq \ker(C-DK)\}$.

Dually, we say that a subspace $\mathcal{T}$ is *input-containing conditioned invariant* if for some matrix L the inclusions

$$(A-LC)\mathcal{T} \subseteq \mathcal{T} \text{ and } \operatorname{im}(B-LD) \subseteq \mathcal{T} \tag{10}$$

hold. As the set of such subspaces is non-empty and closed under subspace intersection, it has a minimal element $\mathcal{T}^*$. Whenever the system Σ is clear from the context, we simply write $\mathcal{T}^*$. The notation $\mathcal{L}(\mathcal{T})$ stands for the set $\{L \mid (A-LC)\mathcal{T} \subseteq \mathcal{T} \text{ and } \operatorname{im}(B-LD) \subseteq \mathcal{T}\}$.

A subspace $\mathcal{R}$ is called an *output-nulling controllability subspace* if for all $x_0, x_1 \in \mathcal{R}$ there exist $T \geqslant 0$ and an input sequence $u \in \mathcal{S}^m$ such that $x^{x_0,u}[0] = x_0$, $x^{x_0,u}[T] = x_1$, and $y^{x_0,u}[k] = 0$ for all $k = 0, 1, \ldots, T$. The set of all such subspaces admits a maximal element. This maximal element is denoted by $\mathcal{R}^*$. It is known, see e.g. [1], that

$$\mathcal{R}^* = \mathcal{V}^* \cap \mathcal{T}^*. \tag{11}$$

The maximal number of steps needed to steer $x_0 \in \mathcal{R}^*$ to $x_1 \in \mathcal{R}^*$ is equal to n (being equal to the state dimension of (1)).

We sometimes write $\mathcal{V}^*(A,B,C,D)$, $\mathcal{T}^*(A,B,C,D)$ and $\mathcal{R}^*(A,B,C,D)$ to make the dependence on (A,B,C,D) explicit.

B. Canonical form

Next, we will transform the system (1) in a canonical form that makes it easier to deal with the constraints (2). Let $\mathcal{V}^*$ and $\mathcal{T}^*$, respectively, denote the largest output-nulling controlled invariant and the smallest input-containing conditioned invariant subspaces of the system $\Sigma(A,B,C,D)$. Also let $K \in \mathcal{K}(\mathcal{V}^*)$. Apply the pre-compensating feedback $u[k] = -Kx[k] + v[k]$, where $v[k]$ is the new input. Then, (1) becomes

$$x[k+1] = (A-BK)x[k] + Bv[k] \tag{12a}$$
$$y[k] = (C-DK)x[k] + Dv[k]. \tag{12b}$$

Obviously, null controllability is invariant under this feedback. Moreover, the systems $\Sigma(A,B,C,D)$ and $\Sigma(A-BK,B,C-DK,D)$ share the same $\mathcal{V}^*$ and $\mathcal{T}^*$ due to Proposition VIII.1 (see Section VIII). Suppose that the transfer matrix $D+C(zI-A)^{-1}B$ is right invertible as a rational matrix (as in Assumption V.1). Proposition VIII.2 implies that the state space $\mathbb{R}^n$ admits the following decomposition

$$\mathbb{R}^n = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3, \tag{13}$$

where $\mathcal{X}_2 = \mathcal{R}^* = \mathcal{V}^* \cap \mathcal{T}^*$, $\mathcal{V}^* = \mathcal{X}_1 \oplus \mathcal{X}_2$, and $\mathcal{T}^* = \mathcal{X}_2 \oplus \mathcal{X}_3$. Let the dimensions of the subspaces $\mathcal{X}_i$ be n_i. Also let the vectors $\{w_1, w_2, \ldots, w_n\}$ be a basis for $\mathbb{R}^n$ such that the first n_1 vectors form a basis for $\mathcal{X}_1$ and the second n_2 for $\mathcal{X}_2$, and the last n_3 for $\mathcal{X}_3$. Also let $L \in \mathcal{L}(\mathcal{T}^*)$. One immediately gets from $\mathcal{V}^* \subseteq \ker(C-DK)$ and $\operatorname{im}(B-LD) \subseteq \mathcal{T}^*$ that

$$B - LD \simeq \begin{bmatrix} 0 \\ B_2' \\ B_3' \end{bmatrix} \tag{14}$$
$$C - DK \simeq \begin{bmatrix} 0 & 0 & C_3 \end{bmatrix}, \tag{15}$$

where $\simeq$ indicates that $B-LD$ is transformed in the coordinates that are adapted to the above basis. Here B_2', B_3', and C_3 are $n_2 \times m$, $n_3 \times m$ and $p \times n_3$ matrices, respectively. Note that $(A-BK-LC+LDK)\mathcal{V}^* \subseteq \mathcal{V}^*$ and $(A-BK-LC+LDK)\mathcal{T}^* \subseteq \mathcal{T}^*$ according to Proposition VIII.1. Therefore, the matrix $(A-BK-LC+LDK)$ is of the form $\left[\begin{smallmatrix} * & 0 & 0 \\ * & * & * \\ 0 & 0 & * \end{smallmatrix}\right]$ in the new coordinates, where the row (column) blocks have n_1, n_2, and n_3 rows (columns), respectively. Let the matrices K and L be partitioned as

$$K = \begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix} \text{ and } L = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix},$$

where K_k and L_k are $m \times n_k$ and $n_k \times m$ matrices, respectively. With these partitions, one gets

$$A - BK \simeq \begin{bmatrix} A_{11} & 0 & L_1C_3 \\ A_{21} & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \tag{16a}$$
$$B \simeq \begin{bmatrix} L_1D \\ B_2 \\ B_3 \end{bmatrix}, \tag{16b}$$

where $A_{\ell k}$ and B_ℓ are matrices of the sizes $n_\ell \times n_k$ and $n_\ell \times m$. Now, one can write (12) in the new coordinates as

$$x_1[k+1] = \quad A_{11}x_1[k] \qquad\qquad + L_1 y[k] \tag{17a}$$

$$x_2[k+1] = \quad A_{21}x_1[k] + A_{22}x_2[k] \quad + A_{23}x_3[k] + B_2 v[k] \tag{17b}$$

$$x_3[k+1] = \qquad\qquad A_{33}x_3[k] + B_3 v[k] \tag{17c}$$

$$y[k] = \qquad\qquad C_3 x_3[k] + Dv[k]. \tag{17d}$$

Hence, after the pre-compensating feedback $u[k] = -Kx[k] + v[k]$ and the similarity transformation $\bar{x} = Sx$, corresponding to the decomposition $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 = \mathbb{R}^n$ (note that we dropped the symbol '$^-$' in $\bar{x}$ in (17) for shortness) we obtained the system description as in (17), which corresponds to the system matrices

$$\left(\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right) = \left(\begin{array}{c|c} S(A-BK)S^{-1} & SB \\ \hline CS^{-1} & \bar{D} \end{array}\right) = \left(\begin{array}{ccc|c} A_{11} & 0 & L_1C_3 & L_1D \\ A_{21} & A_{22} & A_{23} & B_2 \\ 0 & 0 & A_{33} & B_3 \\ \hline 0 & 0 & C_3 & D \end{array}\right) \tag{18}$$

Some interesting observations can be made for this model, based upon Assumption V.1. First of all, Assumption V.1 implies that $[C \;\; D]$ in (1) must have full row rank (see Proposition VIII.2), which in turn implies that $[C_3 \;\; D]$ must have full row rank. Moreover, it holds that

$$\mathcal{T}^*(A_{33}, B_3, C_3, D) = \mathbb{R}^{n_3} \tag{19a}$$

$$\mathcal{V}^*(A_{33}, B_3, C_3, D) = \{0\} \tag{19b}$$

by construction. According to Proposition VIII.2 this implies that $C_3(zI - A_{33})^{-1}B_3 + D$ is right-invertible and allows actually a polynomial inverse $H^0 + H^1 z + \ldots + H^h z^h$ for suitable matrices H^i, $i = 0, \ldots, h$.

Interestingly, the above transformation to (17) reveals already directly some necessary conditions for null controllability. Indeed, (17a) indicates that the null controllability of the x_1-dynamics can only take place via the "control variable" y, which is constrained to be in $\mathcal{Y}$. Hence, this indicates that at least some input-constrained null controllability conditions should hold for the x_1-dynamics as formulated in Theorem IV.4 or Theorem IV.5 (depending on properties of $\mathcal{Y}$) to guarantee controllability for (1) under the constraints (2).

C. Characterization of the set $\mathcal{X}_0$

The applied transformation enables the characterizations of the set $\mathcal{X}_0$.

Theorem V.2 *Consider the system* (1) *with the constraint* (2)*. Suppose that Assumption V.1 holds. Then, the set of initially feasible states can be given by*

$$\begin{aligned} \mathcal{X}_0 = \{x_0 \in \mathbb{R}^n \mid & \textit{there exists } (u^0, u^1, \ldots, u^{n_3-1}) \textit{ such} \\ & \textit{that } (Cx_0 + Du^0,\ CAx_0 + CBu^0 + Du^1, \\ & CA^2x_0 + CABu^0 + CBu^1 + Du^2, \ldots, \\ & CA^{n_3-1}x_0 + CA^{n_3-2}Bu^0 + CA^{n_3-3}Bu^1 + \cdots + \\ & + CBu^{n_3-2} + Du^{n_3-1}) \in \mathcal{Y}^{n_3}\}, \end{aligned} \tag{20}$$

where n_3 denotes the dimension of $\mathcal{X}_3$ (i.e. $n_3 = \dim \mathcal{T}^* - \dim \mathcal{R}^*$). In case $\mathcal{Y}$ satisfies Assumption IV.2 or IV.3, then the set $\mathcal{X}_0$ has a non-empty interior.

Due to space restrictions the proof is omitted.

Interestingly, we only have to check the output conditions on the (discrete) time interval $\{0, 1, \ldots, n_3 - 1\}$ to determine if x_0 lies in $\mathcal{X}_0$ or not. Example III.1 already illustrated this, as $n_3 = 2$ in the first case ($\dim \mathcal{T}^* = 2$, $\dim \mathcal{V}^* = \dim \mathcal{R}^* = 0$), while n_3 is 1 for the second case ($\dim \mathcal{T}^* = \dim \mathcal{V}^* = 1$ and $\dim \mathcal{R}^* = 0$). Note also that in general the maximal output-nulling controlled invariant subspace $\mathcal{V}^*(A, B, C, D)$ ($\mathcal{X}_1 \oplus \mathcal{X}_2$) lies in $\mathcal{X}_0$ and that $\mathcal{X}_0$ is a non-trivial set in the sense that it has a non-empty interior.

D. Main results

The following theorem is the main result of the paper.

Theorem V.3 *Consider the linear system* (1)*. Suppose that Assumption V.1 holds. Then, the system is null controllable under the constraints* (2) *with a conic constraint set as in Assumption IV.2 if, and only if, the following implications hold*

$$\left.\begin{array}{c} \lambda \in \mathbb{C} \setminus \{0\},\ z \in \mathbb{C}^n, \\ z^*A = \lambda z^*,\ z^*B = 0 \end{array}\right\} \Rightarrow\ z = 0 \tag{21a}$$

$$\left.\begin{array}{c} \lambda \in (0, \infty),\ z \in \mathbb{R}^n, \\ w \in \mathcal{Y}^*,\ \begin{bmatrix} z^T & w^T \end{bmatrix} \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} = 0 \end{array}\right\} \Rightarrow\ z = 0. \tag{21b}$$

The system is null controllable under the constraints (2) *with a bounded constraint set as in Assumption IV.3 if, and only if, the following implications hold*

$$\left.\begin{array}{c} \lambda \in \mathbb{C} \setminus \{0\},\ z \in \mathbb{C}^n, \\ z^*A = \lambda z^*,\ z^*B = 0 \end{array}\right\} \Rightarrow\ z = 0 \tag{22a}$$

$$\left.\begin{array}{c} |\lambda| > 1,\ z \in \mathbb{C}^n, \\ w \in \mathbb{C}^n,\ \begin{bmatrix} z^* & w^* \end{bmatrix} \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} = 0 \end{array}\right\} \Rightarrow z = 0 \tag{22b}$$

Interestingly, Kalman's, Hautus', Evans', Nguyen's and Sontag's results are recovered as particular cases of Theorem V.3, as the right-invertibility Assumption V.1 is trivially satisfied in their cases ($C = I$ and $D = 0$).

VI. Examples

Reconsider Example III.1 with $\mathcal{Y} = [0, \infty)$, i.e.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix};\ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix};\ C = \begin{pmatrix} 1 & 0 \end{pmatrix};\ D = 0.$$

Note that the transfer function $\frac{1}{z^2-2z+1}$ for this system is invertible as a rational function. As this system is obviously (null) controllable without any constraints, (21a) is satisfied. To consider (21b) we compute the system matrix

$$\begin{pmatrix} A-\lambda I & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

which is invertible for any λ and thus (21b) is satisfied, which implies that the system is null controllable under the *position* constraint $y[k] = x_1[k] \geqslant 0$, $k \in \mathbb{N}$.

Consider the *velocity* constrained system, i.e. $y = x_2 \geqslant 0$, C becomes $(0\ 1)$ and $\mathcal{Y} = [0, \infty)$. The transfer function, being $\frac{1}{z-1}$, is also invertible and the unconstrained system remains, of course, null controllable. However, null controllability under the output/state constraint $y = x_2 \geqslant 0$ is lost. Indeed,

$$\begin{pmatrix} A-\lambda I & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

and $\lambda = 1$ (an invariant zero of the plant, see e.g. [1]), $z^T = (-1\ 0)$ and $w = 1 \in \mathcal{Y}^* = [0, \infty)$ violate condition (21b). This is also intuitively clear as nonnegative values of x_2 prevent the position x_1 from going to zero, when $x_1[0] > 0$ and thus the system is not null controllable under the constraint $y[k] = x_2[k] \geqslant 0$, $k \in \mathbb{N}$.

VII. Conclusions

This paper characterized the null controllability of discrete-time linear systems subject to input and/or state constraints under the condition of right-invertibility of the transfer matrix. The characterizations are in terms of algebraic conditions that are of a similar nature as the classical results for unconstrained and input-constrained linear systems [3], [5], [8–10], which are recovered as special cases of the main result of this paper. Investigating the possibility of removing of the right-invertibility condition is future work.

VIII. Appendix: Some facts from geometric control theory

We quote some standard facts from geometric control theory (see [1] and [11] for the proofs). The first one presents certain invariants under state feedbacks and output injections. Besides the system (1), which we denote by Σ for shortness, consider the linear system $\Sigma_{K,L}$ given by

$$x[k+1] = (A - BK - LC + LDK)x[k] + (B - LD)v[k] \tag{23a}$$

$$y[k] = (C - DK)x[k] + Dv[k]. \tag{23b}$$

This system can be obtained from (1) by applying both a pre-compensating state feedback $u[k] = -Kx[k] + v[k]$ and output injection $-Ly[k]$.

Proposition VIII.1 *Let $K \in \mathbb{R}^{m\times n}$ and $L \in \mathbb{R}^{n\times p}$ be given. The following statements hold.*

1) $\langle A \mid \operatorname{im} B\rangle = \langle A - BK \mid \operatorname{im} B\rangle$.
2) $\langle \ker C \mid A\rangle = \langle \ker C \mid A - LC\rangle$.
3) $\mathcal{V}^*(\Sigma_{K,L}) = \mathcal{V}^*(\Sigma)$.
4) $\mathcal{T}^*(\Sigma_{K,L}) = \mathcal{T}^*(\Sigma)$.

The right invertibility of the transfer matrix is related to the controlled and conditioned invariant subspaces:

Proposition VIII.2 *The transfer matrix $D+C(zI-A)^{-1}B$ is right invertible if, and only if, $\mathcal{V}^*+\mathcal{T}^* = \mathbb{R}^n$ and $\begin{bmatrix} C & D \end{bmatrix}$ is of full row rank. Futhermore, this right inverse can be chosen polynomial if, and only if, additionally the condition $\langle A \mid \operatorname{im} B\rangle \subseteq \mathcal{T}^* + \langle \ker C \mid A\rangle$ is satisfied.*

Systems that have transfer functions with a polynomial inverse are of particular interest for our treatment. The proof is omitted for brevity.

Proposition VIII.3 *Consider the linear system* (1)*. Suppose that the transfer matrix $G(z) := D + C(zI - A)^{-1}B$ has a polynomial right inverse. Let $H(z) = H^0+zH^1+\cdots+z^hH^h$ be such a right inverse. For any given sequence $\bar{y} \in \mathcal{S}^p$, there exist an initial state x_0 and an input $u \in \mathcal{S}^m$ such that the output $y^{x_0,u}$, corresponding to the initial state $x[0] = x_0$ and the input u, of system* (1) *is identical to $\bar{y}$.*

The proposition below shows what information about the state at a certain time instant can be obtained from the values of the output at the present and future time instants. To prove this result, we will use the fact that one can compute $\mathcal{V}^*$ for a linear system (1) as the limit of the subspaces

$$\mathcal{V}^i = \{x \mid Ax + Bu \in \mathcal{V}^{i-1} \text{ and } Cx + Du = 0 \text{ for some } u\} \tag{24}$$

with $\mathcal{V}^0 = \mathbb{R}^n$. In fact, there exists an index $i \leqslant n$ such that $\mathcal{V}^j = \mathcal{V}^*$ for all $j \geqslant i$ (see [11] for details).

Proposition VIII.4 *Consider the linear system* (1)*. Let the triple $(u, x, y) \in \mathcal{S}^m \times \mathcal{S}^n \times \mathcal{S}^p$ satisfy the equations* (1)*. If for some $\bar{x} \in \mathbb{R}^n$, and $\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{n-1} \in \mathbb{R}^m$*

$$\begin{aligned} y[k] = {} & CA^k\bar{x} + CA^{k-1}B\bar{u}_0 + \\ & + CA^{k-2}B\bar{u}_1 + \cdots + CB\bar{u}_{k-1} + D\bar{u}_k \end{aligned}$$

for $k = 0, 1, \ldots, n-1$, then $x[0] - \bar{x} \in \mathcal{V}^*(A, B, C, D)$.

The proof can be obtained by using Algorithm (24) and showing that $x[0] - \bar{x} \in \mathcal{V}^j$ for all $j = 0, \ldots, n$ and noting that $\mathcal{V}^n = \mathcal{V}^*(A, B, C, D)$.

IX. Appendix: Proof of the main result

This appendix contains the proof of Theorem V.3. However, due to space limitations we will only be able to provide a sketch of the proof.

Proof: Without loss of generality, we can assume that the system is in the form (17) as the null controllability problem is not changed by similarity transformations and pre-compensating feedbacks of the form $u[k] = -Kx[k] + v[k]$. To show the 'if' part, let an initial state $x_0 = \mathrm{col}(x_{01}, x_{02}, x_{03}) \in \mathcal{X}_0$ be given in the coordinates related to $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$ as introduced in Section V-B. The proof is based on the following steps in constructing an input u that steers the state from x_0 to x_f:

Step 1: One can show that the conditions (21a)-(21b) (in case of conic constraint sets) and (22a)-(22b) (in case of bounded constraint sets) imply that the system

$$x_1[k+1] = A_{11}x_1[k] + L_1 y[k], \qquad (25)$$

being (17a), is null controllable, where y is treated as input satisfying the constraint (2).

Step 2: Next, we select the output $(y[0], \ldots, y[n_3 - 1]) \in \mathcal{Y}^{n_3}$ that corresponds to x_0 as in the characterization of $\mathcal{X}_0$ in Theorem V.2. This finite sequence is applied as the "input" to system (25), which steers x_{01} at time 0 to an intermediate state, say, x_{i1} at time n_3.

Step 3: Then, we exploit the null controllability of (25) under the constraints (2) to provide a time $T \in \mathbb{N}$, $T \geqslant n_3$ (for later purposes we also take $T \geqslant n_2$) and an "input sequence" $(y[n_3], \ldots, y[T-1])$ such that it steers the state of system (25) from $x_1[n_3] = x_{i1}$ to $x_1[T] = 0$ and $y[k] \in \mathcal{Y}$, $k = n_3, \ldots, T-1$. By selecting $y[k] = 0$ for $k \in \mathbb{N}$, $k \geqslant T$, we have constructed, together with step 2 an output $y \in \mathcal{S}^p$ with $y[k] \in \mathcal{Y}$ for all $k \in \mathbb{N}$ and $x_1[k] = 0$ for all $k \geqslant T$.

Step 4: Given this output $y \in \mathcal{S}^p$, we construct an input $v \in \mathcal{S}^m$ and a corresponding state trajectory $x \in \mathcal{S}^n$ starting at x_0 such that $y^{x_0,v} = y$. This construction uses the right-invertibility assumption (Ass. V.1), which implies also the right-invertibility of $C_3(zI - A_{33})^{-1}B_3 + D$ (see discussion around (19a)-(19b)). Moreover, (19a)-(19b) and the right-invertibility imply via Proposition VIII.2 and VIII.3 that indeed the output y can be generated by (17c)-(17d) for some x'_{03}. As y was selected in accordance with x_{03} for system (17c)-(17d) (note that this part of the system in (17) generates the output y) it can be shown by using Proposition VIII.4 and $\mathcal{V}^*(A_{33}, B_3, C_3, D) = \{0\}$ that $x_{03} = x'_{03}$.

Step 5: From the above steps we obtain an input $v \in \mathcal{S}^m$ that starts in x_0 at time 0, produces the output $y \in \mathcal{S}^p$ with $y[k] \in \mathcal{Y}$ for all $k \in \mathbb{N}$ and $y[k] = 0$ for $k \in \mathbb{N}$, $k \geqslant T$. Moreover, $x_1[T] = 0$. Since $y[k] = 0$ for $k \geqslant T$, $x_3[T]$ must, by definition, be an element of $\mathcal{V}^*(A_{33}, B_3, C_3, D) = \{0\}$. Hence, only $x_2[T] = 0$ remains to be realized. To do so, observe that $\hat{x} := \mathrm{col}(0, -x_2[T], 0) \in \mathcal{R}^*(A, B, C, D)$. Hence, by definition of $\mathcal{R}^*(A, B, C, D)$ and since $T \geqslant n_2$, there exists an input $v' \in \mathcal{S}^m$ such that $y^{0,v'} = 0$ and $x^{0,v'}[T] = \hat{x}$. By using linearity of the system, we obtain that $x^{x_0,v+v'}[T] = x^{x_0,v}[T] + x^{0,v'}[T] = 0$ and $y^{x_0,v+v'}[k] = y^{x_0,v}[k] + y^{0,v'}[k] = y^{x_0,v}[k] = y[k] \in \mathcal{Y}$ for all $k \in \mathbb{N}$, which concludes the 'if' part of the proof.

The 'only if' part is based on the observation that (17a) must be null controllable under the "input constraints" $y[k] \in \mathcal{Y}$, $k \in \mathbb{N}$ (as already pointed out in Section V-B). Roughly speaking, 'only if' part follows now by translating the corresponding input-constrained null controllability conditions to the original system (1)-(2) before the transformation into the canonical form was applied (although several technical issues have still to be overcome). ■

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