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Assessing Stability of Time-Delay Systems using Rational Systems

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***Abstract*— In this paper we show how stability of an infinite-dimensional linear time-delay system can be assessed by studying the stability of an associated finite-dimensional linear system, constructed after substituting the exponential function in the characteristic equation of the delay-system by a high enough finite power of the bilinear transformation.**

I. Introduction and Motivation

Consider the general class of linear time retarded systems

$$\dot{x}(t) = Ax(t) + Bx(t - \tau), \tag{1}$$

with initial condition $x(t) = \varphi(t)$, $t \in [-\tau,\ 0]$, where $x(t) \in \mathbb{R}^n$, $\tau \geq 0$ and $A,\ B \in \mathbb{R}^{n \times n}$. Stability of system (1) is dictated by the location of the roots of the transcendental function

$$\Delta_\tau(s) := \det(sI - A - Be^{-s\tau}). \tag{2}$$

When $\tau > 0$, the function $\Delta_\tau(s)$ has infinitely many roots. We say that system (1) is stable if all roots of $\Delta_\tau(s)$ lie in the open left half of the complex plane. In this case we say that the roots of $\Delta_\tau(s)$ are stable. Since direct "calculation" of the roots of $\Delta_\tau(s)$ is not practical, the literature is rich in methods that try to indirectly assess the location of the roots. This paper propose one such method.

The idea is to establish an equivalence relationship between stability of the roots of the transcendental function $\Delta_\tau(s)$ with the roots of the rational function

$$\Delta_T^k(s) := \det\left[sI - A - B\left(\frac{1 - Ts}{1 + Ts}\right)^k\right]. \tag{3}$$

We will show that in case $0 \leq \tau < \bar{\tau}$, where $\bar{\tau}$ is finite, then a large enough integer k exists such that studying stability of $\Delta_T^k(s)$ is equivalent to studying stability of $\Delta_\tau(s)$.

For any τ, T and k, let $(\pi, \nu, \delta)_\tau$ and $(\pi, \nu, \delta)_T^k$ be, respectively, the number of roots of $\Delta_\tau(s)$ and $\Delta_T^k(s)$ with negative real part (π), zero real part (ν) and positive real part (δ). The triplet $(\pi, \nu, \delta)_\tau$, also known as *inertia*, completely characterize stability of system (1). We will prove the following theorem.

Theorem 1: Let $0 < \bar{\tau} < \infty$ be given. Let $(\pi, \nu, \delta)_\tau$ and $(\pi, \nu, \delta)_T^k$ be, respectively, the inertia of $\Delta_\tau(s)$ and $\Delta_T^k(s)$ for some k, τ and T. There exists an integer $0 < k^* < \infty$ and a real number $0 < \bar{T} < \infty$ such that for any $k > k^*$ there exists a monotonically increasing continuous function $\phi_k : [0, \bar{\tau}) \to [0, \bar{T})$ so that: (i) $\delta_\tau = \delta_T^k$, (ii) $\nu_\tau = \nu_T^k$ and where $T = \phi_k(\tau)$.

The above theorem essentially establishes that, as far as stability analysis is concerned, one can analyze the roots of the rational function $\Delta_T^k(s)$ instead of those of the transcendental function $\Delta_\tau(s)$. For that to become possible, we had to limit the analysis to a finite interval $\tau \in [0, \bar{\tau})$. This translates into a finite upper bound on an integer k^* above which stability of the roots of $\Delta_\tau(s)$ is simply equivalent to the stability of the roots of $\Delta_T^k(s)$ for any $k > k^*$. The advantage is evident since that analyzing stability of the roots of $\Delta_T^k(s)$, even for large finite k, is a lot simpler than analyzing the stability of the roots of the transcendental function $\Delta_\tau(s)$.

We will show in Section VII that a finite upper bound for k^* can be given as a function of the magnitude of the purely imaginary roots of $\Delta_\tau(s)$ in the interval $[0, \bar{\tau})$ and how spaced the crossing points τ's are. The closer the crossings the larger the k^*. As it is well known (see, for instance, [1]–[3]), if a pair of roots of $\Delta_\tau(s)$ cross the imaginary axis for some finite $\tau_i \geq 0$ at the point $s = j\omega_i$ then other pairs of roots of $\Delta_\tau(s)$ also cross the imaginary axis at $s = j\omega_i$ an infinite number of times for every

$$\tau_{i\ell} = \tau_i + 2\ell\,\omega_i^{-1}\pi, \qquad \ell = 0, 1, 2, \ldots \tag{4}$$

This suggests that $\bar{\tau}$ must indeed be finite for k^* to be finite because the roots of any finite dimensional rational system can only cross the imaginary axis a finite number of times.

Perhaps the main difference between this paper and other works, such as [4]–[7] (see also [2, Chapter 2]) that are also based on the use of the rational function (3) is that those works establish a relationship between $\Delta_\tau(s)$ and $\Delta_T^k(s)$ *only on the imaginary axis*. However, we seek to establish a relationship between stability, hence involving *all roots including the ones that are on the imaginary axis*. Existing works usually consider $k = 1$, as in Rekasius [4] or Olgac and Sipahi [7], or $k = 2$ if only positive values of T are to be analyzed, as in Thowsen [5], [6]. The idea is to show that each time some pair of roots of $\Delta_\tau(s)$ cross the imaginary axis at $s = j\omega$ for some $\tau \geq 0$ and in some direction, some pair of roots of $\Delta_T^1(s)$ (or $\Delta_T^2(s)$) also cross the imaginary axis for some $T \in \mathbb{R}$ (or $T \geq 0$) at exactly $s = j\omega$ and in the same direction. By keeping track of the number and the direction of imaginary axis crossing of both $\Delta_\tau(s)$ and $\Delta_T(s)$ an analysis of stability can be constructed. We revisit such procedures in the numerical example of Section III.

We emphasize that the existing works do not establish a relationship between stability of the roots of $\Delta_\tau(s)$ and those of $\Delta_T^k(s)$. First, even when the roots of $\Delta_\tau(s)$ are stable for

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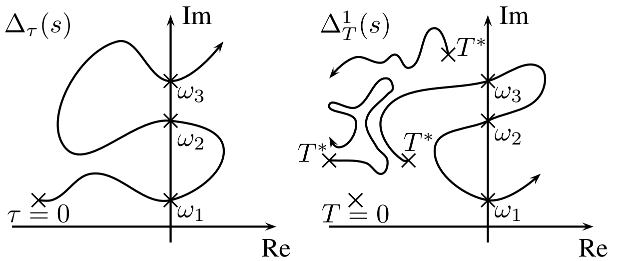

Fig. 1. Root locus branches of $\Delta_\tau(s)$ and $\Delta_T^1(s)$ which are not equivalent, where T^* is a positive small enough value for T.

$\tau = 0$, it is not clear whether the roots of $\Delta_T^k(s)$ should also be stable in any neighborhood of zero, a question we answer positively in Section IV. Furthermore, $\Delta_\tau(s)$ and $\Delta_T^k(s)$ may cross the imaginary axis at $s = j\omega_i$ for τ_i and T_i, respectively, and also at $s = j\omega_{i+1}$ for τ_{i+1} and T_{i+1}. However $\Delta_\tau(s)$ may be stable in the interval (τ_i, τ_{i+1}) while $\Delta_T(s)$ may be unstable in the interval (T_i, T_{i+1}). This is an extremely common phenomenon which will be illustrated by an example in Section III. Another common situation is that depicted in Figure 1.

Notation throughout the paper is standard. When we take norms of a transfer function it is the H_∞ norm. Norms of vectors are the two-norm.

II. PRELIMINARIES

We seek to relate the zeros of $\Delta_\tau(s)$ and $\Delta_T^k(s)$. From (3)

$$\Delta_T^k(s) = \frac{\det\left[(1+Ts)^k(sI-A) - B(1-Ts)^k\right]}{(1+Ts)^{kn}}$$

is a rational function of s of order kn for any $T \neq 0$.

The following are typical results found in the literature.

Lemma 1: Let $k \geq 1$ be a given finite integer. The imaginary number $s = j\omega_c$ is a root of $\Delta_\tau(s)$ for some $\tau_c \geq 0$, i.e. $\Delta_\tau(j\omega_c) = 0$, if and only if it is also a root of $\Delta_T^k(s)$, for some $T_c \in \mathbb{R}$ ($T_c \geq 0$ if $k \geq 2$), i.e. $\Delta_T^k(j\omega_c) = 0$.

A proof for the case $k = 1$ is available in [7] and for $k = 2$ in [6]. The arguments of [6] can be used to extend this result to any $k > 2$ finite.

The following lemma is a slight generalization of [7, Proposition I].

Lemma 2: For any $\tau_c \geq 0$ finite it is true that $\nu_{\tau_c} \leq 2n$. If the imaginary number $s = j\omega_c$ is a root of $\Delta_\tau(s)$ for $\tau = \tau_c$ then ω_c is also finite.

Proof: The first part follows from Lemma 1 and the equivalence of the imaginary roots of $\Delta_\tau(s)$ with those of $\Delta_T^k(s)$, which can only have a finite number of roots on the imaginary axis [1], [2]. Indeed, $\Delta_T^k(s)$ has exactly $(k+1)\,n$ roots for any $T \geq 0$. The bound $2n$ is obtained with $k = 1$.

The second part follows from an extension of argument found in [1] for scalar systems. Note that $\Delta_\tau(s) = 0$ for some $s = \lambda + j\omega$ if and only if there exists $x \in \mathbb{C}^n \neq 0$ such that $[(\lambda + j\omega)I - A]x = Be^{-(\lambda+j\omega)\tau}x$. In particular one can assume $\|x\| = 1$ without loss of generality. Here $\|\cdot\|$ denotes the 2-norm. Therefore, taking norms on both sides $\|[(\lambda+j\omega)I - A]x\| = \|Be^{-\lambda\tau}e^{-j\omega\tau}x\| \leq |e^{-\lambda\tau}|\|B\|$. Then, for purely imaginary roots $\lambda = 0$ and

$$\|(j\omega I - A)x\| \leq \|B\| < \infty.$$

Therefore, if $j\omega$ is not an eigenvalue of A then ω must be finite. If $j\omega$ is an eigenvalue of A then the above inequality may be satisfied regardless of ω if x is also an eigenvector of A. But in this case ω is also finite because A is finite. ∎

Lemma 1 is rooted on the properties of the *bilinear transformation*, which we utilize here after raising the right-hand side to the kth power in the form of the substitution

$$e^{-s\tau} \longrightarrow \left(\frac{1-Ts}{1+Ts}\right)^k, \quad \tau, T \in \mathbb{R}. \tag{5}$$

The bilinear transformation maps the imaginary axis into the unit circle. Evaluation of equation (5) at $s = j\omega$ produces a (non-unique) relationship between τ and T of the form

$$\tau = 2\,k\,\omega^{-1}\arctan(\omega T). \tag{6}$$

Unfortunately the substitution (5) is exact *only on the imaginary axis*, which limits the scope of results such as Lemmas 1 and 2 to imaginary roots. Not much can be said about the roots which are not purely imaginary without invoking new ideas. An analysis of stability is possible however on the basis of arguments similar to the one in [7]. For that we need to define the notion of root tendency. First define the *root locus*

$$\psi(\tau) = \{s \in \mathbb{C} : \quad \Delta_\tau(s) = 0, \quad \tau \geq 0\}. \tag{7}$$

As usual, $\psi(\tau)$ is a collection of curves parametrized by τ. Any point $s_0 \in \psi(\tau)$ is said to be *regular* if the underlying curve passing by s_0 is a differentiable function of τ. We identify a particular curve in $\psi(\tau)$ passing by s_0 at τ_0 by writing $\psi(\tau, \tau_0, s_0)$ and a particular point as $s_0 \in \psi(\tau_0)$. A condition sufficient for regularity is that s_0 be a single (non-repeated) root of $\Delta_\tau(s)$. On a regular point on the imaginary axis $j\omega_c \in \psi(\tau)$ we can define the quantity

$$RT_\tau(\omega_c, \tau_c) := \operatorname{sign}\left(\operatorname{Re}\left\{\left.\frac{d\psi(\tau, \tau_c, j\omega_c)}{d\tau}\right|_{\tau=\tau_c}\right\}\right) \tag{8}$$

which is an indicator of the direction of crossing of the imaginary root $j\omega_c$. If $RT_\tau(\omega_c, \tau_c) = 1$ roots of $\Delta_\tau(s)$ cross the imaginary axis at $j\omega_c$ from left (stable) to right (unstable); conversely, if $RT_\tau(\omega_c, \tau_c) = -1$ roots of $\Delta_\tau(s)$ cross the imaginary axis at $j\omega_c$ from right to left. We may similarly define $RT_T^k(\omega_c, T_c)$ by substituting $\psi(\tau)$ by

$$\psi_k(T) = \{s \in \mathbb{C} : \quad \Delta_T^k(s) = 0, \quad T \geq 0\} \tag{9}$$

and replacing the derivative with respect to τ by a derivative with respect to T in the previous discussion.

The work [7] does not explicitly discuss regularity but regularity is implicitly assumed in the definitions and proofs. For a non-regular point $j\omega_c \in \psi(\tau)$ the notion of root tendency can still be defined but should require a more elaborate setup. The idea is that singular points in $\psi(\tau)$ are isolated and therefore, even though $\psi(\tau)$ is not differentiable

TABLE I
IMAGINARY AXIS CROSSINGS OF ZEROS OF $\Delta_T^1(s)$

i	ω_i	T_i	tendency	τ_ℓ	ℓ
1	3.0352	0.0829	LR	0.1624	1
2	2.9124	0.0953	RL	0.1859	2
3	2.1109	0.6233	LR	0.8725	5
4	15.5032	-0.4269	LR	0.2220	3
5	0.8407	-0.1332	LR	7.2105	30

TABLE II
IMAGINARY AXIS CROSSINGS OF ZEROS OF $\Delta_\tau(s)$

ℓ	i	τ_ℓ	tendency	ℓ	i	τ_ℓ	tendency
1	1	0.1624	LR	16	4	3.8695	LR
2	2	0.1859	RL	17	4	4.2748	LR
3	4	0.2220	LR	18	1	4.3026	LR
4	4	0.6272	LR	19	2	4.5007	RL
5	3	0.8725	LR	20	4	4.6801	LR
6	4	1.0326	LR	21	4	5.0854	LR
7	4	1.4378	LR	22	4	5.4907	LR
8	4	1.8431	LR	23	4	5.8959	LR
9	1	2.2325	LR	24	4	6.3012	LR
10	4	2.2484	LR	25	1	6.3727	LR
11	2	2.3433	RL	26	2	6.6581	RL
12	4	2.6537	LR	27	4	6.7065	LR
13	4	3.0590	LR	28	3	6.8253	LR
14	4	3.4642	LR	29	4	7.1118	LR
15	3	3.8489	LR	30	5	7.2105	LR

on some point $j\omega_c$, one can still effectively compute the root tendency by evaluating the directional derivatives on both sides as τ approaches τ_c. We plan to formalize these notions in a future paper.

The following result is from [7, Proposition II].

Lemma 3: Let $s = j\omega_i \in \psi(\tau_i)$. Hence $s = j\omega_i \in \psi_k(T_i)$ for some $T_i \in \mathbb{R}$ ($T_i \geq 0$ for $k \geq 2$). As τ reaches τ_i or any one of the infinitely many values of $\tau_{i\ell}$ given in (4) the function $RT_\tau(\omega_i, \tau_i) = RT_\tau(\omega_c, \tau_{i\ell})$ for any $\ell = 0, 1, 2, \ldots$ Furthermore $RT_\tau(\omega_i, \tau_i) = RT_T^k(\omega_i, T_i)$ for any $k \geq 1$.

In short, this lemma states that the root tendency on an imaginary root $s = j\omega_c$ is the same for $\Delta_\tau(s)$ or $\Delta_T^k(s)$ regardless of the particular τ or T which makes $s \to j\omega_c$.

The next example should give a hint on how this can be used to asses stability of $\Delta_\tau(s)$ from $\Delta_T^k(s)$.

III. EXAMPLE (PART A)

Consider the delay system (1) with

$$A = \begin{pmatrix} -1 & 13.5 & -1 \\ -3 & -1 & -2 \\ -2 & -1 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} -5.9 & 7.1 & -70.3 \\ 2 & -1 & 5 \\ 2 & 0 & 6 \end{pmatrix}, \quad (10)$$

as in [7]. For this example, the function $\Delta_T^1(s)$ is a rational function of order $(k+1)n = 6$. We start by finding all values of T and ω at which the roots of the function $\Delta_T^1(s)$ cross the imaginary axis. These are listed in Table I. We also list the direction of crossing, from left to right (LR) or right to left (RL). The evaluation of these quantities is Step **A**) in [7]. Step **B**) is the construction of the Table II, in which we use (4) discarding negative values in order to compute all values of τ crossings. In Table II we list all crossing times until all crossing frequencies ω_i appear at least once. The index i in the second column corresponds to column i in Table I, from where the values of root tendency and crossing frequencies ω_i, $i = 1, \ldots, 5$ can be obtained. Conversely, the columns τ_ℓ and ℓ in Table I correspond to the first values of ℓ and τ_ℓ for which some root ω_i appears in Table II.

It is looking at Table II that stability can be analyzed. Note that the roots of $\Delta_\tau(s)$ are stable at $\tau = 0$ and invoking continuity of the roots of $\Delta_\tau(s)$ in τ then we can conclude that the system (10) is stable in the intervals $\tau \in [0, \tau_1) \cup (\tau_2, \tau_3)$ and unstable otherwise, because after τ_3 there will always be more roots crossing the imaginary axis from left to right (LR) than from right to left (RL), which prevent the system from becoming stable ever again.

This example leaves little doubt that one cannot conclude about the stability of the roots of $\Delta_\tau(s)$ (Table II) by looking at the stability of the roots of $\Delta_T^1(s)$ (Table I). At least not in the whole interval $[0, \tau_{30})$ considered in Table II. The difficulties are various:

a) Most τ's in Table II are generated by repetitions through formula (4) and are not directly represented in Table I.
b) One needs to pay attention to the roots of $\Delta_T^k(s)$ as they emerge from $T = 0$, since this is a point of discontinuity. Note that, even though the first 5 root crossing tendencies in Tables I and II are the same (in this case by mere coincidence), one cannot generate the same sequence of crossings as the one displayed in Table II, that is matching both frequency of crossing and root tendency, by sweeping T continuously and monotonically from 0 along the root locus $\psi_1(T)$.
c) One cannot even analyze the root locus $\psi_1(T)$ in a single direction with $T \geq 0$ as Table I contains entries with negative T's. One of the rows with a negative value of T, $i = 4$, is responsible for generating most of the entries in Table II and cannot be safely ignored. One extra complication is the fact that formula (6) will generate a negative value of τ if T is negative.

In the next sections we will show how increasing k may solve all of the above problems. As mentioned in the introduction, the key will be to limit the analysis for $\tau \in [0, \bar{\tau})$ for some finite given $\bar{\tau}$. Note that even in this complicated example one will achieve equivalence between stability of the roots of $\Delta_\tau(s)$ and $\Delta_T^1(s)$ for any $\bar{\tau} \leq \tau_3 = 0.2220$. In fact, one can verify that the roots of $\Delta_T^1(s)$ are indeed stable in the interval $T \in [0, T_1) \cup (T_2, T_2 + \epsilon)$, for some small $\epsilon > 0$. Again, by continuity of the root locus $\psi_1(T)$ at any $T > 0$ and the fact that $RT_T^1(\omega_1, T_1) = RT_\tau(\omega_1, \tau_1)$ and $RT_T^1(\omega_2, T_2) = RT_\tau(\omega_2, \tau_2)$ we will show how one can infer that the roots of $\Delta_\tau(s)$ will be stable in the interval $\tau \in [0, \tau_1) \cup (\tau_2, \bar{\tau}), \tau_2 < \bar{\tau} < \tau_3$. The key will be to look at the ordering of the τ_i's and T_i's. In the next sections we will elaborate on the technical requirements that make such conclusions possible.

IV. STABILITY FOR SMALL T

For $T = 0$, and only there, $\Delta_{\tau=0}(s) = \Delta_{T=0}^k(s)$ is a polynomial of degree n, as opposed to a transcendental or rational function when $\tau, T > 0$. Note that in this case Theorem 1 is trivial because $(\pi,\nu,\delta)_{\tau=0} = (\pi,\nu,\delta)^k_{T=0}$. Most of our attention is therefore devoted to the case $T > 0$. The first obstacle to be overcome is the discontinuity of the roots of $\Delta^k_T(s)$ as they emerge from $T = 0$. In this section we will show that if the roots of $\Delta_{\tau=0}(s) = \Delta^k_{T=0}(s)$ are stable then the roots of $\Delta^k_T(s)$ are also stable in the interval $[0, T^*)$ for some small enough T^*. We start with the following auxiliary result.

Lemma 4: Consider the transfer function

$$G^k(s) := 1 - \left(\frac{\sigma - s}{\sigma + s}\right)^k.$$

Then $s^{-1}G^k(s) \in H_\infty$ and $\|s^{-1}G^k(s)\| = 2k\sigma^{-1}$ for all $0 < \sigma < \infty$.

Proof: That $s^{-1}G^k(s)$ is in H_∞ for all $\sigma > 0$ comes from the fact that all the poles of $G^k(s)$ are at $\lambda_i = -\sigma < 0$ and that $G^k(0) = 0$ so that $s^{-1}G^k(s)$ has the exact same poles as $G^k(s)$. Now note that

$$s^{-1}G^k(s) = -s^{-1}[1 - z^k(s)], \qquad z(s) := \frac{\sigma - s}{\sigma + s},$$

where $\|z^i(s)\| = 1$ for any integer i. Using the identity

$$1 - z^k = (1 - z)\left(1 + z + \cdots + z^{k-1}\right)$$

it follows that

$$\begin{aligned}\|s^{-1}G^k(s)\| &\le \|s^{-1}[1 - z(s)]\| \left(1 + \cdots + \|z^{k-1}(s)\|\right), \\ &\le k \left\|\frac{1}{s}\left(1 - \frac{\sigma - s}{\sigma + s}\right)\right\| = 2k\sigma^{-1}.\end{aligned}$$

However

$$\lim_{s\to 0} -s^{-1}G^k(s) = \lim_{s\to 0} \frac{2k\sigma}{(\sigma + s)^2}\left(\frac{\sigma - s}{\sigma + s}\right)^{k-1} = 2k\sigma^{-1},$$

so that $\|s^{-1}G^k(s)\| \ge \lim_{\omega\to 0} \|(j\omega)^{-1}G^k(j\omega)\| = 2k\sigma^{-1}$. This proves that the equality must hold. ∎

Lemma 4 can be combined with standard robust stability analysis to prove our first point in the next lemma.

Lemma 5: Assume that the roots of $\Delta_\tau(s)$ are stable at $\tau = 0$. Then, for any integer $k \ge 1$ there exists a sufficiently small $T^* > 0$ such that the roots of $\Delta^k_T(s)$ are stable for all $T \in [0, T^*)$.

Proof: If the roots of $\Delta_{\tau=0}(s)$ are stable then the roots of $\Delta^k_T(s)$ are stable for $T = 0$. Furthermore, the transfer function $H(s) = s\,(sI - A - B)^{-1}\,B$ is in H_∞, then $\|H(s)\| = \mu < \infty$. From the small gain theorem [8], the feedback connection of $H(s)$ with any $F(s) \in H_\infty$ such that $\|F(s)\| < \mu^{-1}$ is also stable. For instance, with $F(s) = s^{-1}G^k(s)$ where $G^k(s)$ is as in Lemma 4 and all $\sigma > 2k\mu$ we have $\|s^{-1}G^k(s)\| < \mu^{-1}$. Therefore the feedback connection

$$y(s) = H(s)\,w(s) \qquad w(s) = s^{-1}G^k(s)[r(s) - y(s)]$$

is stable. Eliminating $w(s)$ we obtain

$$\left(sI - A - B\left[1 - G^k(s)\right]\right) y(s) = BG^k(s)\,r(s),$$

which reveals that

$$y(s) = \left[sI - A - B\left(\frac{\sigma - s}{\sigma + s}\right)^k\right]^{-1} BG^k(s)\,r(s),$$

so that the roots of $\Delta^k_T(s)$ are stable for all $\sigma \in (2k\mu, \infty)$, that is for all $T = \sigma^{-1} \in (0, T^*)$ with $T^* = (2k\mu)^{-1}$. ∎

In above, a fundamental assumption is the fact that $T \ge 0$ (from $T = \sigma^{-1} > 0$). Indeed, one can always find a root of $\Delta^k_T(s)$ that is not stable for some $T < 0$.

V. A Condition in Terms of Ordered Sets

Inspired by the previous example let us start by defining two discrete sets

$$\Psi_{\bar\tau} := \{(\tau_i) : \quad j\omega_i \in \psi(\tau_i),\ 0 \le \tau_i < \tau_{i+1} \le \bar\tau\}, \quad (11)$$

$$\Psi_k := \left\{ \left(2\,k\,\omega_i^{-1} \arctan(\omega_i T_i)\right) : \right. \quad (12)$$
$$\left. j\omega_i \in \psi_k(T_i),\ 0 \le T_i < T_{i+1},\ 1 \le i \le k \right\}.$$

By now it should be clear that both sets have a finite number of elements when k and $\bar\tau$ are finite. Note also that $\Psi_{\bar\tau}$ is a *totally ordered set* in that the elements are sorted according to the usual inequality '$\le$' of real numbers. Unlike $\Psi_{\bar\tau}$, the set Ψ_k may not be an ordered set. Of interest here is $\bar\Psi_k$, the largest ordered subset of Ψ_k. Define also the sets

$$\Omega_{\bar\tau} := \{(\tau_i, \omega_i) : \quad j\omega_i \in \psi(\tau_i),$$
$$0 \le \tau_i < \tau_{i+1} \le \bar\tau\}, \quad (13)$$
$$\Omega_k := \{(T_i, \omega_i) : \quad j\omega_i \in \psi_k(T_i),$$
$$0 \le T_i < T_{i+1}, \quad 1 \le i \le k\}, \quad (14)$$

with pairs (τ, ω) and (T, ω).

The next lemma establishes conditions in terms of the above sets under which stability of the roots of $\Delta_\tau(s)$ is equivalent to stability of the roots of $\Delta^k_T(s)$.

Lemma 6: Let $0 < \bar\tau < \infty$ be given. Consider the transcendental function $\Delta_\tau(s)$ and the rational function $\Delta^k_T(s)$ for some finite integer $k \ge 1$ as defined in (2) and (3). Assume that the roots of $\Delta_\tau(s)$ at $\tau = 0$ are stable. Define the associated sets ψ, and Ψ_k as in (11) and (12). Let $\bar\Psi_k \subseteq \Psi_k$ be the largest ordered subset of Ψ_k. There exists $\bar T > 0$ and a monotonically increasing continuous function $\phi_k : [0, \bar\tau) \to [0, \bar T)$ so that: (i) $\delta_\tau = \delta^k_T$, (ii) $\nu_\tau = \nu^k_T$; where $T = \phi_k(\tau)$ for all $\tau \in [0, \bar\tau)$ if and only if $\Psi_{\bar\tau} \subseteq \bar\Psi_k$.

Proof: Consider first the case when $\Psi_{\bar\tau}$ is empty. Then trivially $\Psi_{\bar\tau} \subseteq \bar\Psi_k$. Furthermore the roots of $\Delta_\tau(s)$ never cross the imaginary axis. By Lemma 1 the roots of $\Delta^k_T(s)$ will also never cross the imaginary axis so that $\delta_\tau = \delta^k_T = \nu_\tau = \nu^k_T = 0$ for all $\tau, T \ge 0$. The case when $\Psi_{\bar\tau}$ is not empty is the interesting one.

That $\nu_\tau = \nu^k_T$ is a consequence of Lemma 1. If $\emptyset \ne \Psi_{\bar\tau} \subseteq \bar\Psi_k$ then each time the roots of $\Delta_\tau(s)$ cross the imaginary axis at $(\tau_i, \omega_i) \in \Omega_{\bar\tau}$ the roots of $\Delta^k_T(s)$ cross the imaginary axis at $(T_i, \omega_i) \in \Omega_k$ *for the same index* i. This is a consequence of the ordering of both $\Psi_{\bar\tau}$ and $\bar\Psi_k$.

Using Lemma 5, the roots of $\Delta^k_T(s)$ are stable for all $T \in [0, T^*)$ for some small enough $T^* > 0$. That is $\delta^k_T = 0$ for all $T \in [0, T^*)$. We can then increase T from 0 passing

TABLE III
IMAGINARY AXIS CROSSINGS OF ZEROS OF $\Delta_T^2(s)$

i	ω_i	T_i	tendency	τ_ℓ	ℓ
1	3.0352	0.0408	LR	0.1624	1
2	2.9124	0.0468	RL	0.1859	2
3	15.5032	0.0750	LR	0.2220	3
4	2.1109	0.2350	LR	0.8725	5
5	0.8407	21.3076	LR	7.2105	20

through T^* continuously and monotonically until the roots of $\Delta_T^k(s)$ cross the imaginary axis at ω_1, $T_1 > 0$. By Lemma 1, at the same time the roots of $\Delta_\tau(s)$ should cross the imaginary axis at ω_1, $\tau_1 = 2k\omega_1^{-1}\arctan(\omega_1 T_1) > 0$. As T increases continuously and monotonically, the roots of $\Delta_\tau(s)$ and $\Delta_T^k(s)$ cross the imaginary axis at all ω_i, T_i, $\tau_i = 2k\omega_i^{-1}\arctan(\omega_i T_i) < \bar{\tau}$. This ensures the existence of a continuous and monotone map ϕ_k from $\tau \in [0, \bar{\tau})$ to $T \in [0, \bar{T})$. Without loss of generality the value of $\bar{T}$ can be chosen as $\bar{T} = T_\iota + \epsilon$ for some sufficiently small $\epsilon > 0$, where

$$\iota = \arg\max_i \{T_i : \ (T_i, \omega_i) \in \Omega_k, \ 2k\arctan(\omega_i T_i) \le \omega_i \bar{\tau}\}.$$

This ordering of the imaginary axis crossing allows one to conclude that for $T \in [T_i, T_{i+1})$ and $\tau \in [\tau_i, \tau_{i+i})$ for any $i \le \iota$ such that $\tau_i \le \bar{\tau}$ the number of roots with nonnegative real part of $\Delta_T^k(s)$ and $\Delta_\tau(s)$ are always the same, as the same number of roots cross the imaginary axis in the same order, hence implying that $\delta_\tau = \delta_T^k$.

In order to prove that the condition $\Psi_{\bar{\tau}} \subseteq \bar{\Psi}_k$ is also necessary it suffices to invoke Lemma 1 to show that if $\Psi_{\bar{\tau}} \not\subseteq \bar{\Psi}_k$ then there will necessarily exist $\tau_i, T_i > 0$ for which $\Delta_T^k(s)$ crosses the imaginary axis at $\omega_i \neq \omega_j$, hence $\tau_i \neq 2k\omega_j^{-1}\arctan(\omega_j T_j)$ so that the map ϕ_k cannot be continuous and monotonic. ■

In the next section these conditions on ordered sets will be illustrated by the example in Section III.

VI. Example (part b)

Consider now the example introduced in Section III, and start with $\bar{\tau} = 0.2$. From Table II we obtain the ordered set

$$\Psi_{\bar{\tau}=0.2} = \{0.1624,\, 0.1859\}.$$

For $k = 1$, Table I provides

$$\Psi_{k=1} = \{0.1624,\, 0.1859,\, 0.8725\}. \tag{15}$$

Note that $\bar{\Psi}_{k=1} = \Psi_{k=1}$ is the largest ordered subset of $\Psi_{k=1}$. Because $\Psi_{\bar{\tau}=0.2} \subset \bar{\Psi}_{k=1}$ then $\delta_\tau = \delta_T^k$ and $\nu_\tau = \nu_T^k$ for all $\tau \in [0, 0.2)$ and all $T \in [0, \bar{T})$, in this case for some $0.0953 = T_2 < \bar{T} < T_3 = 0.6233$.

Now let $\bar{\tau} = 1.0$. We build from Table II the ordered set

$$\Psi_{\bar{\tau}=1.0} = \{0.1624,\, 0.1859,\, 0.2220,\, 0.6272,\, 0.8725\}.$$

For $k = 1$ the sets $\Psi_{k=1} = \bar{\Psi}_{k=1}$ are the same as given in (15). In this case however $\Psi_{\bar{\tau}=1.0} \not\subseteq \bar{\Psi}_{k=1}$ and indeed at $\tau = \tau_4 = 0.6272 < 1.0$ we have $\delta_\tau = 4$ whereas from Table I we have $\delta_T^k \le 2$ for all $T \ge 0$.

Keeping $\bar{\tau} = 1.0$ we now let $k = 2$ and build Table III showing only the crossings for $T \ge 0$. Indeed, for $k \ge 2$ we need not consider $T < 0$, as noticed by Thowsen [5], [6]. From this table we build

$$\Psi_{k=2} = \{0.1624,\, 0.1859,\, 0.2220,\, 0.8725,\, 7.2105\}.$$

As for $k = 1$, coincidently $\bar{\Psi}_{k=2} = \Psi_{k=2}$. Note that $\Psi_{\bar{\tau}=0.2} \subset \bar{\Psi}_{k=1} \subset \bar{\Psi}_{k=2}$ but that $\Psi_{\bar{\tau}=1.0} \not\subseteq \bar{\Psi}_{k=2}$. Indeed, at $T_3 = 0.2350$ we have $\Psi_{k=2,i=3} = \tau_3 = 0.2220$ and at $T_4 = 0.2350$ we have $\tau_4 = 0.6272$ whereas the fourth element of $\Psi_{k=2,i=4} = 0.8725$, so that no continuous and monotonic mapping ϕ_k can exist between $\tau \in [0, 1.0)$ and T. For this example one needs $k \ge 6$ for $\Psi_{\bar{\tau}=1.0} \subseteq \bar{\Psi}_{k=6}$. This will be illustrated after the next section.

VII. Conditions for Ordering

We now discuss the issue of ensuring the ordering of the sets required by Lemma 6 by increasing k. We will show that given $0 < \bar{\tau} < \infty$ there exists a sufficiently large yet finite k^* such that for any $k > k^*$ the ordering condition $\Psi_{\bar{\tau}} \subseteq \bar{\Psi}_k$ is always satisfied. This will be achieved in two steps. Note that the results in this section do not require the hypothesis that the roots of $\Delta_\tau(s)$ be stable at $\tau = 0$.

Lemma 7: Let $0 < \bar{\tau} < \infty$ be given. Consider the transcendental function $\Delta_\tau(s)$ and the rational function $\Delta_T^k(s)$ for some integer $k \ge 1$ as defined in (2) and (3). Define the associated sets $\Psi_{\bar{\tau}}$, and $\Omega_{\bar{\tau}}$ as in (11) and (13). Compute

$$\bar{k} := \pi^{-1}\max\{(\omega_i \tau_i) : \quad (\tau_i, \omega_i) \in \Omega_{\bar{\tau}}\}. \tag{16}$$

If $k \ge \bar{k}$ then $\Psi_{\bar{\tau}} \subseteq \Psi_k$ where Ψ_k is as in (12).

Proof: The condition $\Psi_{\bar{\tau}} \subseteq \Psi_k$ is essentially a requirement that the relation

$$f : \Omega_k \to \Psi_{\bar{\tau}}, \quad f(T, w) = 2\,k\,\omega^{-1}\arctan(\omega T)$$

be surjective, that is, that each crossing time $\tau_i \in \Omega_{\bar{\tau}}$ be mapped by some $(T_j, \omega_j) \in \Omega_k$ not necessarily with $j = i$ (see next lemma). Correspondingly in $\Psi_{\bar{\tau}}$ and Ψ_k. In view of Lemma 1, all that is needed is that

$$0 \le \max\{(\omega_i \tau_i) : (\tau_i, \omega_i) \in \Omega_{\bar{\tau}}\} \le 2k\arctan(\omega_i T).$$

Since $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$ it suffices that $k \ge \bar{k}$ for f to be surjective. ■

The condition $k \ge \bar{k}$ will thus ensure that $\bar{\Psi}_{\bar{\tau}} \subseteq \Psi_k$. This is a necessary condition for $\Psi_{\bar{\tau}} \subseteq \bar{\Psi}_k$ in Lemma 6 to hold. That is because $\bar{\Psi}_k \subseteq \Psi_k$. The next lemma provides a sufficient condition for $\Psi_{\bar{\tau}} \subseteq \bar{\Psi}_k$.

Lemma 8: Let $0 < \bar{\tau} < \infty$ be given. Consider the transcendental function $\Delta_\tau(s)$ and the rational function $\Delta_T^k(s)$ for some integer $k \ge \bar{k}$ as defined in (2), (3) and (16). Define the associated sets $\Psi_{\bar{\tau}}$ and $\Omega_{\bar{\tau}}$ as in (11) and (13). Compute

$$k^* = \max\left\{ \frac{\omega_j \tau_j}{2\arccos\left(\sqrt{\tau_j/\tau_i}\right)} : \ (\omega_i, \tau_i), (\omega_j, \tau_j) \in \Omega_{\bar{\tau}} \right\}.$$

If $k > k^*$ then $\Psi_{\bar{\tau}} \subseteq \bar{\Psi}_k$, where $\bar{\Psi}_k$ is the largest ordered subset of Ψ_k defined in (12).

TABLE IV

$\bar{\tau}$	$\bar{k}$	k^*	$k^{*\dagger}$
0.20	0.2	0.7	—
1.00	3.0	13.4	8.7
2.24	9.0	39.3	33.2
5.00	23.0	496.5	412.1
7.22	35.0	754.4	470.0

Proof: Because $k \geq \bar{k}$ then each crossing time $\tau_i \in \Omega_{\bar{\tau}}$ is mapped by some $T_i \in \Omega_k$. We want to show that if $k > k^*$ then $\Psi_{\bar{\tau}} \subseteq \bar{\Psi}_k$. With that in mind, assume that $k > k^*$ but $\Psi_{\bar{\tau}} \not\subseteq \bar{\Psi}_k$. This implies that there exists at least one pair of indices i and j for which $0 < \tau_j < \tau_i < \bar{\tau}$ but with $T_j > T_i$. Now recall that the trigonometric tangent is continuous and differentiable in the interval $(-\pi/2, \pi/2)$ and invoke zero-order Taylor's formula [9], it yields: $\tan(x) = x/\cos^2(\xi)$, where $0 \leq \xi \leq x < \pi/2$. This equality holds for some $\xi \in [0, x]$. Because tangent is also bijective (invertible) in the interval $(-\pi/2, \pi/2)$ we have

$$T_i = \frac{1}{\omega_i} \tan\left(\frac{\omega_i \tau_i}{2k}\right) = \frac{\tau_i}{2k\cos^2(\xi_i)}, \quad 0 \leq \xi_i \leq \frac{\omega_i \tau_i}{2k}.$$

From Lemma 7, if $k \geq \bar{k}$ then $\omega_i \tau_i/(2k) \in [0, \pi/2)$. Likewise: $T_j = \dfrac{\tau_j}{2k\cos^2(\xi_j)}, 0 \leq \xi_j \leq \dfrac{\omega_j \tau_j}{2k}$, and

$$T_i - T_j = \frac{\tau_i - \rho_{ij}^2 \tau_j}{2k\cos^2(\xi_i)}, \qquad \rho_{ij} := \frac{\cos(\xi_i)}{\cos(\xi_j)}. \tag{17}$$

Now recall that: $k > k^* \geq \dfrac{\omega_j \tau_j}{2\arccos\left(\sqrt{\tau_j/\tau_i}\right)}$, which implies: $\cos\left(\dfrac{\omega_j \tau_j}{2k}\right) > \left(\dfrac{\tau_j}{\tau_i}\right)^{1/2}$. On the other hand

$$\rho_{ij} \leq \left[\cos\left(\frac{\omega_j \tau_j}{2k}\right)\right]^{-1},$$

which leads to the conclusion that $\rho_{ij}^2 < \tau_i/\tau_j$. But from (17)

$$T_i - T_j = \frac{\tau_i - \rho_{ij}^2 \tau_j}{2k\cos^2(\xi_i)} > 0$$

which contradicts the hypothesis that $T_j > T_i$. ■

VIII. Proof of Theorem 1

A proof of Theorem 1 can now be simply constructed by combining Lemmas 6, 7 and 8.

IX. Example (part c)

Back to our example, we now assemble Table IV where we list the values of $\bar{k}$ and k^* as computed in Lemmas 7 and 8 for various values of $\bar{\tau}$. The value of $k^{*\dagger}$ has been computed essentially by the same formula used to compute k^* except that the indices i, j were taken only over the subset of entries which appear unordered at $k = \bar{k}$. The proof of Lemma 8 remains unaltered if that restriction is added. As Table IV reveals, it should not bring much improvement except for values of $\bar{\tau}$'s with many crossings.

We then build Table V, which was created after evaluating all imaginary axis zero crossing for $\Delta_T^k(s)$ for the values of k immediately larger then the least value $\bar{k}$ computed in Table IV. What we list in Table V is the index i corresponding to a particular τ_ℓ. The condition $\Psi_{\bar{\tau}} \subseteq \bar{\Psi}_k$ in Lemma 6 is fulfilled when a particular column in Table V appears completely ordered and with no gaps. In Table V we show in boldface the largest ordered subset of indices, which is associated with the sets $\bar{\Psi}_k$. Note how conservative the values of the upper bound k^* can be. For instance, total ordering of the crossings for $\tau < \bar{\tau} = 2.24$ already happens at $k = 24$ as opposed to the least upper bound $k^{*\dagger} = 34$. Likewise, total ordering for $\bar{\tau} < 1.00$ happens for $k \geq 6$ (not shown in the table).

TABLE V

Imaginary axis crossings of zeros of $\Delta_\tau(s)$

i				τ_ℓ
$\bar{\tau} = 0.20$, $k = 1$	$\bar{\tau} = 1.00$, $k = 4$	$\bar{\tau} = 2.24$, $k = 10$	$\bar{\tau} = 5.00$, $k = 24$	
1	**1**	**1**	**1**	0.1624
2	**2**	**2**	**2**	0.1859
	3	**3**	**3**	0.2220
	5	**4**	**4**	0.6272
	4	**5**	**5**	0.8725
		6	**6**	1.0326
		9	**7**	1.4378
		15	**8**	1.8431
		7	**9**	2.2325
			11	2.2484
			10	2.3433
			12	2.6537
			16	3.0590
			17	3.4642
			13	3.8489
			22	3.8695
			30	4.2748
			14	4.3026
			15	4.5007
			46	4.6801

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