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# On output sampling based sliding mode control for discrete time systems

Srinath Govindaswamy, Thierry Floquet, and Sarah K. Spurgeon

*Abstract*— **This paper considers the problem of designing an output feedback sliding mode controller for discrete time systems. A minimal set of current and past outputs are identified to determine an extended output signal. This augmented system permits the design of a sliding manifold based upon output information only, which renders the sliding mode dynamics stable. It is shown that any transmission zeros of the augmented system will also be among the transmission zeros of the original plant. The class of discrete time systems which can be stabilised by output feedback sliding mode control is broadened by the proposed method. The theoretical results are constructive and are demonstrated via a motivational example which could not previously be solved by static output feedback.**

## I. INTRODUCTION

In continuous time, a sliding mode is generated by means of discontinuities in the control signals about a surface in the state space [27]. It is required that the discontinuity surface, usually called the sliding surface, is attained from any initial condition in a finite time interval. For an appropriately selected controller, the motion on the surface, or *sliding mode* is completely insensitive to any matched uncertainty in the system [27], [6]. In a discrete control implementation, the control signal is held constant during the sample period and hence it is not possible, in general, to attain a sliding mode which requires the control to switch at infinite frequency. As a result, the invariance properties of continuous time sliding-mode control can be lost. The obvious solution of sampling at high frequency, which will closely approximate continuous time, may not be possible for given hardware specifications. This has led to interest in the idea of discrete time sliding-mode control (DSMC). For the case of uncertain discrete systems, it is not possible to ensure the states remain on a surface within the state space and for this reason much of the early DSMC literature focused on establishing a discrete time counterpart to the (continuous time) reachability condition [21], [11], [3]. A comprehensive overview of these early developments is given in [18]. One distinctive feature is that DSMC does not necessarily require the use of a discontinuous control strategy [24]. The results presented in [24], [13] show that an appropriate choice of sliding surface, used with the equivalent control required to ensure sliding, can guarantee a bounded motion about the surface in the presence of bounded matched uncertainty and that the use of a relay/switch in the control law is detrimental to performance.

Early contributions in sliding mode control were developed in a framework in which all the system states are available. This may not be realistic for practical engineering problems and has motivated the need for output feedback controllers. A number of algorithms have been developed for robust stabilization of uncertain systems which are based on sliding surfaces and output feedback control schemes [28], [5]. In [28], a geometric condition is developed to guarantee the existence of the sliding surface and the stability of the reduced order sliding motion. Edwards and Spurgeon derived an algorithm [5], [6] which is convenient for practical use. Based on the work in [28], some dynamic feedback sliding mode controllers have been proposed [15], [22]. In all the above output feedback sliding mode control schemes, it is an *a priori* requirement that the system under consideration is minimum phase and relative degree one and that a particular sub-system must be output feedback stabilisable [6].

Compared with continuous time sliding-mode strategies, the design problem in discrete time is much less mature. Other than early work in [23], much of the literature assumes all states are available [13], [10], [26]. Discrete sliding mode control schemes which have restricted themselves to output measurements alone have often been observer based schemes with or without disturbance estimation [17], [25]. Recent exceptions have been the work in [12] which considers both static and dynamic output feedback problems, and the discrete time versions of certain higher-order sliding-mode control schemes [1], [2].

For continuous time systems, it was shown in [9] that the relative degree condition associated with the solution of the existence problem can be weakened if a classical sliding mode observer is combined with sliding mode exact differentiators to generate additional independent output signals from the available measurements. In this paper, it will be shown that by using the output signal at the current time instant together with a limited amount of information from previous sample instants, the class of systems for which an output feedback based sliding mode controller can be developed is significantly broadened.

The paper is structured as follows. Section 2 presents the problem motivation. The existence problem is considered in Section 3 and a solution to the reachability problem is given in Section 4. A motivational example, from the class of discrete systems which could not previously be stabilised by output feedback sliding mode control, is presented to

Srinath Govindaswamy and Sarah Spurgeon are with the Instrumentation, Control & Embedded Systems Research Group, Department of Electronics, University of Kent, Canterbury CT2 7NT, UK. S.K.Spurgeon@kent.ac.uk

T. Floquet is with LAGIS UMR CNRS 8146, École Centrale de Lille, BP 48, Cité Scientifique, 59651 Villeneuve d'Ascq and with Équipe Projet ALIEN, INRIA Lille-Nord Europe, France. thierry.floquet@ec-lille.fr

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illustrate the approach.

## II. Motivation

Consider the discrete, linear, time invariant state space system representation given below:

$$x_{k+1} = Ax_k + Bu_k \tag{1}$$
$$y_k = [(y_1)_k ... (y_p)_k]^T = Cx_k, (y_i)_k = C_i x_k \tag{2}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $y_k \in \mathbb{R}^p$ is the output vector and $u_k \in \mathbb{R}^m$ is the control input. It is assumed that $m \leq p$, the pair $(A,B)$ is controllable and without loss of generality, that $\text{rank}(C) = p$ and that $\text{rank}(B) = m$.

Consider the development of a control law based on output measurements only which will induce an ideal sliding motion on the surface

$$s = \{x \in \mathbb{R}^n : FCx_k = 0\} \tag{3}$$

for some selected matrix $F \in \mathbb{R}^{m\times p}$. It is well known that for a unique equivalent control to exist, the matrix $FCB \in \mathbb{R}^{m\times m}$ must have full rank. As

$$\text{rank}(FCB) \leq min\{\text{rank}(F), \text{rank}(CB)\} \tag{4}$$

it follows that both $F$ and $CB$ must have full rank. As $F$ is a design parameter, it can be chosen to be full rank. A necessary condition for $FCB$ to be full rank, and thus for solvability of the output feedback sliding mode design problem, thus becomes that $CB$ must have rank $m$. If this rank condition holds and any invariant zeros of the triple $\{A,B,C\}$ lie in the unit disk, then the existence of a matrix $F$ defining the surface (3), which provides a stable sliding motion with a unique equivalent control is determined from the stabilizability by output feedback of a specific, well-defined subsystem of the plant [6].

The aim here is to extend the existing results such that a static output feedback based sliding mode controller can be designed for the system (1)-(2) in cases where the subsystem at the heart of the existence problem is not output feedback stabilisable. The central idea is similar to that used for continuous time systems in [9]. In continuous time, sliding mode differentiators were used to extend the output. Here it will be shown that output extension using past output values of the sampled data model is constructive. It should be noted that this parallels in some respect the work of [14], where fast output sampling is used to design a discrete sliding mode controller. However, in [14] the output is extended to the dimension of the state and the invertibility property of the sampled system is not employed. In this paper, the output is extended so that the core triple used for sliding surface design is output feedback stabilisable.This is also achieved without taking additional samples of the measured output signals; instead both current and past output measurements are used by the proposed discrete time sliding mode controller.

## III. The Existence Problem

Consider the continuous time linear time invariant system

$$\dot{z}(t) = Fz(t) + Gu(t) \tag{5}$$

If the system (1)-(2) is the discretized form of the continuous time system (5) under sampling i.e,

$$A = e^{(Ft)}, B = \int_0^T e^{Ft} d\tau G \tag{6}$$

then it is shown in [20] (page 386) that the state space matrix $A$ is invertible.

Thus, the system can be rewritten as:

$$x_k = A^{-1}(x_{k+1} - Bu_k) \tag{7}$$
$$y_k = Cx_k \tag{8}$$

### A. Generation of the extended output

Consider system (1)-(2) without any *a priori* assumptions relating to either the stability of the invariant zeros or the fulfillment of the matching condition. The main idea here is to construct a matrix $\tilde{C}$:

$$\tilde{C} = \begin{bmatrix} C_1 \\ \vdots \\ C_1 A^{-\mu_1+1} \\ \vdots \\ C_p \\ \vdots \\ C_p A^{-\mu_p+1} \end{bmatrix} \tag{9}$$

such that $\tilde{C}$ is full rank, $\text{rank}(\tilde{C}B) = \text{rank}(B)$, and any invariant zeros of the triple $\{A,B,C\}$ lie inside the unit disk. Also, the $\mu_i$ are chosen such that $\tilde{p} = \sum_{i=1}^{p} \mu_i$ is minimal.

Note that $\tilde{C} = C$ means that the original system is output feedback stabilizable using existing methods (see e.g. [6]).

It can also be shown that any invariant zeros of the triple $\{A,B,\tilde{C}\}$ are amongst the invariant zeros of the triple $\{A,B,C\}$. Indeed, let $z_o \in \mathbb{C}$ be an invariant zero of $\{A,B,\tilde{C}\}$. Consequently $\tilde{P}(z)|_{z=z_o}$ loses rank, where $\tilde{P}(z)$ is Rosenbrock's system matrix defined by:

$$\tilde{P}(z) = \begin{bmatrix} zI - A & B \\ \tilde{C} & 0 \end{bmatrix}$$

Since by assumption $\tilde{p} \geq m$, this implies $\tilde{P}(z)$ loses column rank and therefore there exists non zero vectors $\eta_1$ and $\eta_2$ such that:

$$\begin{aligned} (z_o I - A)\eta_1 + B\eta_2 &= 0 \\ \tilde{C}\eta_1 &= 0 \end{aligned}$$

From the definition of $\tilde{C}$, $\tilde{C}\eta_1 = 0 \Rightarrow C\eta_1 = 0$. Consequently

$$\begin{aligned} (z_o I - A)\eta_1 + B\eta_2 &= 0 \\ C\eta_1 &= 0 \end{aligned}$$

and so $P(z)|_{z=z_o}$ loses rank, where

$$P(z) = \begin{bmatrix} zI - A & B \\ C & 0 \end{bmatrix}$$

Therefore any invariant zero of the triple $\{A,B,\tilde{C}\}$ is also an invariant zero of the triple $\{A,B,C\}$. It follows trivially that if the invariant zeros of $\{A,B,C\}$ lie inside the unit disk, then the invariant zeros of $\{A,B,\tilde{C}\}$ are also stable.

### *B. Design of the sliding manifold and analysis of the equivalent dynamics*

Assume that it is possible to construct a matrix $\tilde{C}$ as defined in (9). Then, the minimum requirements for solvability of the output feedback sliding mode design problem are satisfied by the triple $\{A,B,\tilde{C}\}$. It remains to find a suitable sliding variable that depends on the available measurements only.

For this, extend the original outputs with delayed ones as shown below:

$$\tilde{y}_k = \begin{bmatrix} (y_1)_k \\ \vdots \\ (y_1)_{k-\mu_1+1} \\ \vdots \\ (y_p)_k \\ \vdots \\ (y_p)_{k-\mu_p+1} \end{bmatrix}$$

From the system (7)-(8), it can be computed that:

$$\begin{aligned} y_k &= Cx_k \\ y_{k-1} &= CA^{-1}(x_k - Bu_{k-1}) \\ y_{k-2} &= CA^{-1}(x_{k-1} - Bu_{k-2}) \\ &= CA^{-1}(A^{-1}(x_k - Bu_{k-1}) - Bu_{k-2}) \\ &= CA^{-2}x_k - CA^{-2}Bu_{k-1} - CA^{-1}Bu_{k-2} \\ &\vdots \\ y_{k-j} &= CA^{-j}x_k - CA^{-j}Bu_{k-1} - \cdots - CA^{-1}Bu_{k-j} \\ &= CA^{-j}x_k - \sum_{l=1}^{j} CA^{-l}Bu_{k-j+l-1} \end{aligned}$$

Thus, the extended output matrix $\tilde{y}$ can be expressed as follows:

$$\tilde{y}_k = \tilde{C}x_k - M \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_p \end{bmatrix}$$

with, for $i = 1,...,p$:

$$\bar{u}_i = \begin{bmatrix} u_{k-1} \\ u_{k-2} \\ \vdots \\ u_{k-\mu_i+1} \end{bmatrix}$$

$$\mathbf{M} = diag\left\{\mathbf{M}_1,...,\mathbf{M}_p\right\}$$

$$\mathbf{M}_i = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ C_iA^{-1}B & 0 & \cdots & \cdots & 0 \\ C_iA^{-2}B & C_iA^{-1}B & \ddots & & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ \vdots & & & C_iA^{-1}B & 0 \\ CA^{-\mu_i+1}B & \cdots & \cdots & C_iA^{-2}B & C_iA^{-1}B \end{pmatrix} \in^{\mu_i\times(\mu_i-1)}$$

The following class of sliding manifolds, that only depend on known variables, can then be defined

$$\begin{aligned} s_k &= F\tilde{y}_k + FM \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_p \end{bmatrix} && (10) \\ &= F\tilde{C}x_k && (11) \\ & && (12) \end{aligned}$$

where $F \in^{m\times\tilde{p}}$ is a design parameter.

To analyse the stability of the resulting sliding motion, it is now convenient to introduce a coordinate transformation to the usual regular form, making the final $\tilde{p}$ states of the system depend directly on the extended outputs [6]:

$$\begin{aligned} A &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \tilde{C} = \begin{bmatrix} 0 & T \end{bmatrix} && (13) \\ x &= \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \end{aligned}$$

where $T \in \mathbb{R}^{\tilde{p}\times\tilde{p}}$ is an orthogonal matrix, $A_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$ and the remaining sub-blocks in the system matrix are partitioned accordingly. The corresponding switching surface parameter is given by

$$\begin{bmatrix} \overset{\overset{\tilde{p}-m}{\leftrightarrow}}{F_1} & \overset{\overset{m}{\leftrightarrow}}{F_2} \end{bmatrix} = FT \qquad (14)$$

where $T$ is the matrix from equation (13). As a result

$$F\tilde{C} = \begin{bmatrix} F_1C_f & F_2 \end{bmatrix} \qquad (15)$$

where

$$C_f = \begin{bmatrix} 0_{(\tilde{p}-m)\times(n-\tilde{p})} & I_{(\tilde{p}-m)} \end{bmatrix} \qquad (16)$$

Therefore $F\tilde{C}B = F_2B_2$ and the square matrix $F_2$ is non-singular. The canonical form in (13) can be viewed as a special case of the regular form normally used in sliding mode controller design, and thus the reduced-order sliding motion is governed by a free motion with system matrix

$$A_{11}^s = A_{11} - A_{12}F_2^{-1}F_1C_f \qquad (17)$$

which must therefore be stable. If $K \in \mathbb{R}^{m\times(\tilde{p}-m)}$ is defined as $K = F_2^{-1}F_1$ then

$$A_{11}^s = A_{11} - A_{12}KC_f \qquad (18)$$

and the problem of hyperplane design is equivalent to a *static output feedback problem* for the system $(A_{11},A_{12},C_f)$. In order to utilize the existing literature it is necessary that the pair $(A_{11},A_{12})$ is controllable and $(A_{11},C_f)$ is observable. The former is ensured as $(A,B)$ is controllable. The observability of $(A_{11}, C_f)$, is not so straightforward, but can be investigated by considering the canonical form below.

***Lemma 1***

*Let $(A, B, \tilde{C})$ be a linear system with $\tilde{p} > m$ and rank$(\tilde{C}B) = m$. Then a change of coordinates exists so that the system triple with respect to the new coordinates has the following structure:*

- *The system matrix can be written as*
$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$
*where $A_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$ and the sub-block $A_{11}$ when partitioned has the structure*
$$A_{11} = \left[\begin{array}{cc|c} A_{11}^o & A_{12}^o & \\ 0 & A_{22}^o & A_{12}^m \\ \hline 0 & A_{21}^o & A_{22}^m \end{array}\right]$$
*where $A_{11}^o \in \mathbb{R}^{r\times r}$, $A_{22}^o \in \mathbb{R}^{(n-\tilde{p}-r)\times(n-\tilde{p}-r)}$ and $A_{21}^o \in \mathbb{R}^{(\tilde{p}-m)\times(n-\tilde{p}-r)}$ for some $r \geq 0$ and the pair $(A_{22}^o, A_{21}^o)$ is completely observable.*
- *The input distribution matrix $B$ and the output distribution matrix $\tilde{C}$ have the structure in (13).*

For a proof and a constructive algorithm to obtain this canonical form see [5].

In the case where $r > 0$, the intention is to construct a new system $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_f)$ which is both controllable and observable with the property that
$$\lambda(A_{11}^s) = \lambda(A_{11}^o) \cup \lambda(\tilde{A}_{11} - \tilde{B}_1 K \tilde{C}_f).$$
To this end, as in [5], partition the matrices $A_{12}$ and $A_{12}^m$ as
$$A_{12} = \begin{bmatrix} A_{121} \\ A_{122} \end{bmatrix} \quad \text{and} \quad A_{12}^m = \begin{bmatrix} A_{121}^m \\ A_{122}^m \end{bmatrix}$$
where $A_{122} \in \mathbb{R}^{(n-m-r)\times m}$ and $A_{122}^m \in \mathbb{R}^{(n-\tilde{p}-r)\times(\tilde{p}-m)}$ and form a new sub-system represented by the triple $(\tilde{A}_{11}, A_{122}, \tilde{C}_1)$ where
$$\begin{aligned} \tilde{A}_{11} &= \begin{bmatrix} A_{22}^o & A_{122}^m \\ A_{21}^o & A_{22}^m \end{bmatrix} \\ \tilde{C}_f &= \begin{bmatrix} 0_{(\tilde{p}-m)\times(n-\tilde{p}-r)} & I_{(\tilde{p}-m)} \end{bmatrix} \end{aligned} \qquad (19)$$
It follows that the spectrum of $A_{11}^s$ decomposes as
$$\lambda(A_{11} - A_{12}KC_f) = \lambda(A_{11}^o) \cup \lambda(\tilde{A}_{11} - A_{122}K\tilde{C}_f)$$

***Lemma 2 [5]***

*The spectrum of $A_{11}^o$ represents the invariant zeros of $(A, B, \tilde{C})$.*

It follows directly that for a stable sliding motion, the invariant zeros of the system $(A, B, \tilde{C})$ must lie inside the unit disk and the triple $(\tilde{A}_{11}, A_{122}, \tilde{C}_f)$ must be stabilisable with respect to output feedback.

The matrix $A_{122}$ is not necessarily full rank. Suppose rank$(A_{122}) = m'$ then, as in [5], it is possible to construct a matrix of elementary column operations $T_{m'} \in \mathbb{R}^{m\times m}$ such that
$$A_{122}T_{m'} = \begin{bmatrix} \tilde{B}_1 & 0 \end{bmatrix} \qquad (20)$$
where $\tilde{B}_1 \in \mathbb{R}^{(n-m-r)\times m'}$ and is of full rank. If $K_{m'} = T_{m'}^{-1}K$ and $K_{m'}$ is partitioned compatibly as
$$K_{m'} = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \begin{array}{l} \updownarrow m' \\ \updownarrow m-m' \end{array}$$
then
$$\begin{aligned} \tilde{A}_{11} - A_{122}K\tilde{C}_f &= \tilde{A}_{11} - \begin{bmatrix} \tilde{B}_1 & 0 \end{bmatrix} K_{m'}\tilde{C}_f \\ &= \tilde{A}_{11} - \tilde{B}_1 K_1 \tilde{C}_f \end{aligned}$$
and $(\tilde{A}_{11}, A_{122}, \tilde{C}_f)$ is stabilizable by output feedback if and only if $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_f)$ is stabilizable by output feedback. It follows that:

***Lemma 3***

*The pair $(\tilde{A}_{11}, \tilde{B}_1)$ is completely controllable and $(\tilde{A}_{11}, \tilde{C}_f)$ is completely observable.*

## IV. Control Law Design

Here the control action necessary to maintain an ideal sliding motion is defined. For one step reaching, the reaching condition yields:
$$s_{k+1} = 0 \qquad (21)$$
From (11), it is obtained that
$$s_{k+1} = F\tilde{C}x_{k+1} = 0$$
Substituting for $(\bar{x}_1)_{k+1}$, $(\bar{x}_2)_{k+1}$ from (13), using (15) and rearranging gives:
$$\begin{aligned} u_k = &-(F_2B_2)^{-1}(F_1C_f(A_{11}(\bar{x}_1)_k + A_{12}(\bar{x}_2)_k) + \\ &F_2(A_{21}(\bar{x}_1)_k + A_{22}(\bar{x}_2)_k)) \end{aligned}$$
Because $s_k = 0$, $(\bar{x}_2)_k = -F_2^{-1}F_1C_f(\bar{x}_1)_k$ and $u_k$ can be written as
$$\begin{aligned} u_k = &-(F_2B_2)^{-1}(F_1C_f(A_{11} - A_{12}F_2^{-1}F_1C_f) + \\ &F_2(A_{21} - A_{22}F_2^{-1}F_1C_f)(\bar{x}_1)_k \end{aligned} \qquad (22)$$

## V. Motivational Example

Consider the state space representation of the plant given below:
$$x_{k+1} = \begin{bmatrix} a & 0 & 1 \\ 0 & b & 1 \\ 0 & 1 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_k \qquad (23)$$
$$y_k = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x_k \qquad (24)$$
It is assumed that $|a| < 1$ and $|b| > 1$. Note that this system is not output feedback stabilisable using a static output feedback control law and the existence problem cannot be solved for the plant with the given output. Note that the invariant zeros for the system (23)-(24) are at $a$ and $b$. The original system thus possesses an unstable transmission zero.

To construct an extended output so that the system can be output feedback stabilized, choose
$$\tilde{C} = \begin{bmatrix} C \\ CA^{-1} \end{bmatrix}$$

and the associated output extension

$$\tilde{y}_k = \begin{bmatrix} y_k \\ y_{k-1} \end{bmatrix}$$

The invariant zero for the triple $(A,B,\tilde{C})$ is at $a$ which is again an invariant zero of the triple $(A,B,C)$. The eigenvalue of $A_{11}^o$ is thus $a$ and $|a|<1$ implies $a$ lies inside the unit disk and hence is stable. It is seen that the output extension has enabled the unstable transmission zero in the original triple to be removed. The inclusion of a delayed output signal provides an additional degree of freedom to design a stable sliding surface, which is not possible for the original system (23)-(24).

Now consider the class of sliding surfaces given in (11). For the above plant (23)-(24), $s_k$ will have the form as shown below:

$$s_k = \begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} y_k \\ y_{k-1} \end{bmatrix} - bF_2 u_{k-1}$$

$$s_k = F_1(x_3)_k + F_2(x_2)_k - bF_2(x_3)_k \tag{25}$$

Thus, in the sliding mode $s_k = 0$, $(x_3)_k = -(F_1 - bF_2)^{-1}F_2(x_2)_k$ and the reduced order sliding surface dynamics can be represented as:

$$\begin{bmatrix} (x_1)_{k+1} \\ (x_2)_{k+1} \end{bmatrix} = \begin{bmatrix} a & -(F_1-bF_2)^{-1}F_2 \\ 0 & b-(F_1-bF_2)^{-1}F_2 \end{bmatrix} \begin{bmatrix} (x_1)_k \\ (x_2)_k \end{bmatrix} \tag{26}$$

For one step reaching, the reaching condition is that:

$$s_{k+1} = 0$$

or

$$F_1(x_3)_{k+1} + F_2(x_2)_{k+1} - bF_2(x_3)_{k+1} = 0.$$

After substituting for $(x_2)_{k+1}$, $(x_3)_{k+1}$ and $(x_3)_k$, one can get $u(k)$ as:

$$u_k = \left[((F_1-bF_2)^{-1}F_2)^2 - (F_1-bF_2)^{-1}F_1\right](x_2)_k \tag{27}$$

To check the stability of the triple $(A,B,s)$, consider:

$$s_{k+1} = F_1(x_3)_{k+1} + F_2(x_2)_{k+1} - bF_2(x_3)_{k+1} \tag{28}$$

From (25), one can obtain $x_3(k) = (F_1 - bF_2)^{-1}(s_k - F_2x_2(k))$. Substituting $x_2(k+1)$, $x_3(k+1)$ and $x_3(k)$ in (28), one gets:

$$\Longrightarrow \quad s_{k+1} = (F_1 - F_2^2(F_1-bF_2)^{-1})x_2(k) + (F_1-bF_2)u(k) + (F_1-bF_2)^{-1}F_2 s_k$$

Choosing $u_k$ as given in equation (27) gives

$$s_{k+1} = (F_1-bF_2)^{-1}F_2 s_k \tag{29}$$

Equation (29) along with equation (26), represents the dynamics of the triple $(A,B,s)$. For $(A,B,s)$ to be stable we can choose $b-(F_1-bF_2)^{-1}F_2 < 1$ and $(F_1-bF_2)^{-1}F_2 < 1$.

Simulations were performed for the above system with $F_1 = -2.2$, $F_2 = 1$, $a = 0.1$ and $b = -1.5$. The simulation results are shown in the figures below. It is seen that the proposed methodology renders the process (23)-(24) stable using output measurements only.

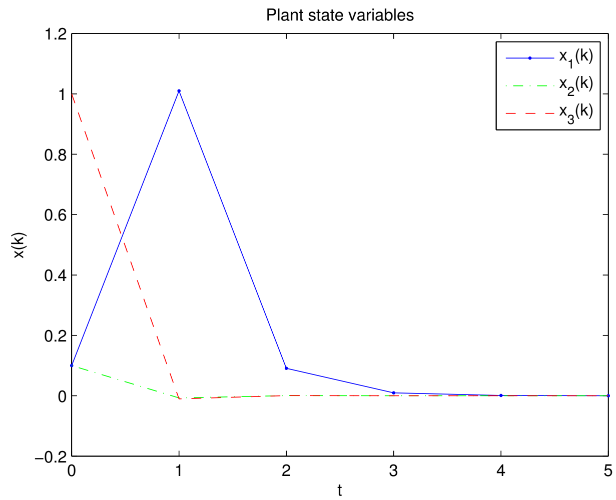


Fig. 1. Response of the states $(x_1)_k, (x_2)_k, (x_3)_k$

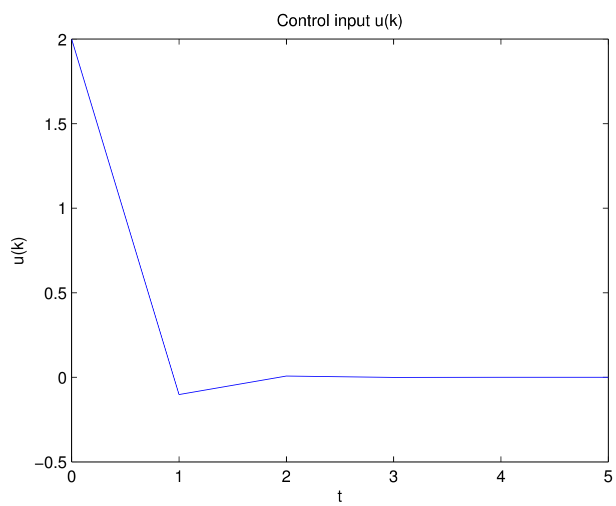


Fig. 2. The Control Input $u_k$ applied to the plant

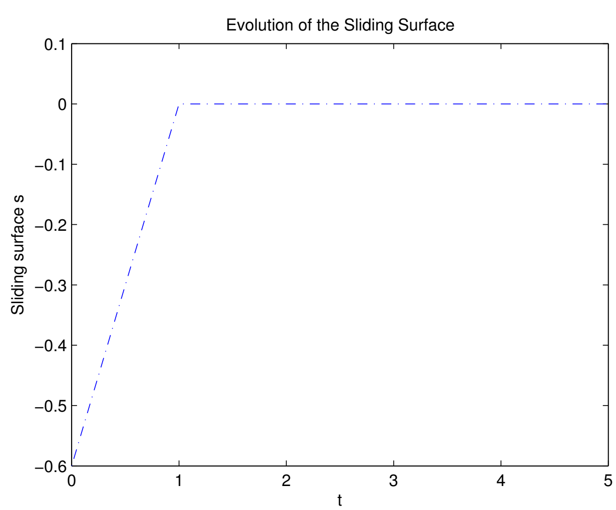


Fig. 3. Evolution of the sliding surface $s_k$ with time t

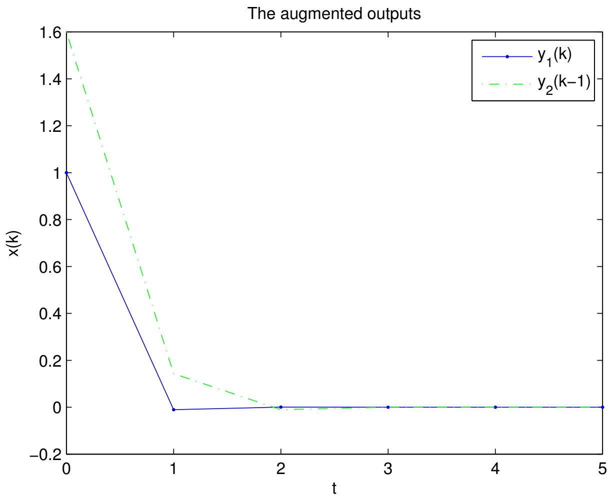


Fig. 4. The augmented plant output $y_k$ and $y_{k-1}$ with time t

## VI. Conclusions

An output feedback sliding mode controller has been developed which uses a minimal set of current and past output signals to determine an extended signal. It has been shown that the augmented system permits the design of a sliding manifold based upon the output information, which renders the sliding mode dynamics stable. It has also been shown that the invariant zeros of the augmented system are contained within the invariant zeros of the original plant. A design was performed on an example which is not stabilisable using a static output feedback control law, to show the efficacy of the method.

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