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Global uniform asymptotic Lyapunov stabilization of a vectorial chained-form system with a smooth time-varying control law

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Abstract— **This work presents two continuous time-varying control laws that globally uniformly asymptotically stabilize the origin of a vectorial generalization of the basic chained-form system. A strict Lyapunov function for the controlled system is developed, and the results verified through numerical simulation.**

I. INTRODUCTION

Chained-form systems are classical nonholonomic systems, and several control laws have been made for them.

For the general chained-form system, an algorithmic approach using sinusoids was developed in [1] that gave global uniform asymptotic stability of the origin.

In [2], a globally uniformly asymptotically stabilizing time-varying control law was developed, in addition to discontinuous time-varying control laws. This, however, was for a subsystem and not the most general chained-form system.

Discontinuous control laws have been developed in [3], [4], [5] and [6], among others.

While not explicitly stating so, [7] developed a continuous time-varying control law that globally asymptotically stabilized the lowest-order chained form system.

In this paper we present two continuous time-varying control laws that globally uniformly asymptotically stabilize the origin of a vectorial generalization of the basic chained-form system. To the authors' best knowledge, no previous control laws have been presented for this specific generalization of the chained form system.

The paper is organized as follows: In Section II, the model is presented. In Sections III and IV, the control laws are presented. Section V presents simulation results. Conclusions are given in Section VI. The Appendix provides some further details into the proof in Section III.

II. THE MODEL

We look at a system on the form

$$\dot{x}_1 = u_1 \in \mathbb{R}^m \tag{1}$$

$$\dot{x}_2 = u_2 \in \mathbb{R} \tag{2}$$

$$\dot{x}_3 = x_2 u_1 \in \mathbb{R}^m \tag{3}$$

for any $m \in \mathbb{N}$, which is a vectorial generalization of the basic chained-form system (where $m = 1$). The control objective is to asymptotically stabilize the origin of (1)–(3).

According to [8] and [9], this is impossible to do with any time-invariant control law. This paper presents two smooth, time-varying globally uniformly asymptotically stabilizing control laws.

The control laws are developed using vectorial backstepping, the method of finding strict Lyapunov functions developed in [10] and the cascade theorems from [11].

III. STABILIZING THE REDUCED SYSTEM

We first look at the subsystem

$$\dot{x}_1 = u_1 \tag{4}$$

$$\dot{x}_3 = \phi_x u_1 \tag{5}$$

where $\phi_x \in \mathbb{R}$ and $u_1 \in \mathbb{R}^m$ are considered control inputs.

We define $\chi_1 \triangleq x_1 + c_3 \cos(\omega t) x_3 \in \mathbb{R}^m$, whose time derivative is given by

$$\begin{aligned} \dot{\chi}_1 &= \dot{x}_1 + c_3 \cos(\omega t)\dot{x}_3 - c_3\omega \sin(\omega t) x_3 \\ &= (1 + c_3 \cos(\omega t)\phi_x) u_1 - c_3 \omega \sin(\omega t) x_3. \end{aligned} \tag{6}$$

We note that $\chi_1 \equiv 0$ iff $x_1 \equiv x_3 \equiv 0$. Thus we need to stabilize the origin of the (χ_1, x_3) system.

Lemma 1: The origin of the system (4)–(5) is globally uniformly asymptotically stabilized by the control law

$$\phi_x(t, x_3) = -\frac{\sin(\omega t)\|x_3\|^2}{2(\epsilon + \|x_3\|^2)} \tag{7}$$

$$u_1(t, \chi_1, x_3) = \frac{-c_1 \chi_1 + c_3 \omega \sin(\omega t) x_3}{1 + c_3 \cos(\omega t)\phi_x} \tag{8}$$

where

$$\chi_1 = x_1 + c_3 \cos(\omega t) x_3 \in \mathbb{R}^m \tag{9}$$

for some $\epsilon \in (0, 1)$, $c_3 \in (0, 1)$, $c_3\omega \geq c_1 > 0$ and $\omega > 0$.

Furthermore,

$$U_2(t, \chi_1, x_3) = \left(\frac{c_1}{16\omega} + 2\right) U_1 - \frac{c_1 U_1^2 \sin(2\omega t)}{16\omega(1 + U_1)}, \tag{10}$$

with $U_1(\chi_1, x_3) \triangleq 2\|\chi_1\|^2 + \|x_3\|^2$, is a strict Lyapunov function for the controlled system.

Proof: We start by noting that $|\phi_x| \leq 1/2$, so that the denominator in (8) is always strictly positive. The time derivative of χ_1 with the control law (8) is given by

$$\dot{\chi}_1 = -c_1 \chi_1. \tag{11}$$

The rest of the proof is a generalization of the proof of Theorem 3 in [7].

Using the Lyapunov function candidate

$$U_1(\chi_1, x_3) \triangleq 2\|\chi_1\|^2 + \|x_3\|^2 \tag{12}$$

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we see that its time derivative along the trajectories of the system (5)–(6) with the control input (7)–(8) is given by

$$\begin{aligned}\dot{U}_1 &= \frac{2\phi_x}{1+c_3\cos(\omega t)\phi_x} x_3^{\mathrm{T}}(-c_1\chi_1 + c_3\omega\sin(\omega t)x_3) \\ &\quad - 4c_1\|\chi_1\|^2 \\ &= \frac{c_1\sin(\omega t)\|x_3\|^2 x_3^{\mathrm{T}}\chi_1 - c_3\omega\sin^2(\omega t)\|x_3\|^4}{\left(1-\frac{c_3}{4}\sin(2\omega t)\frac{\|x_3\|^2}{\epsilon+\|x_3\|^2}\right)(\epsilon+\|x_3\|^2)} \\ &\quad - 4c_1\|\chi_1\|^2 \qquad (13) \\ &\le 2c_1|\sin(\omega t)|\|\chi_1\|\frac{\|x_3\|^3}{\epsilon+\|x_3\|^2} \qquad (14) \\ &\quad - \frac{c_3\omega\sin^2(\omega t)\|x_3\|^4}{\left(1-\frac{c_3}{4}\sin(2\omega t)\frac{\|x_3\|^2}{\epsilon+\|x_3\|^2}\right)(\epsilon+\|x_3\|^2)} - 4c_1\|\chi_1\|^2 \\ &\le 2c_1|\sin(\omega t)|\|\chi_1\|\frac{\|x_3\|^3}{\epsilon+\|x_3\|^2} \\ &\quad - \frac{2c_3\omega\sin^2(\omega t)\|x_3\|^4}{3(\epsilon+\|x_3\|^2)} - 4c_1\|\chi_1\|^2 \qquad (15) \\ &\le -\frac{c_3\omega\sin^2(\omega t)\|x_3\|^4}{3(\epsilon+\|x_3\|^2)} - c_1\|\chi_1\|^2 \qquad (16)\end{aligned}$$

where it has been used that $c_3 < 1$ for the step (14) to (15). See Appendices A and B for further details into the other steps of this part of the proof.

We define the positive definite functions

$$W_1(\chi_1, x_3) \triangleq \frac{c_3\omega\|x_3\|^4}{3(\epsilon+\|x_3\|^2)} + \frac{c_1}{2}\|\chi_1\|^2 \qquad (17)$$

$$\lambda(U_1) \triangleq \frac{c_1 U_1^2}{8(1+U_1)}. \qquad (18)$$

W_1 and λ satisfy

$$W_1(\chi_1, x_3) \ge \lambda(U_1) \qquad (19)$$

if $\epsilon \in (0,1)$, $c_3 \in (0,1)$, $c_3\omega \ge c_1 > 0$ and $\omega > 0$ (see Appendix C for proof).

This gives

$$\begin{aligned}\dot{U}_1 &\le -\sin^2(\omega t)W_1(x_3,\chi_1) - \frac{c_1}{2}\|\chi_1\|^2 \\ &\le -\sin^2(\omega t)\lambda(U_1) \qquad (20)\end{aligned}$$

if $\epsilon \in (0,1)$, $c_3 \in (0,1)$, $c_3\omega \ge c_1 > 0$ and $\omega > 0$.

It is worth noting that

$$\begin{aligned}\lambda(U_1) &\le \frac{c_1}{8}U_1 \\ \frac{\partial\lambda}{\partial U_1} &= \frac{c_1}{8}\frac{2U_1+U_1^2}{(1+U_1)^2} \in \left[0, \frac{c_1}{8}\right].\end{aligned}$$

Using the technique of constructing strict Lyapunov functions for a class of time-varying systems developed in [10, Theorem 1] and used in [7], we choose the function

$$U_2(t,\chi_1,x_3) \triangleq \left(\frac{c_1}{16\omega}+2\right)U_1 - \frac{1}{2\omega}\sin(2\omega t)\lambda(U_1) \qquad (21)$$

which satisfies

$$2U_1 \le U_2 \le 2\left(\frac{c_1}{16\omega}+1\right)U_1. \qquad (22)$$

The time derivative of U_2 is given by

$$\begin{aligned}\dot{U}_2 &= \left(\frac{c_1}{16\omega}+2\right)\dot{U}_1 - \frac{\sin(2\omega t)}{2\omega}\frac{\partial\lambda}{\partial U_1}\dot{U}_1 - \cos(2\omega t)\lambda(U_1) \\ &= 2\dot{U}_1 + \left(2\sin^2(\omega t)-1\right)\lambda(U_1) \\ &\quad + \frac{1}{2\omega}\left(\frac{c_1}{8} - \sin(2\omega t)\frac{\partial\lambda}{\partial U_1}\right)\dot{U}_1 \\ &\le -2\sin^2(\omega t)\lambda(U_1) + \left(2\sin^2(\omega t)-1\right)\lambda(U_1) \\ &\quad + \frac{1}{2\omega}\left(\frac{c_1}{8} - \sin(2\omega t)\frac{\partial\lambda}{\partial U_1}\right)\dot{U}_1 \\ &= -\lambda(U_1) + \frac{1}{2\omega}\left(\frac{c_1}{8} - \sin(2\omega t)\frac{\partial\lambda}{\partial U_1}\right)\dot{U}_1 \\ &\le -\lambda(U_1) \qquad (23)\end{aligned}$$

since

$$\frac{c_1}{8} - \sin(2\omega t)\frac{\partial\lambda}{\partial U_1} \ge 0 \quad \text{and} \quad \dot{U}_1 \le 0.$$

According to [12, Theorem 4.9], the origin of the (χ_1, x_3) system, and thus the origin of the system (4)–(5), is then globally uniformly asymptotically stable. ■

IV. Stabilizing the Entire System

We define

$$\chi_2 \triangleq x_2 - \phi_x \qquad (24)$$

as per the techniques of backstepping in [13] and [12].

The time derivative of χ_2 is given by

$$\dot{\chi}_2 = \dot{x}_2 - \dot{\phi}_x = u_2 - \dot{\phi}_x \qquad (25)$$

where $\dot{\phi}_x$ can be found analytically to be

$$\dot{\phi}_x = -\frac{\omega\cos(\omega t)\|x_3\|^2}{2(\epsilon+\|x_3\|^2)} - \frac{\epsilon\sin(\omega t)x_2 x_3^{\mathrm{T}}u_1}{(\epsilon+\|x_3\|^2)^2}. \qquad (26)$$

Choosing $v_2 \triangleq u_2 - \dot{\phi}_x$, we get the new system

$$\begin{aligned}\dot{\chi}_1 &= -c_1\chi_1 + c_3\cos(\omega t)\chi_2 u_1 \in \mathbb{R}^m \qquad (27) \\ \dot{\chi}_2 &= v_2 \in \mathbb{R} \qquad (28) \\ \dot{x}_3 &= (\chi_2 + \phi_x)u_1 \in \mathbb{R}^m \qquad (29)\end{aligned}$$

where ϕ_x and u_1 are as in (7) and (8), respectively.

The goal then becomes to find a v_2 such that the origin of the system (27)–(29) is globally uniformly asymptotically stable.

A. Control Law A

Theorem 1: The origin of the system (1)–(3) is globally uniformly asymptotically stabilized by the control law

$$u_1(t,\chi_1,x_3) = \frac{-c_1\chi_1 + c_3\omega\sin(\omega t)x_3}{1+c_3\cos(\omega t)\phi_x} \qquad (30)$$

$$\begin{aligned}u_2(t,\chi_1,\chi_2,x_3) &= \dot{\phi}_x - c_2\chi_2 - \gamma(U_1)x_3^{\mathrm{T}}u_1 \\ &\quad - 2c_3\gamma(U_1)\cos(\omega t)\chi_1^{\mathrm{T}}u_1 \qquad (31)\end{aligned}$$

where

$$\gamma(U_1) \triangleq 2\left(\frac{c_1}{16\omega} + 2 - \frac{c_1}{16\omega}\sin(2\omega t)\frac{2U_1+U_1^2}{(1+U_1)^2}\right) \qquad (32)$$

for some $\epsilon \in (0,1)$, $c_3 \in (0,1)$, $c_3\omega \geq c_1 > 0$ and $c_2, \omega > 0$, where U_1, ϕ_x and $\dot{\phi}_x$ are given by (12), (7) and (26), respectively.

Furthermore,

$$U_3(t,\chi_1,\chi_2,x_3) = U_2(t,\chi_1,x_3) + \chi_2^2/2, \tag{33}$$

where U_2 is given by (10), is a strict Lyapunov function for the controlled system.

Proof: We concentrate our efforts on the system (27)–(29). We define the Lyapunov function candidate U_3 as

$$U_3(t,\chi_1,\chi_2,x_3) \triangleq U_2(t,\chi_1,x_3) + \frac{1}{2}\chi_2^2 \tag{34}$$

which satisfies

$$2U_1 + \frac{1}{2}\chi_2^2 \leq U_3 \leq 2\left(\frac{c_1}{16\omega}+1\right)U_1 + \frac{1}{2}\chi_2^2.$$

The time derivative of U_3 is given by

$$\begin{aligned}\dot{U}_3 &= \frac{\partial U_2}{\partial t} + \frac{\partial U_2}{\partial \chi_1}\dot{\chi}_1 + \frac{\partial U_2}{\partial x_3}\dot{x}_3 + \chi_2\dot{\chi}_2 \\ &= \frac{\partial U_2}{\partial t} + \frac{\partial U_2}{\partial \chi_1}\left(-c_1\chi_1 + c_3\cos(\omega t)\chi_2 u_1\right) \\ &\quad + \frac{\partial U_2}{\partial x_3}(\chi_2+\phi_x)u_1 + \chi_2 v_2 \\ &= \frac{\partial U_2}{\partial t} - \frac{\partial U_2}{\partial \chi_1}c_1\chi_1 + \frac{\partial U_2}{\partial x_3}\phi_x u_1 \\ &\quad + \frac{\partial U_2}{\partial \chi_1}c_3\cos(\omega t)\chi_2 u_1 + \frac{\partial U_2}{\partial x_3}\chi_2 u_1 + \chi_2 v_2.\end{aligned}$$

Based on the proof of Lemma 1, we know that the first three terms on the right-hand side of the above expression are collectively less than or equal to $-\lambda(U_1)$. We thus get that

$$\begin{aligned}\dot{U}_3 &\leq -\lambda(U_1) + \frac{\partial U_2}{\partial \chi_1}c_3\cos(\omega t)\chi_2 u_1 + \frac{\partial U_2}{\partial x_3}\chi_2 u_1 + \chi_2 v_2 \\ &= -\lambda(U_1) \\ &\quad + \chi_2\left(2c_3\gamma(U_1)\cos(\omega t)\chi_1^{\mathrm{T}}u_1 + \gamma(U_1)x_3^{\mathrm{T}}u_1 + v_2\right) \\ &= -\lambda(U_1) - c_2\chi_2^2 \triangleq -W_3(\chi_1,\chi_2,x_3)\end{aligned} \tag{35}$$

with the feedback $v_2 = -c_2\chi_2 - 2c_3\gamma(U_1)\cos(\omega t)\chi_1^{\mathrm{T}}u_1 - \gamma(U_1)x_3^{\mathrm{T}}u_1$. Thus, according to [12, Theorem 4.9], the origin of the system (27)–(29) is globally uniformly asymptotically stable.

Knowing that $\chi_1 \equiv 0$ iff $x_1 \equiv x_3 \equiv 0$ and that $u_2 = \dot{\phi}_x + v_2$, we see that the control law (30)–(31) globally uniformly asymptotically stabilizes the origin of the system (1)–(3). ∎

B. Control Law B

Theorem 2: The origin of the system (1)–(3) is globally uniformly asymptotically stabilized by the control law

$$u_1(t,\chi_1,x_3) = \frac{-c_1\chi_1 + c_3\omega\sin(\omega t)x_3}{1+c_3\cos(\omega t)\phi_x} \tag{36}$$

$$u_2(t,\chi_1,\chi_2,x_3) = \dot{\phi}_x - c_2\chi_2 \tag{37}$$

for some $\epsilon \in (0,1)$, $c_3 \in (0,1)$, $c_3\omega \geq c_1 > 0$ and $c_2, \omega > 0$, where ϕ_x and $\dot{\phi}_x$ are given by (7) and (26), respectively. .

Proof: According to [11, Theorem 2], the origin of the non-linear time-varying cascade system

$$\dot{z}_1 = f_1(t,z_1) + g(t,z)z_2 \tag{38}$$

$$\dot{z}_2 = f_2(t,z_2) \tag{39}$$

is globally uniformly asymptotically stable if

1) The origin of the system $\dot{z}_1 = f_1(t,z_1)$ is globally uniformly asymptotically stable.
2) The function $g(t,z)$ satisfies
$$\|g(t,z)\| \leq \theta_1(\|z_2\|) + \theta_2(\|z_2\|)\|z_1\| \tag{40}$$
for some continuous functions $\theta_1, \theta_2 : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$.
3) The origin of the system $\dot{z}_2 = f_2(t,x_2)$ is globally exponentially stable.

These points are satisfied for the system (1)–(3) with $z_1 \triangleq [\chi_1^{\mathrm{T}}, x_3^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{2m}$ and $z_2 \triangleq \chi_2 \in \mathbb{R}$ using the control (36)–(37):

1) Proven in Lemma 1.
2) In this case, the function g can be found to be
$$g(t,z) = \frac{c_3\omega\sin(\omega t)x_3 - c_1\chi_1}{1+c_3\cos(\omega t)\phi_x}\begin{bmatrix} c_3\cos(\omega t) \\ 1\end{bmatrix}.$$
Using the 2-norm gives
$$\begin{aligned}\|g\|^2 &= \frac{\|-c_1\chi_1 + c_3\omega\sin(\omega t)x_3\|^2}{(1+c_3\cos(\omega t)\phi_x)^2}\left(c_3^2\cos^2(\omega t)+1\right) \\ &\leq \frac{32}{9}\left(c_3^2+1\right)\left(\|c_1\chi_1\|^2 + \|c_3\omega\sin(\omega t)x_3\|^2\right) \\ &\leq \frac{32}{9}\left(c_3^2+1\right)\max(c_1^2, c_3^2\omega^2)\left(\|\chi_1\|^2+\|x_3\|^2\right) \\ &= \frac{32}{9}\left(c_3^2+1\right)c_3^2\omega^2\left\|\begin{bmatrix}\chi_1 \\ x_3\end{bmatrix}\right\|^2 = k^2\|z_1\|^2\end{aligned}$$
since $c_1 < c_3\omega$ by design.
3) $\dot{\chi}_2 = -c_2\chi_2 \Rightarrow \chi_2(t) = \chi_2(t_0)e^{-c_2(t-t_0)}$.

Thus the origin of the system (1)–(3), with the control input (36)–(37) is globally uniformly asymptotically stable. ∎

V. Simulations

To verify the results in the previous section, numerical simulations were carried out on the system (1)–(3) with the control laws of Theorems 1 and 2. The following (arbitrary) simulation parameters were used:

$$\begin{aligned} m &= 2 & c_1 &= 0.8 \\ t_0 &= 0 & c_2 &= 1 \\ x_1(t_0) &= [3,\ -1]^{\mathrm{T}} & c_3 &= 0.9 \\ x_2(t_0) &= 1 & \epsilon &= 0.01 \\ x_3(t_0) &= [-0.5,\ 2]^{\mathrm{T}} & \omega &= 1 \end{aligned}$$

All simulations were in accordance with the theoretical results.

Three metrics were used for the comparison of the control laws: The smallest value of T such that $\|x_i(t)\| \leq 0.05\|x_i(t_0)\|$, $\forall\ t \geq T, i \in \{1,2,3\}$, was used as convergence time. The simulations were halted at $t = T$. As a measure of actuator use, the RMS value $u_{1,\mathrm{RMS}}^2 = \frac{1}{T-t_0}\int_{t_0}^{T}\|u_1(t)\|^2\ \mathrm{d}t$ of u_1 and $u_{2,\mathrm{RMS}}^2 = \frac{1}{T-t_0}\int_{t_0}^{T}|u_2(t)|^2\ \mathrm{d}t$ of u_2 were used.

A. *Control Law A*

Simulation results with Control Law A can be seen in Figs. 1–6. The origin is, as expected, attractive.

According to Theorem 1, the time derivative of the Lyapunov function U_3 along the trajectories of the system (27)–(29) should satisfy the property $\dot{U}_3 + W_3(\chi_1, \chi_2, x_3) \leq 0$ for all $t \geq t_0$, where W_3 is given by (35). As can be seen from Fig. 4, this is the case.

With the parameters used in this simulation, the convergence time was $T \approx 29.3$. The actuator use was $u_{1,\mathrm{RMS}} \approx 0.209$ and $u_{2,\mathrm{RMS}} \approx 1.84$.

B. *Control Law B*

Simulation results with Control Law B can be seen in Figs 7–11. The origin is, as expected, attractive.

With the parameters used in this simulation, the convergence time was $T \approx 41.8$. The actuator use was $u_{1,\mathrm{RMS}} \approx 0.522$ and $u_{2,\mathrm{RMS}} \approx 0.263$.

C. *Analysis*

From the figures it is seen that, in this case, the use of control input u_2 is significantly less with Control Law B than with Control Law A. However, the convergence time is longer for Control Law B than A, and the use of u_1 higher.

While only the results of one simulation is included in this paper, simulations with different parameters and different initial conditions indicates that the above is the case in general.

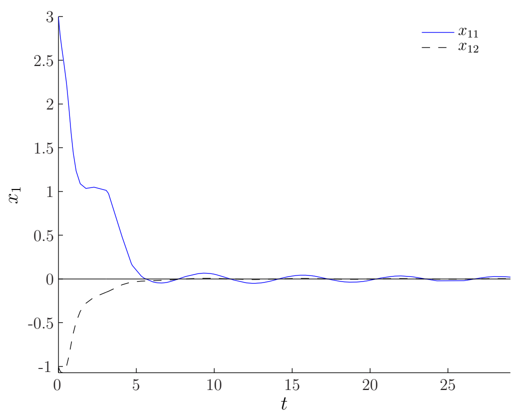

Fig. 1. Control Law A: $x_1(t)$.

VI. Conclusions

Two control laws that globally asymptotically stabilize the origin of the nonholonomic chained-form system (1)–(3) were presented, and a strict Lyapunov function developed.

The control laws were tested in simulation. As can be seen in Figs. 1–11, the simulation results are in accordance with the theoretical results.

Simulation results show some differences between Control Laws A and B. Control Law A is faster than Control Law B. Control Law A uses less control input u_1 but more u_2 than Control Law B. Control Law B is the simpler of the pair, but only for Control Law A is there a strict Lyapunov function.

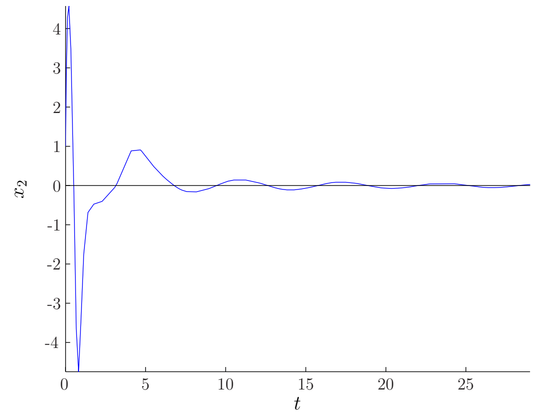

Fig. 2. Control Law A: $x_2(t)$.

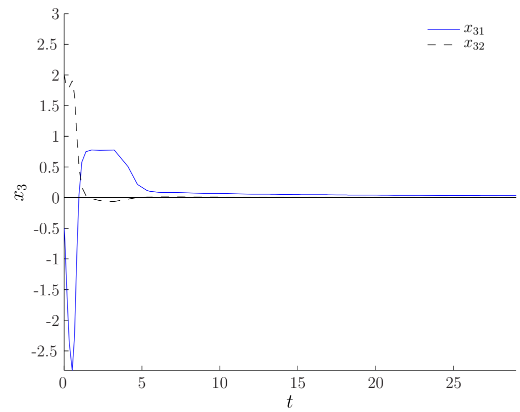

Fig. 3. Control Law A: $x_3(t)$.

Appendix

A. *From* (13) *to* (14)

To get from (13) to (14), we need to prove that

$$\frac{c_1 \sin(\omega t)\|x_3\|^2 x_3^{\mathrm{T}} \chi_1}{\left(1 - \frac{c_3}{4}\sin(2\omega t)\frac{\|x_3\|^2}{\epsilon+\|x_3\|^2}\right)(\epsilon + \|x_3\|^2)} \leq 2c_1 |\sin(\omega t)| \|\chi_1\| \frac{\|x_3\|^3}{\epsilon + \|x_3\|^2}. \tag{41}$$

We start by noting that

$$\sin(\omega t) x_3^{\mathrm{T}} \chi_1 \leq |\sin(\omega t)| \|\chi_1\| \|x_3\| < \frac{3}{2} |\sin(\omega t)| \|\chi_1\| \|x_3\| < 2|\sin(\omega t)| \|\chi_1\| \|x_3\| \left(1 - \frac{c_3}{4}\sin(2\omega t)\frac{\|x_3\|^2}{\epsilon + \|x_3\|^2}\right)$$

since

$$1 - \frac{c_3}{4}\sin(2\omega t)\frac{\|x_3\|^2}{\epsilon + \|x_3\|^2} \in (3/4, 5/4).$$

Multiplying with

$$\frac{c_1 \|x_3\|^2}{\left(1 - \frac{c_3}{4}\sin(2\omega t)\frac{\|x_3\|^2}{\epsilon+\|x_3\|^2}\right)(\epsilon + \|x_3\|^2)} > 0$$

on both sides of the inequality gives (41). This concludes this part of the proof.

B. From (15) *to* (16)

To get from (15) to (16), we need to prove that

$$2c_1|\sin(\omega t)|\|\chi_1\|\frac{\|x_3\|^3}{\epsilon+\|x_3\|^2} - \frac{2c_3\omega\sin^2(\omega t)\|x_3\|^4}{3(\epsilon+\|x_3\|^2)} - 4c_1\|\chi_1\|^2 \leq -\frac{c_3\omega\sin^2(\omega t)\|x_3\|^4}{3(\epsilon+\|x_3\|^2)} - c_1\|\chi_1\|^2. \quad (42)$$

We start by noting that

$$\begin{aligned} 0 &\leq c_1\|x_3\|^2\left(\frac{1}{\sqrt{3}}|\sin(\omega t)|\|x_3\| - \sqrt{3}\|\chi_1\|\right)^2 \\ &= \frac{c_1}{3}\sin^2(\omega t)\|x_3\|^4 + 3c_1\|\chi_1\|^2\|x_3\|^2 \\ &\quad - 2c_1|\sin(\omega t)|\|\chi_1\|\|x_3\|^3 \\ &\leq \frac{c_3\omega}{3}\sin^2(\omega t)\|x_3\|^4 - 2c_1|\sin(\omega t)|\|\chi_1\|\|x_3\|^3 \\ &\quad + 3c_1\|\chi_1\|^2(\epsilon+\|x_3\|^2) \end{aligned}$$

since $c_3\omega \geq c_1$ by design.

Dividing by $(\epsilon+\|x_3\|^2) > 0$ on both sides of the above inequality and rearranging the terms gives the inequality (42). This concludes this part of the proof.

C. Equation (19)

We need to prove (19), which reads

$$W_1(\chi_1, x_3) \geq \lambda(U_1)$$

or

$$\frac{c_3\omega\|x_3\|^4}{3(\epsilon+\|x_3\|^2)} + \frac{c_1}{2}\|\chi_1\|^2 \geq \frac{c_1U_1^2}{8(1+U_1)}.$$

We start by noting that

$$\begin{aligned} &(8c_3\omega - 3c_1)\|x_3\|^6 + (8c_3\omega - 3\epsilon c_1)\|x_3\|^4 \\ &+(16c_3\omega + 24c_1)\|x_3\|^4\|\chi_1\|^2 + (24\epsilon c_1 + 12c_1)\|x_3\|^2\|\chi_1\|^2 \\ &+12c_1\|x_3\|^2\|\chi_1\|^4 + 12\epsilon c_1\|\chi_1\|^4 + 12\epsilon c_1\|\chi_1\|^2 \geq 0 \end{aligned}$$

since $c_3\omega \geq c_1$ and $\epsilon \in (0,1)$.

Dividing by $24(\epsilon+\|x_3\|^2)(1+\|x_3\|^2+2\|\chi_1\|^2) > 0$ on both sides of the inequality and rearranging the terms gives

$$\begin{aligned} &\frac{8c_3\omega(1+\|x_3\|^2+2\|\chi_1\|^2)\|x_3\|^4}{24(\epsilon+\|x_3\|^2)(1+\|x_3\|^2+2\|\chi_1\|^2)} \\ &-\frac{3c_1(\|x_3\|^2+2\|\chi_1\|^2)^2(\epsilon+\|x_3\|^2)}{24(\epsilon+\|x_3\|^2)(1+\|x_3\|^2+2\|\chi_1\|^2)} \\ &+\frac{4c_1\|\chi_1\|^2(\epsilon+\|x_3\|^2)(1+\|x_3\|^2+2\|\chi_1\|^2)}{24(\epsilon+\|x_3\|^2)(1+\|x_3\|^2+2\|\chi_1\|^2)} \geq 0. \end{aligned}$$

Using $U_1 = 2\|\chi_1\|^2 + \|x_3\|^2$ and rearranging the terms gives (19). This concludes this part of the proof.

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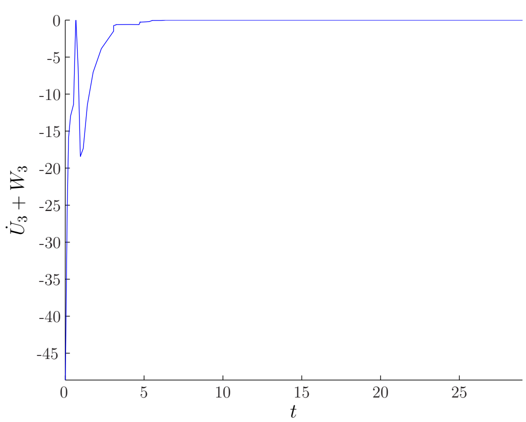

Fig. 4. Control Law A: $\dot{U}_3 + W_3$.

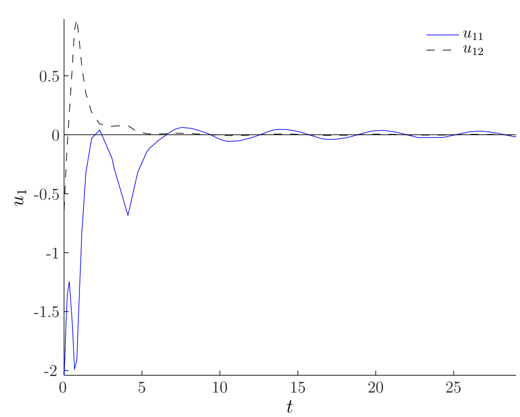

Fig. 5. Control Law A: $u_1(t)$.

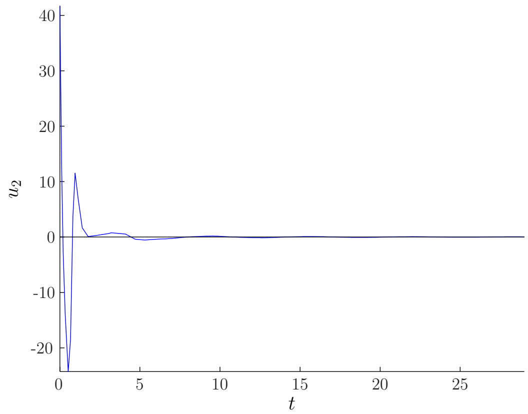

Fig. 6. Control Law A: $u_2(t)$.

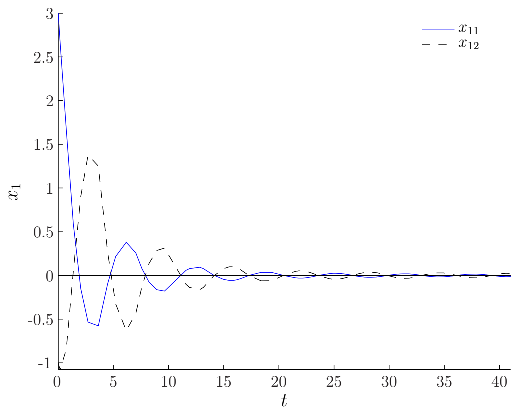

Fig. 7. Control Law B: $x_1(t)$.

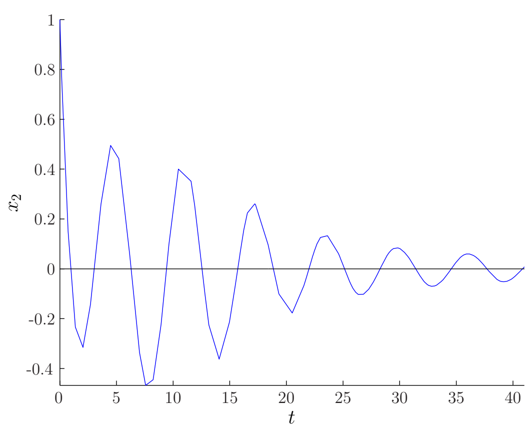

Fig. 8. Control Law B: $x_2(t)$.

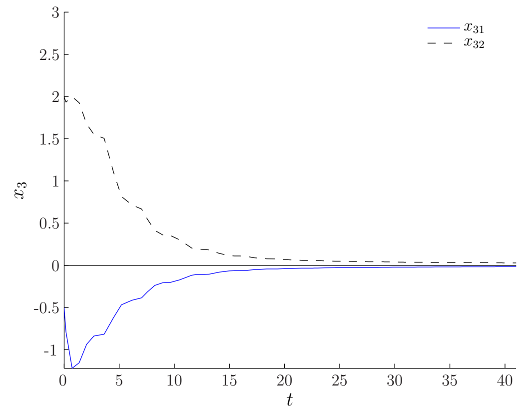

Fig. 9. Control Law B: $x_3(t)$.

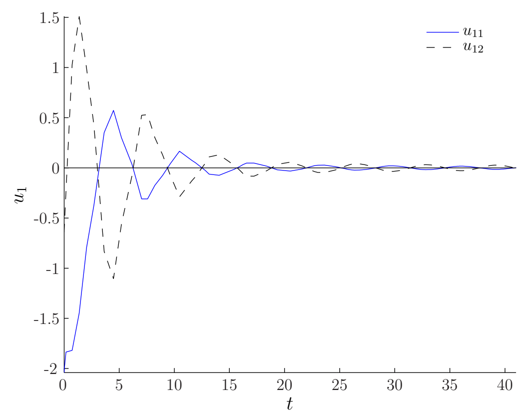

Fig. 10. Control Law B: $u_1(t)$.

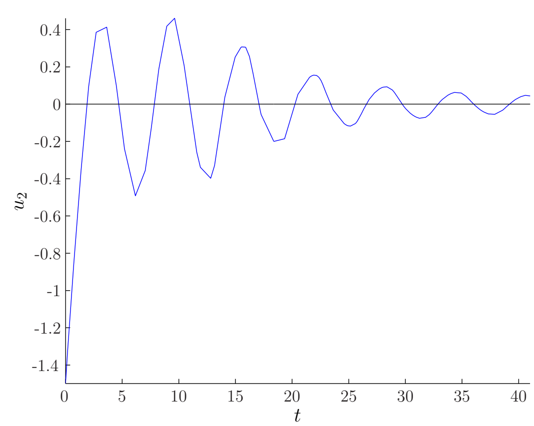

Fig. 11. Control Law B: $u_2(t)$.