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$\mathcal{L}_2$-stability of hinging hyperplane models via integral quadratic constraints

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Abstract— **This paper is concerned with $\mathcal{L}_2$-stability analysis of hinging hyperplane autoregressive models with exogenous inputs (HHARX). The proposed approach relies on analysis results for systems with repeated nonlinearities based on the use of integral quadratic constraints. An equivalent linear fractional representation of HHARX models is firstly derived. In this representation, an HHARX model is seen as the feedback interconnection of a linear system and a diagonal static block with repeated scalar nonlinearity. This makes it possible to exploit the aforementioned analysis results. The corresponding sufficient condition for $\mathcal{L}_2$-stability can be checked via a linear matrix inequality. A numerical example shows that the proposed approach is effective in practice.**

I. Introduction

This paper addresses the problem of $\mathcal{L}_2$-stability analysis for hinging hyperplane autoregressive models with exogenous inputs (HHARX), i.e. nonlinear regression models based on the use of hinging hyperplane (HH) functions. An HH function is the sum of a given number of hinge functions, each consisting in either the maximum or minimum of two affine functions [1]. It turns out that HH functions are a special class of piecewise affine (PWA) functions for which the mapping is continuous. In fact, the class of HH functions is equivalent to the class of continuous PWA functions that can be expressed in Chua's canonical representation [2], [3]. Though it is not a representation of all continuous PWA functions, the class of HH functions is an approximant of all continuous functions on a compact set [4]. A valuable property of HH functions is that the number of parameters required for accurate approximation is relatively small compared to PWA functions [5]. Hence, HHARX models form a suitable black-box model structure for nonlinear identification [6]. Typically, their identification is addressed by minimizing a suitable cost function (see, e.g., [7]–[9]).

A. Motivation

While a wide range of methods are available for analyzing Lyapunov stability of PWA systems (see the survey paper [10] and references therein), to the best of the authors' knowledge $\mathcal{L}_2$-stability analysis of HHARX models is to a large extent an open field. In principle, a possible indirect approach is to build a state-space realization of the HHARX model, and then look for a suitable piecewise quadratic storage function guaranteeing the boundedness of the $\mathcal{L}_2$-gain of the system. However, this approach may be hindered by the following facts:

i) When an HHARX model is equivalently represented as a piecewise affine ARX (PWARX) model, i.e. through a polyhedral partition of the regressor domain and affine ARX submodels in each region of the partition, the number of regions is typically much larger than the number of hinges.

ii) Since a complete realization theory for PWARX models is not available yet (in [11] only autonomous systems are considered; see also [12]), in order to build a state-space realization of the PWARX model one has to resort to the trivial approach of defining the state equal to the regression vector of the PWARX model. The PWA state-space realization thus obtained is typically not minimal with respect to both the dimension of the state and the number of regions (equal to the number of regions of the PWARX model).

iii) The computation of piecewise quadratic storage functions for PWA models can be tackled via LMI techniques similar to those proposed in [13] in the context of passivity. However, the overall computation time correlates directly to the number of regions and, more importantly, to the number of transitions that occur between regions.

In summary, $\mathcal{L}_2$-stability analysis of HHARX models via state-space techniques may be seriously limited by computational complexity due to the large number of regions and possible transitions that may show up in the equivalent PWA state-space model.

B. Paper contribution

The first contribution of this paper is to derive an equivalent linear fractional representation (LFR) of HHARX models. In this representation, a given HHARX model is decomposed into the feedback interconnection of a linear system and a diagonal static block with repeated scalar nonlinearity. Provided that the linear system is stable, this makes it possible to apply analysis results for systems with repeated nonlinearities based on integral quadratic constraints (IQCs) [14]. The second contribution of the paper is hence to provide a sufficient condition for $\mathcal{L}_2$-stability of HHARX models and a computational procedure for checking this sufficient condition via a single linear matrix inequality (LMI) whose dimension grows linearly with the number of hinges.

The paper is structured as follows. In Section II a linear fractional representation of HHARX models is derived.

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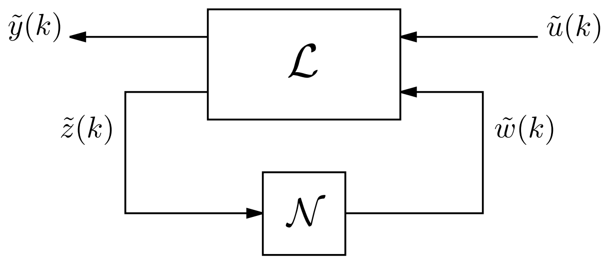


Fig. 1. LFR of an interconnected system consisting of a finite number of linear time-invariant systems and static nonlinear maps.

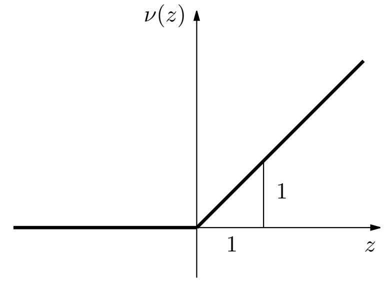


Fig. 2. Scalar nonlinearity that is repeated in the diagonal static block $\mathcal{N}$ of the LFR of an HHARX model.

Section III describes the proposed approach to $\mathcal{L}_2$-stability analysis of HHARX models. A numerical example is illustrated in Section IV, while Section V draws the conclusions and foreshadows topics for future research.

C. Notation

The sets of real, integer and nonnegative integer numbers are denoted by $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{Z}^{\geq 0}$, respectively. The absolute value of a scalar $x \in \mathbb{R}$ is denoted by $|x|$, while $\binom{n}{k}$ denotes the binomial coefficient. A matrix A with elements a_{ij} is sometimes introduced as $A = \{a_{ij}\}$. An $m \times n$ matrix with 0 everywhere is denoted by $0_{m\times n}$. The Kronecker product of two matrices A and B is denoted by $A \otimes B$, while H^* denotes the conjugate transpose of a complex matrix H. For $p \geq 1$, the set $\mathcal{L}_p$ is formed by all discrete-time signals u such that $\|u\|_{\mathcal{L}_p} = (\sum_{k=0}^{\infty} \|u(k)\|^p)^{\frac{1}{p}} < \infty$, where $\|\cdot\|$ denotes the p-norm of a vector. The extended set $\mathcal{L}_{p,e}$ is defined by $\mathcal{L}_{p,e} = \{u : u_{[0,\tau]} \in \mathcal{L}_p,\ \forall \tau \in \mathbb{Z}^{\geq 0}\}$, where $u_{[0,\tau]}$ is the truncation of u at time τ, i.e.

$$u_{[0,\tau]}(k) = \begin{cases} u(k) & \text{if } 0 \leq k \leq \tau \\ 0 & \text{if } k > \tau. \end{cases} \quad (1)$$

For a real discrete-time signal $h(k)$, $\hat{h}(z) = \mathcal{Z}\{h(k)\}$ denotes its zeta transform. The set of proper rational matrices with real coefficients and all poles inside the open unit circle is denoted by $\mathcal{RH}_\infty$. If $\varepsilon > 0$, a scalar function $\nu : \mathbb{R} \to \mathbb{R}$ is said to belong to the sector $[0, \varepsilon]$ if $\nu(x)(\varepsilon x - \nu(x)) \geq 0$ for all $x \in \mathbb{R}$.

II. LINEAR FRACTIONAL REPRESENTATION OF HHARX MODELS

Interconnected systems consisting of any finite number of linear time-invariant systems and static nonlinear maps can be always represented as in Fig. 1 (see, e.g., [15]). The block $\mathcal{L}$ describes the overall linear dynamics of the system, while all the nonlinearities are pulled out in the block $\mathcal{N}$. Signals $\tilde{u}(k)$ and $\tilde{y}(k)$ are the system input and output at time $k \in \mathbb{Z}$, while $\tilde{z}(k)$ and $\tilde{w}(k)$ are internal signals representing the inputs and outputs of the nonlinear part. This section describes a systematic procedure for representing a given HHARX model as an LFR of the type shown in Fig. 1.

A. HHARX models

Let us consider a single-input single-output system described by the HHARX model [1], [7]:

$$y(k) = \theta_0^T \begin{bmatrix} \varphi(k) \\ 1 \end{bmatrix} + \sum_{i=1}^{M} \sigma_i \max\{\theta_i^T \begin{bmatrix} \varphi(k) \\ 1 \end{bmatrix}, 0\}, \quad (2)$$

where, for fixed orders n_a and n_b, the regression vector is

$$\begin{aligned} \varphi(k) = [\ & y(k-1)\ \dots\ y(k-n_a) \\ & u(k)\ u(k-1)\ \dots\ u(k-n_b)\ ]^T, \end{aligned} \quad (3)$$

and $y \in \mathbb{R}$ and $u \in \mathbb{R}$ are the system output and input, respectively. In (2), $\theta_i \in \mathbb{R}^{n_a+n_b+2}$, $i = 0, 1, \dots, M$, are the model parameters, while the coefficients $\sigma_i \in \{-1, 1\}$, $i = 1, \dots, M$, are needed to allow for both convex and non-convex functions. The functions $h_i(\varphi) = \max\{\theta_i^T \begin{bmatrix} \varphi \\ 1 \end{bmatrix}, 0\}$, $i = 1, \dots, M$, are called *hinge* functions and have the shape of "open books," being formed by two half-planes joined together at the hinge. The number of regions of an equivalent PWARX representation of (2) is bounded by the quantity $\sum_{j=0}^{n} \binom{M}{j}$, depending on the length $n = n_a + n_b + 1$ of the regression vector and on the number M of hinge functions.

Remark 1: It is worthwhile to notice that the representation (2) is not unique, i.e. the same system can be described by several different sets of parameter values. This is due to the fact that the following property holds:

$$x + \max\{-x, 0\} = \max\{x, 0\}, \quad \forall x \in \mathbb{R}. \quad (4)$$

Hence, for instance, $y(k) = \theta_0^T \begin{bmatrix} \varphi(k) \\ 1 \end{bmatrix} + \max\{\theta_1^T \begin{bmatrix} \varphi(k) \\ 1 \end{bmatrix}, 0\}$ and $y(k) = (\theta_0 + \theta_1)^T \begin{bmatrix} \varphi(k) \\ 1 \end{bmatrix} + \max\{-\theta_1^T \begin{bmatrix} \varphi(k) \\ 1 \end{bmatrix}, 0\}$ are alternative representations of the same system.

B. Converting HHARX models into LFR

In order to represent (2) through an LFR, let us introduce a fictitious constant input $v(k) = 1$, $\forall k$. Then, by defining

$$\phi(k) = [\ \varphi(k)^T\ v(k)\ ]^T, \quad (5)$$

the HHARX model (2) can be rewritten as

$$y(k) = \theta_0^T \phi(k) + \sum_{i=1}^{M} \sigma_i \max\{\theta_i^T \phi(k), 0\}. \quad (6)$$

If we define $\tilde{z}(k)$ and $\tilde{w}(k)$ componentwise as

$$\tilde{z}_i(k) = \theta_i^T \phi(k), \quad i = 1, \dots, M, \quad (7)$$

$$\tilde{w}_i(k) = \max\{\tilde{z}_i(k), 0\}, \quad i = 1, \dots, M, \quad (8)$$

TABLE I

TRANSFER FUNCTIONS OF THE LINEAR PART $\mathcal{L}$ IN THE LFR OF HHARX MODELS

$$Y(z) = \frac{\beta_0 + \beta_1\, z^{-1} + \ldots + \beta_{n_b}\, z^{-n_b}}{d(z^{-1})}\, U(z) + \frac{\gamma_0}{d(z^{-1})}\, V(z) + \frac{\Sigma}{d(z^{-1})}\, \tilde{W}(z) \triangleq L_{yu}(z)U(z) + L_{yv}(z)V(z) + L_{y\tilde{w}}(z)\tilde{W}(z)$$

$$\tilde{Z}(z) = \Theta^T \begin{bmatrix} z^{-1}L_{yu}(z) \\ \vdots \\ z^{-n_a}L_{yu}(z) \\ 1 \\ z^{-1} \\ \vdots \\ z^{-n_b} \\ 0 \end{bmatrix} U(z) + \Theta^T \begin{bmatrix} z^{-1}L_{yv}(z) \\ \vdots \\ z^{-n_a}L_{yv}(z) \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} V(z) + \Theta^T \begin{bmatrix} z^{-1}L_{y\tilde{w}}(z) \\ \vdots \\ z^{-n_a}L_{y\tilde{w}}(z) \\ 0_{1\times M} \\ 0_{1\times M} \\ \vdots \\ 0_{1\times M} \\ 0_{1\times M} \end{bmatrix} \tilde{W}(z) \triangleq L_{\tilde{z}u}(z)U(z) + L_{\tilde{z}v}(z)V(z) + L_{\tilde{z}\tilde{w}}(z)\tilde{W}(z)$$

and let $\tilde{u}(k) = [\ u(k)\ v(k)\ ]^T$ and $\tilde{y}(k) = y(k)$, then (6) can be represented as in Fig. 1. The equations of the linear part $\mathcal{L}$ are given by

$$\tilde{y}(k) = \theta_0^T \phi(k) + \Sigma\, \tilde{w}(k) \quad (9)$$
$$\tilde{z}(k) = \Theta^T \phi(k), \quad (10)$$

where $\Sigma = [\ \sigma_1\ \ldots\ \sigma_M\ ]$ and $\Theta = [\ \theta_1\ \ldots\ \theta_M\ ]$, while $\mathcal{N}$ is a diagonal static block described by the equation:

$$\tilde{w}(k) = \mathcal{N}(\tilde{z}(k)) = [\ \nu(\tilde{z}_1(k))\ \ldots\ \nu(\tilde{z}_M(k))\ ]^T, \quad (11)$$

where $\nu(z) = \max\{z, 0\}$. The repeated scalar nonlinearity $\nu(z)$, plotted in Fig. 2, is non-odd and belongs to the sector $[0, 1]$. Expressions for the transfer functions of the linear part $\mathcal{L}$ are reported in Table I, where we have decomposed the parameter vector θ_0 as

$$\theta_0 = [\ -\alpha_1\ \ldots\ -\alpha_{n_a}\ \beta_0\ \beta_1\ \ldots\ \beta_{n_b}\ \gamma_0\ ]^T, \quad (12)$$

and we have defined the polynomial:

$$d(z^{-1}) = 1 + \alpha_1\, z^{-1} + \ldots + \alpha_{n_a}\, z^{-n_a}. \quad (13)$$

It is stressed that all the input-output behaviors of the original HHARX model (2) are reproduced by the constructed LFR model, provided that the additional input $v(k)$ is taken constant and equal to 1.

III. $\mathcal{L}_2$-STABILITY ANALYSIS OF HHARX MODELS

The following definition of $\mathcal{L}_2$-stability is considered in this paper [16]:

Definition 1: A system S with input $\tilde{u}$ and output $\tilde{y}$ is said to be *finite-gain $\mathcal{L}_2$-stable* from $\tilde{u}$ to $\tilde{y}$ if there exists a nonnegative constant γ such that, for all $\tilde{u} \in \mathcal{L}_{2,e}$,

$$\|\tilde{y}_{[0,\tau]}\|_{\mathcal{L}_2} \leq \gamma\, \|\tilde{u}_{[0,\tau]}\|_{\mathcal{L}_2}, \quad \forall \tau = 0, 1, \ldots \quad (14)$$

In the above definition, it is implicitly assumed that the output vanishes at $\tilde{u} = 0$. Otherwise, a bias term should be added to the right-hand side of (14).

In order to address $\mathcal{L}_2$-stability for the HHARX model (2), the idea is to exploit the linear fractional representation derived in Section II. The following result is straightforward.

Theorem 1: If the LFR model (9)-(11) depicted in Fig. 1 is finite-gain $\mathcal{L}_2$-stable from $\tilde{u} = [\ u\ \ v\ ]^T$ to $\tilde{y}$, then the HHARX model (2) is finite-gain $\mathcal{L}_2$-stable from u to y.

Proof. It follows immediately by recalling that all the input-output behaviors of the HHARX model (2) are obtained from the LFR model (9)-(11) with $v(k) = 1$, $\forall k$, and by observing that $v \in \mathcal{L}_{2,e}$. □

The subsequent analysis will be conducted under the following assumption.

Assumption 1: All the transfer functions $L_{y,u}$, $L_{y,v}$, $L_{y,\tilde{w}}$, $L_{\tilde{z},u}$, $L_{\tilde{z},v}$, $L_{\tilde{z},\tilde{w}}$ defined in Table I are elements of $\mathcal{RH}_\infty$.

For the transfer functions defined in Table I, the above assumption simply corresponds to require that the polynomial $d(z^{-1})$ in (13) is Schur. Notice that Assumption 1 may not be so restrictive as it seems. As discussed in Remark 1, the same system admits different equivalent HHARX representations, some of which may satisfy Assumption 1, while others may not. An example is provided in Section IV.

A. $\mathcal{L}_2$-stability of feedback interconnections

Let us consider the feedback interconnection depicted in Fig. 3, where $L_{\tilde{z},\tilde{w}}$ is the transfer function matrix from $\tilde{w}$ to $\tilde{z}$ of the linear block $\mathcal{L}$ in Fig. 1, defined in Table I, and $\mathcal{N}$ is the static nonlinear block defined in (11). The following is a standard result in stability theory (see, e.g, [16]).

Theorem 2: Under Assumption 1, if the loop in Fig. 3 is finite-gain $\mathcal{L}_2$-stable from $\eta = [\ \eta_1^T\ \eta_2^T\ ]^T$ to $\zeta = [\ \zeta_1^T\ \zeta_2^T\ ]^T$, then the LFR model in Fig. 1 is finite-gain $\mathcal{L}_2$-stable from $\tilde{u}$ to $\tilde{y}$.

Based on Theorems 1 and 2, we will henceforth focus on the $\mathcal{L}_2$-stability of the loop in Fig. 3. To this aim, the IQC analysis in [14] can be used, since the nonlinear block $\mathcal{N}$ defined in (11) is composed of repeated scalar monotone and sector-bounded functions. Since these functions are also

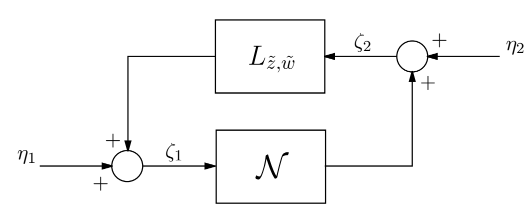


Fig. 3. Feedback configuration for stability analysis.

non-odd, further constraints have to be imposed with respect to the IQC analysis for the more common case of repeated odd nonlinearities. The following theorem specializes to the problem at hand the IQC analysis originally developed in [14] for the case of continuous-time systems and repeated non-odd scalar nonlinearities.

Theorem 3: Consider the loop in Fig. 3, with $L_{\tilde{z},\tilde{w}}$ defined in Table I and $\mathcal{N}$ defined in (11), and assume $L_{\tilde{z},\tilde{w}} \in \mathcal{RH}_\infty$. If there exist an $M \times M$ symmetric matrix-valued sequence $H(k) = \{h_{ij}(k)\}$, $i, j = 1, \ldots, M$, with entries in $\mathcal{L}_1$, and an $M \times M$ real symmetric matrix $G = \{g_{ij}\}$ satisfying the following conditions:

$$\begin{aligned}
&g_{ii} \geq \sum_{\substack{j=1\\ j\neq i}}^{M} |g_{ij}| + \sum_{j=1}^{M} \|h_{ij}(k)\|_{\mathcal{L}_1}, \ \forall i = 1, \ldots, M \\
&g_{ij} \leq 0, \ \forall i, j = 1, \ldots, M, \ i \neq j \\
&h_{ij}(k) \geq 0, \ \forall i, j = 1, \ldots, M \\
&\begin{bmatrix} L_{\tilde{z},\tilde{w}}(e^{j\omega}) \\ I \end{bmatrix}^* \Pi(e^{j\omega}) \begin{bmatrix} L_{\tilde{z},\tilde{w}}(e^{j\omega}) \\ I \end{bmatrix} < 0, \quad \forall \omega \in [0, 2\pi],
\end{aligned} \tag{15}$$

where

$$\Pi(e^{j\omega}) = \begin{bmatrix} 0 & G - \hat{H}^*(e^{j\omega}) \\ G - \hat{H}(e^{j\omega}) & -2G + \hat{H}(e^{j\omega}) + \hat{H}^*(e^{j\omega}) \end{bmatrix}, \tag{16}$$

then the loop in Fig. 3 is finite-gain $\mathcal{L}_2$-stable from η to ζ.

Combined with Theorems 1 and 2, Theorem 3 provides a sufficient condition for $\mathcal{L}_2$-stability of the HHARX model (2). To clarify why Assumption 1 is required in Theorem 3, we observe that open-loop stability of $L_{\tilde{z},\tilde{w}}$ is necessary to apply the IQC analysis in [14], which is developed for all repeated scalar monotone nondecreasing nonlinearities belonging to a finite sector $[0, \varepsilon]$. In this respect, a degree of conservativeness is introduced in the proposed approach by treating the static nonlinearity in Fig. 2 as a general sector nonlinearity when applying Theorem 3.

For computation, it is convenient to restrict the sequence $H(k)$ introduced in Theorem 3 to a linear combination of basis functions, i.e.

$$H(k) = \sum_{q=1}^{r} \Lambda_q \, l_q(k), \tag{17}$$

where $\Lambda_q \in \mathbb{R}^{M\times M}$, $q = 1, \ldots, r$, are symmetric matrices with positive entries $\lambda_{q_{ij}}$, and l_q, $q = 1, \ldots, r$, are positive scalar sequences with finite $\mathcal{L}_1$ norm $\rho_q = \|l_q\|_{\mathcal{L}_1}$. With the parameterization (17), the $\mathcal{L}_2$-stability condition of Theorem 3 can be recast into a set of LMIs by using a Kalman-Yacubovich-Popov lemma argument [14], [17].

Theorem 4: Consider the loop in Fig. 3, with $L_{\tilde{z},\tilde{w}}$ defined in Table I and $\mathcal{N}$ defined in (11), and assume $L_{\tilde{z},\tilde{w}} \in \mathcal{RH}_\infty$. Let $\Lambda_q = \{\lambda_{q_{ij}}\}$ and l_q be defined in (17), and (A, B, C, D) be a state space realization of the transfer function matrix $\Psi(z) \begin{bmatrix} L_{\tilde{z},\tilde{w}}(z) \\ I \end{bmatrix}$, where

$$\Psi(z) = \begin{bmatrix} I & -I \\ 0 & I \\ I & -I \\ 0 & I \\ \Upsilon(z) & -\Upsilon(z) \\ 0 & I \end{bmatrix}, \tag{18}$$

and $\Upsilon(z) = [\ \hat{l}_1(z) \ \ldots \ \hat{l}_r(z)\ ]^T \otimes I$. If there exist $M \times M$ real symmetric matrices $G^+ = \{g^+_{ij}\}$, $G^- = \{g^-_{ij}\}$ and P satisfying the following conditions:

$$\begin{aligned}
&g^+_{ij} \geq 0, \ g^-_{ij} \geq 0, \ \forall i, j = 1, \ldots, M \\
&g^-_{ii} = 0, \ \forall i = 1, \ldots, M \\
&g^+_{ij} - g^-_{ij} \leq 0, \ \forall i, j = 1, \ldots, M, \ i \neq j \\
&g^+_{ii} \geq \sum_{\substack{j=1\\ j\neq i}}^{M} (g^+_{ij} + g^-_{ij}) + \sum_{j=1}^{M} \sum_{q=1}^{r} \rho_q \lambda_{q_{ij}}, \ \forall i = 1, \ldots, M \\
&\begin{bmatrix} A^TPA - P & A^TPB \\ B^TPA & B^TPB \end{bmatrix} + [\, C \ D\,]^T W [\, C \ D\,] < 0, \ \ P > 0,
\end{aligned} \tag{19}$$

where

$$W = \begin{bmatrix} 0 & G^+ & 0 & 0 & 0 & 0 \\ G^+ & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -G^- & 0 & 0 \\ 0 & 0 & -G^- & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -X^T \\ 0 & 0 & 0 & 0 & -X & 0 \end{bmatrix} \tag{20}$$

and $X = [\ \Lambda_1 \ \ldots \ \Lambda_r\ ]$, then the loop in Fig. 3 is finite-gain $\mathcal{L}_2$-stable from η to ζ.

For fixed basis functions l_q, $q = 1, \ldots, r$, the sufficient condition for $\mathcal{L}_2$-stability given in Theorem 4 can be checked via the solution of the LMI problem (19) in the unknowns G^+, G^-, P and Λ_q, $q = 1, \ldots, r$.

IV. Numerical Example

In this section, the sufficient condition for $\mathcal{L}_2$-stability provided by Theorem 4 is applied to a system described by a HHARX model (2) with $M = 4$ hinges, orders $n_a = 3$, $n_b = 1$, and parameters

$$\theta_0 = \begin{bmatrix} -0.8 & -0.5 & -0.3 & -0.1 & -0.7 & 0.1 \end{bmatrix}^T, \tag{21}$$

$$\Theta = \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 \end{bmatrix} = \begin{bmatrix} \alpha & 0.1 & 0.3 & 0.3 \\ 0.4 & 0.2 & 0.3 & 0.7 \\ 0.3 & 0.5 & 0.3 & 0.3 \\ -0.2 & 0.3 & -0.1 & 0.2 \\ 1 & -0.1 & 0.3 & -0.4 \\ 0.3 & -0.5 & 0.1 & 0.3 \end{bmatrix}, \tag{22}$$

$$\Sigma = \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix}, \tag{23}$$

where $\alpha > 0$ is an uncertain coefficient. The model thus defined satisfies Assumption 1, since all the zeros of the polynomial

$$d(z^{-1}) = 1 + 0.8\, z^{-1} + 0.5\, z^{-2} + 0.3\, z^{-3} \tag{24}$$

lie inside the open unit circle.

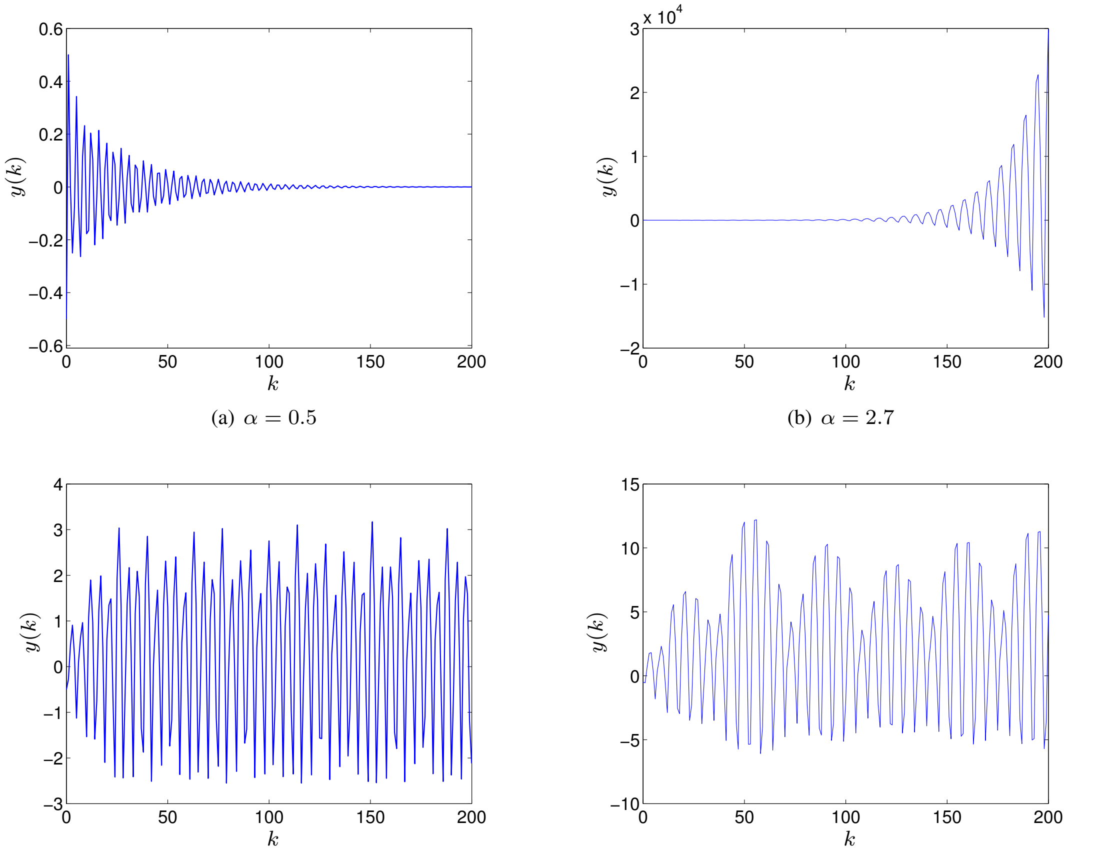
To test the effectiveness of the proposed approach, we provide a guaranteed estimate of the set of positive α that yield an $\mathcal{L}_2$-stable system. More specifically, we compute an interval contained in the $\mathcal{L}_2$-stability domain of the system with the aid of an alternating projection-like heuristic based on Theorem 4. We consider the following choices for $H(k)$:

a) $H(k) = 0$;

b) $H(k)$ parameterized as in (17) with the basis functions

$$l_q(k) = (0.1q)^k, \quad q = 1, \ldots, 9, \tag{25}$$

having the properties that $l_q(k) \geq 0$ for all k, and $\rho_q = \|l_q\|_{\mathcal{L}_1} = (1 - 0.1q)^{-1}$.



(a) $\alpha = 0.5$ (b) $\alpha = 2.7$

(c) $\alpha = 2.05$ (d) $\alpha = 2.68$

Fig. 4. Example of Section IV: Impulse responses of the HHARX model for different values of the uncertain parameter α.

With the choice a), we are able to prove $\mathcal{L}_2$-stability for $\alpha \in [0, 0.535]$, while the choice b) yields the less conservative estimate $\alpha \in [0, 2.08]$. In the latter case, the solution of the LMI problem of Theorem 4 took about 34 seconds CPU time on a 2.2 GHz AMD Athlon 64 processor using the SeDuMi optimization package [18].

Figure 4 reports simulations of the impulse response $y(k)$ of the HHARX model for different values of α. For $\alpha = 0.5$, which is contained in the estimated stability domain, the $\mathcal{L}_2$-stability condition is clearly satisfied, and moreover the output $y(k)$ converges to zero. This is the consequence of choosing the affine terms in such a way that y vanishes at $u = 0$. For $\alpha = 2.05$, i.e. very close to the boundary of the estimated stability domain, the system output exhibits an oscillatory behaviour which is still compatible with $\mathcal{L}_2$-stability. In particular, simulations show that the system trajectories converge to a stable limit cycle. The system finally goes unstable for $\alpha \approx 2.7$. It is interesting to note that the way to instability in this system is a sequence of bifurcations that also leads to complex non-periodic behaviour for values of α close to 2.7. Figures 4(d) and 5 report the impulse response $y(k)$ and the corresponding pairs of internal signals $(\tilde{z}_1(k), \tilde{z}_2(k))$ for $\alpha = 2.68$. These simulations clearly show that complex non-periodic solutions occur.

If (4) is repeatedly applied to the HHARX model (2) defined by the parameters (21)-(23), the following equivalent representation can be obtained:

$$\begin{aligned} y(k) &= (\theta_0 + \theta_1 - \theta_3 - \theta_4)^T \begin{bmatrix} \varphi(k) \\ 1 \end{bmatrix} \\ &+ \max\{-\theta_1^T \begin{bmatrix} \varphi(k) \\ 1 \end{bmatrix}, 0\} + \max\{\theta_2^T \begin{bmatrix} \varphi(k) \\ 1 \end{bmatrix}, 0\} \\ &- \max\{-\theta_3^T \begin{bmatrix} \varphi(k) \\ 1 \end{bmatrix}, 0\} - \max\{-\theta_4^T \begin{bmatrix} \varphi(k) \\ 1 \end{bmatrix}, 0\}. \end{aligned} \tag{26}$$

For (26), the polynomial (13) becomes

$$d(z^{-1}) = 1 + (1.4 - \alpha)\, z^{-1} + 1.1\, z^{-2} + 0.6\, z^{-3}, \tag{27}$$

that is not Schur, e.g., for the value $\alpha = 1.5$ contained in the estimated $\mathcal{L}_2$-stability domain. This example shows that the same system may admit HHARX representations that satisfy Assumption 1, while others do not. Nevertheless, the system can be proven to be $\mathcal{L}_2$-stable by applying the sufficient condition of Theorem 4 to an HHARX representation satisfying Assumption 1.

Finally, we note that an equivalent PWARX representation of the HHARX model of this example, computed using [19], has $s = 16$ regions. A state space realization obtained by defining the state equal to the regression vector –excluding $u(k)$– has order $n = 4$ and $s = 16$ modes.

V. CONCLUSIONS

$\mathcal{L}_2$-stability analysis of HHARX models has been addressed in this paper. The proposed approach is based on a linear fractional representation of the HHARX model, and

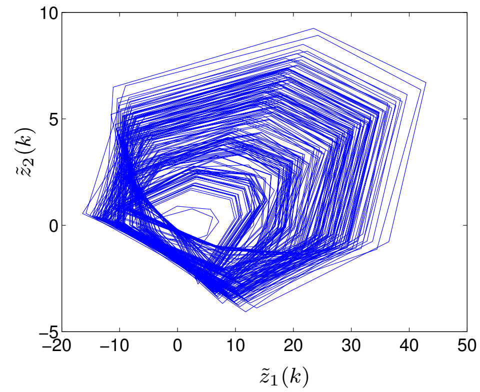


Fig. 5. Example of Section IV: Pairs $(\tilde{z}_1(k), \tilde{z}_2(k))$ corresponding to the impulse response of the HHARX model for $\alpha = 2.68$.

the use of IQCs for repeated static monotone nondecreasing nonlinearities. A sufficient condition for $\mathcal{L}_2$-stability in terms of an LMI is obtained.

Ongoing work aims at finding less conservative IQCs, as well as convex relaxations for guaranteeing robust stability in the face of parametric uncertainties. Moreover, further investigation is needed to extend the approach to more general model classes, such as those based on nested canonical PWA functions, and discontinuous PWA regression models.

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