Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Model Reduction for a Class of Input-Quantized Systems in the Max-Plus Algebra

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Abstract— **We present a systematic method for model reduction of a class of input-quantized systems in the max-plus algebra. We consider a generalization of the flow shop with finite intermediate storage. These systems are useful in modeling chemical processes and manufacturing systems, including pharmaceutical manufacturing, construction, propellant manufacturing and assembly lines. The makespan minimization problem we consider is $\mathcal{NP}$-complete. Our method of model reduction reduces the number of states that a system can reach, thus reducing the search space for the optimization problem. This allows us to construct a smaller $\mathcal{NP}$-complete problem to approximate the solution to the larger problem. We show that the error of the approximation is bounded and that as the approximated system approaches the true system, that the error of the approximation goes to zero.**

I. Introduction

We consider a method of model reduction for a class of input-quantized systems motivated by batch manufacturing systems. These systems are discussed in more detail in [4], [6], [7], [10], [11], [12]. This model can be used for several applications including flowshops, chemical processing plants, and services. Some examples of relevant applications are pharmaceutical production, propellant manufacturing, building construction, and assembly lines. We consider a restriction on these systems that simplifies the decision to a sequencing problem. The input to these systems is the job type to process in the manufacturing system.

In [12], a max-plus algebra representation for a subset of batch manufacturing systems is given and it is shown that the systems exhibit a particular structure. In this paper we approach the problem of finite horizon makespan minimization over these systems. Due to the quantized input, this is a combinatorial optimization problem that is $\mathcal{NP}$-complete in the length of the horizon. Optimization in the face of quantization has been studied extensively. Recently, in [2], [3] the authors present a method of partitioning the state space offline to allow fast online decision making. The more recent paper, [3], shows how to do this for discrete-time switched linear autonomous systems with a finite number of switches. We consider an open loop optimal control problem of a switched linear autonomous system in the max-plus algebra.

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Typically, model reduction attempts to find a lower order model such that the error between the lower order system and the higher order system is minimized with respect to some norm. Model reduction is used to lower the complexity of the plant, and hence the controller, to ease the computational effort of control. Model reduction for continuous linear time-invariant systems is described in [5]. A discrete-time version of model reduction based on LMI methods is given in [1]. We present a model reduction method to reduce the complexity of an open loop optimal control problem. Because the complexity of our problem is due to the length of the horizon, we reduce our system by allowing the current state to only be affected by a small number of previous inputs. This effectively reduces the length of the horizon to reduce the complexity of the problem we wish to solve. Furthermore, we show that the distance of the solution obtained using this method from the optimal solution has a bound that can be computed a priori. This allows a decision maker to determine the complexity of the reduced system based on the amount of acceptable error. Once the sub-optimal solution has been obtained, a tighter bound is easily computed using our solution.

This paper is organized as follows. We first present the structure of the class of systems we consider. Then, we pose a problem over these systems related to makespan minimization in manufacturing. To aid in the development of a good approximation method, we detail some results of these systems. We then present a method of model reduction to reduce the number of states to be considered in the optimal control problem. We show that the error due to the approximation is bounded and that as the approximated problem approaches the true problem, the error bound goes to zero.

II. Max-Plus Algebra Preliminaries

We will briefly discuss the max-plus algebra. A more thorough treatment is given in [9]. The max-plus algebra is defined over $\mathbb{R}_{max} = \mathbb{R} \cup \{-\infty\}$. We will define three

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binary operations for scalars:

$$\begin{aligned} &\forall a, b \in \mathbb{R}_{max} \\ &\quad a \oplus b \quad =: \quad \max(a, b), \\ &\quad a \otimes b \quad =: \quad a + b, \\ &\quad a \oslash b \quad =: \quad a - b. \end{aligned}$$

The zero element is defined as $\epsilon =: -\infty$, and the unit element is defined as $e =: 0$. Throughout this paper we will use the convention that for any $y \in \mathbb{R}_{max}$, the indeterminate form $y \oplus (\epsilon \oslash \epsilon) = y$.

Matrix arithmetic is also defined. For matrices, $A, B \in \mathbb{R}_{max}^{n\times l}$, $C \in \mathbb{R}_{max}^{l\times m}$ we define the operations:

$$\begin{aligned} [A \oplus B]_{ij} &=: a_{ij} \oplus b_{ij} \\ [B \otimes C]_{ik} &=: \bigoplus_{j=1}^{l} b_{ij} \otimes c_{jk}. \end{aligned}$$

The zero vector and the unit vector are given by

$$\boldsymbol{\epsilon} =: \begin{bmatrix} \epsilon \\ \vdots \\ \epsilon \end{bmatrix}, \quad \mathbf{e} =: \begin{bmatrix} e \\ \vdots \\ e \end{bmatrix},$$

where e and ϵ are as defined above. The identity matrix is

$$I_{max} =: \begin{bmatrix} e & \epsilon & \dots & \epsilon \\ \epsilon & e & & \epsilon \\ \vdots & & \ddots & \vdots \\ \epsilon & \epsilon & \dots & e \end{bmatrix}.$$

We can now define a linear state-space system in the max-plus algebra. For $\mathbf{x}_k \in \mathbb{R}_{max}^n$, $k = 1, \dots$ and $A \in \mathbb{R}_{max}^{n\times n}$, there is a linear autonomous system,

$$\mathbf{x}(k+1) = A \otimes \mathbf{x}(k).$$

Definition 2.1: We say a max-plus autonomous system is stable if

$$\forall i, \exists v \in \mathbb{R} \text{ such that } \lim_{k\to\infty} x_i(k) \oslash x_1(k) = v.$$

This means that a system is stable if, as the system is evolves in time, the distance between all elements of the vector x reach a finite constant.

We will define the 1-norm in the max-plus algebra.

Definition 2.2: The 1-norm of a max-plus vector, $b \in \mathbb{R}_{max}^n$ is

$$||b||_{1_{max}} = \bigoplus_{i=1}^{n} b_i = \mathbf{e}^T \otimes b$$

Note that this norm is the maximum element of the vector b. This norm induces a norm on a matrix.

Definition 2.3: The max-plus 1-induced norm of an operator $A \in \mathbb{R}_{max}^{n\times m}$ is

$$||A||_{1_{max}} = \max_x (||A \otimes x||_{1_{max}} \oslash ||x||_{1_{max}}).$$

Lemma 2.1: Given a matrix $A \in \mathbb{R}_{max}^{n\times m}$, the max-plus 1-induced norm of A is

$$||A||_{1_{max}} = \max_{ij} a_{ij}.$$

Proof: Let $A \in \mathbb{R}_{max}^{n\times m}$ be given. Without loss of generality, we will say that $||x||_{1_{max}} = e$. Note that $||A \otimes x||_{1_{max}} = \mathbf{e}^T \otimes A \otimes x$. The vector $v^T =: \mathbf{e}^T \otimes A$ is the vector containing the max element of each column of A. Therefore, we want to maximize $v^T \otimes x$. Because $||x||_{1_{max}} = e$, the largest element in x is e. To maximize $v^T \otimes x$, we want to make each element of x as large as possible; this means we set $x = \mathbf{e}$ which gives $v^T \otimes x = \max_{ij} a_{ij}$. ■

By this theorem, we see that the max-plus 1-induced norm of a matrix is very simple to compute. We are also interested in a similar quantity, $\min_x \|A \otimes x\|_{1_{max}} \oslash \|x\|_{1_{max}}$. This quantity is also simple to compute.

Lemma 2.2: Given a matrix $A \in \mathbb{R}_{max}^{n\times m}$,

$$\min_x ||A \otimes x||_{1_{max}} \oslash ||x||_{1_{max}} = \min_i [\mathbf{e}^T \otimes A]_i.$$

Proof: Let $A \in \mathbb{R}_{max}^{n\times m}$ be given. Without loss of generality, let $||x||_1 = e$ and consider $v^T \otimes x$ with $v^T =: \mathbf{e} \otimes A$. Now we want to minimize $v^T \otimes x$, so we want each element of x as small as possible. However, having $||x||_{1_{max}} = e$ requires at least one element of x equal to e. Thus, we need only consider each $\mathbf{e}_i$ where $\mathbf{e}_{ii} = e$ and $\mathbf{e}_{ij} = \epsilon$ for $j \neq i$. So

$$\begin{aligned} \min_x v^T \otimes x &= \min_i (v^T \otimes \mathbf{e}_i) \\ &= \min_i (v_i) \\ &= \min_i [\mathbf{e}^T \otimes A]_i. \end{aligned}$$

■

III. PROBLEM FORMULATION

Definition 3.1: We will say that a matrix $A \in \mathcal{M}^n \subset \mathbb{R}_{max}^{n\times n}$ if:

$$a_{ij} \le a_{i+1,j},\ \forall i \le n-1, j \le n, \quad (1)$$

$$a_{ij} \ge a_{i,j+1},\ \forall i \le n, j \le n-1, \quad (2)$$

$$\xi_{1i}(A) \ge \dots \ge \xi_{ni}(A) \ge 0,\ \forall i \le n, \quad (3)$$

$$a_{ij} > -\infty,\ \forall j \le i+1, i \le n, j \le n, \quad (4)$$

where

$$\xi_{ij}(A) = a_{ij} - a_{i,j+1}.$$

Let $\mathcal{A}$ be a set of m distinct matrices in $\mathcal{M}^n$ indexed by the set $\mathcal{U} = \{1, \dots, m\}$. We consider a class of input quantized systems of the form

$$\begin{aligned} x_{k+1} &= A_{u_k} \otimes x_k \\ y_k &= \|A_{u_k} \otimes x_k\|_{1_{max}} \oslash \|x_k\|_{1_{max}}. \end{aligned} \quad (5)$$

Where $x_k \in \mathbb{R}^n$, $y_k \in \mathbb{R}$, $u_k \in \mathcal{U}$, and $A_{u_k} \in \mathcal{A}$. It is shown in [12] that a class of batch manufacturing systems can be represented as systems of this form.

Given a vector $\mathbf{q} \in \mathbb{N}^m$, we say that a sequence $u = (u_0, \dots, u_{|\mathbf{q}|_1 - 1})$, with $u_i \in \mathcal{U}$ is *admissible* if

$$\sum_{i=0}^{|\mathbf{q}|_1 - 1} I_j(u_i) = q_j \quad \forall\, 1 \le j \le m,$$

where $I_j(k)$ is the indicator function:

$$I_j(k) = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

Characterizing admissible inputs to the system then leads to the following problem:

$$\begin{array}{ll} \min\limits_{U \text{ admissible}} & \sum\limits_{k=0}^{|\mathbf{q}|_1 - 1} y_k \\ \text{subject to} & x_{k+1} = A_{u_k} \otimes x_k \\ & y_k = \|A_{u_k} \otimes x_k\|_{1_{max}} \oslash \|x_k\|_{1_{max}}. \end{array} \tag{6}$$

When each $A \in \mathcal{A}$ represents the recipe in a batch flow shop, and $\mathbf{q}$ is interpreted as a fixed quota, this problem is equivalent to the makespan minimization problem with respect to a quota. We will now show that this problem is $\mathcal{NP}$-complete.

Proposition 3.1: The problem given in (6) is $\mathcal{NP}$-complete.

Proof: First we must show that our problem is in $\mathcal{NP}$. This is trivial since an admissible sequence can be easily constructed, and checking the solution is done by calculating $x_{|q|_1}$ and then $\|x_{|q|_1}\|_{1_{max}} \oslash \|x_0\|_{1_{max}}$. These can all be done in polynomial time.

To show our problem is $\mathcal{NP}$-complete, we will reduce $F_3|block|C_{max}$, which is the optimal sequencing of the 3-machine flowshop with blocking with respect to makespan, to our problem. This problem is shown to be $\mathcal{NP}$-complete in [8]. This problem can be represented as a 3 machine batch flowshop with machine capacities all equal. A polynomial time algorithm to transform a batch flowshop into our problem is presented in [12]. Thus $F_3|block|C_{max}$ is reducible to our problem in polynomial time. ■

IV. Max-plus Systems Generated by $\mathcal{M}^n$

A. Properties of $\mathcal{M}^n$

Because of the structure of matrices in $\mathcal{M}^n$, it has many useful properties. We will show that the set $\mathcal{M}^n$ is closed under multiplication and that all max plus operators in $\mathcal{M}^n$ exhibit some particular input-output properties.

Lemma 4.1: Suppose $A, B \in \mathcal{M}^n$. If

$$a_{ir} \otimes b_{rj} = \bigoplus_{s=1}^{n} (a_{is} \oslash b_{sj}),$$

$$a_{il} \otimes b_{l,j+1} = \bigoplus_{s=1}^{n} (a_{is} \oslash b_{s,j+1},$$

then $r \le l$.
The proof is given in [13].

Lemma 4.2: Suppose $A, B \in \mathcal{M}^n$. If

$$a_{ir} \otimes b_{rj} = \bigoplus_{s=1}^{n} (a_{is} \otimes b_{sj}),$$

$$a_{i+1,l} \otimes b_{l,j} = \bigoplus_{s=1}^{n} (a_{i+1,s} \otimes b_{sj}),$$

then $r \le l$.
The proof is given in [13].

We will now show that the set $\mathcal{M}^n$ is closed under max-plus matrix multiplication.

Theorem 4.1: Suppose $A, B \in \mathcal{M}^n$. Then $A \otimes B \in \mathcal{M}^n$.

Proof: Let $A, B \in \mathcal{M}^n$ be given. We will write $C = A \otimes B$. To show that $C \in \mathcal{M}^n$, we must show that all four equations in Definition 3.1 hold. We will show each individually.

Equation (1), $c_{ij} \le c_{i+1,j}$: Let $i, j \le n$ be given. We will pick κ such that $c_{ij} = a_{i\kappa} \otimes b_{\kappa j}$. Then,

$$\begin{aligned} c_{ij} \oslash c_{i+1,j} &\le a_{i\kappa} \otimes b_{\kappa j} \oslash (a_{i+1,\kappa} \otimes b_{\kappa j}) \\ &= a_{i\kappa} \oslash a_{i+1,\kappa} \\ &\le e. \end{aligned}$$

Equation (2), $c_{ij} \ge c_{i,j+1}$: Let $i, j \le n$ be given. We will pick κ such that $c_{ij+1} = a_{i\kappa} \otimes b_{\kappa,j+1}$. Then,

$$\begin{aligned} c_{ij} \oslash c_{i,j+1} &\ge a_{i\kappa} \otimes b_{\kappa j} \oslash (a_{i\kappa} \otimes b_{\kappa,j+1}) \\ &= b_{\kappa j} \oslash b_{\kappa,j+1} \\ &\ge e. \end{aligned}$$

Equation (3), $c_{ij} \oslash c_{i,j+1} \ge c_{i+1,j} \oslash c_{i+1,j+1}$: Let $i, j < n$ be given. We will pick κ, l, r, s such that

$$c_{ij} = a_{i\kappa} \otimes b_{\kappa j} \tag{7}$$

$$c_{i,j+1} = a_{il} \otimes b_{l,j+1} \tag{8}$$

$$c_{i+1,j} = a_{i+1,r} \otimes b_{rj} \tag{9}$$

$$c_{i+1,j+1} = a_{i+1,s} \otimes b_{s,j+1}. \tag{10}$$

From (7, 10) we can derive the following inequalities

$$a_{i\kappa} \oslash a_{il} \ge b_{lj} \oslash b_{\kappa j} \tag{11}$$

$$a_{i+1,s} \oslash a_{i+1,r} \ge b_{r,j+1} \oslash b_{s,j+1} \tag{12}$$

$$b_{s,j+1} \oslash b_{l,j+1} \ge a_{i+1,l} \oslash a_{i+1,s} \tag{13}$$

$$b_{\kappa j} \oslash b_{rj} \ge a_{ir} \oslash a_{i\kappa}. \tag{14}$$

Now consider

$$\begin{aligned} \omega =& c_{ij} \oslash c_{i,j+1} \oslash (c_{i+1,j} \oslash c_{i+1,j+1}) \\ =& a_{i\kappa} \otimes b_{\kappa j} \oslash a_{il} \oslash b_{l,j+1} \\ & \oslash a_{i+1,r} \oslash b_{rj} \otimes a_{i+1,s} \otimes b_{s,j+1} \\ =& (a_{i\kappa} \oslash a_{il}) \otimes (a_{i+1,s} \oslash a_{i+1,r}) \\ & \otimes (b_{s,j+1} \oslash b_{l,j+1}) \otimes (b_{\kappa j} \oslash b_{rj}). \end{aligned} \tag{15}$$

From this equation we will consider two cases.

Suppose $l \leq r$. Then we write

$$\begin{aligned}\omega \geq & b_{lj} \oslash b_{\kappa j} \otimes b_{\kappa j} \oslash b_{l,j+1} && (16)\\ & \otimes b_{r,j+1} \oslash b_{s,j+1} \oslash b_{rj} \otimes b_{s,j+1} \\ = & b_{lj} \oslash b_{l,j+1} \oslash (b_{rj} \oslash b_{r,j+1}) && (17)\\ \geq & e. && (18)\end{aligned}$$

Where we obtain (16) by substituting (11) and (12) into (15), (17) by canceling and rearranging terms, and (18) by (3).

Suppose $l > r$. Then we write

$$\begin{aligned}\omega \geq & a_{i\kappa} \oslash a_{i+1,r} \otimes a_{ir} \oslash a_{i\kappa} && (19)\\ & \otimes a_{i+1,s} \oslash a_{il} \otimes a_{i+1,l} \oslash a_{i+1,s} \\ = & a_{ir} \oslash a_{il} \oslash (a_{i+1,r} \oslash a_{i+1,l}) && (20)\\ \geq & e. && (21)\end{aligned}$$

Where we obtain (19) by substituting (13) and (14) into (15), (20) by canceling and rearranging terms, and (21) by (3).

Equation (4): Let i, j such that $j \leq i+1$, $i, j \leq n$. Then

$$\begin{aligned}c_{ij} &= \bigoplus_{\kappa=1}^{n} a_{i\kappa} \otimes b_{\kappa j} \\ &\geq a_{ii} \otimes b_{ij} \\ &> -\infty.\end{aligned}$$

■

Definition 4.1: For some $A \in \mathcal{M}^n$, we define

$$\begin{aligned}Z_i(A) &= a_{in} \oslash a_{i-1,n} \\ z_i(A) &= a_{i1} \oslash a_{i-1,1}.\end{aligned}$$

The following Lemma, taken from [12] is useful to our analysis.

Lemma 4.3 ([12]): For some $A \in \mathcal{M}^n$, for any $x \in \mathbb{R}^n_{max}$, if we let $y = A \otimes x$, then

$$z_i(A) \leq y_i \oslash y_{i-1} \leq Z_i(A).$$

This Lemma gives us a range on the "spread" that can occur between the elements of the state vector after applying any input. We are now equipped to make some statements related to input-output properties of these max-plus operators.

Proposition 4.1: Suppose $A \in \mathcal{M}^n$. Then

$$\min_{x \neq \epsilon} \left(\|A \otimes x\|_{1_{max}} \oslash \|x\|_{1_{max}} \right)$$

is satisfied with $x = \begin{bmatrix} \epsilon & \ldots & \epsilon & e \end{bmatrix}^T$. Furthermore, if $B \in \mathcal{M}^n$, then

$$\min_{x \neq \epsilon} \left(\|A \otimes B \otimes x\|_{1_{max}} \oslash \|B \otimes x\|_{1_{max}} \right) \qquad (22)$$

is satisfied with the same choice of x.

Proof: Let $A \in \mathcal{M}^n$ be given. By Lemma 2.2 and Definition 3.1, we know that

$$\begin{aligned}\min_{x \neq \epsilon} \left(\|A \otimes x\|_{1_{max}} \oslash \|x\|_{1_{max}} \right) &= \min_i \left[\mathbf{e}^T \otimes A \right]_i \\ &= a_{nn}.\end{aligned}$$

Consider $x = \begin{bmatrix} \epsilon & \ldots & \epsilon & e \end{bmatrix}^T$, then $\|x\|_{1_{max}} = e$, and

$$\begin{aligned}\|A \otimes x\|_{1_{max}} &= \left\| \begin{bmatrix} a_{n1} & \ldots & a_{nn} \end{bmatrix} \right\|_{1_{max}} \\ &= a_{nn}\end{aligned}$$

To show (22), we will also suppose that $B \in \mathcal{M}^n$. Let $\tilde{y} = B \otimes \tilde{x}$ with $\tilde{x} = \begin{bmatrix} \epsilon & \cdots & e \oslash b_{nn} \end{bmatrix}^T$ and suppose that there is some $y^* = B \otimes x^*$ with $\|y^*\|_{1_{max}} = e$ such that $\|A \otimes y^*\|_{1_{max}} < \|A \otimes \tilde{y}\|_{1_{max}}$.

We will pick j, κ such that $\|A \otimes \tilde{y}\|_{1_{max}} = a_{nj} \otimes \tilde{y}_j$ and $\|A \otimes y^*\|_{1_{max}} = a_{n\kappa} \otimes y^*_\kappa$. Then we have the following inequalities

$$\begin{aligned}a_{nj} \otimes \tilde{y}_j &> a_{n\kappa} \otimes y^*_\kappa \\ a_{n\kappa} \otimes y^*_\kappa &\geq a_{nj} \otimes y^*_j.\end{aligned}$$

These can be combined to get

$$y^*_j < \tilde{y}_j.$$

But, by Lemma 4.3,

$$\begin{aligned}y^*_n &\leq y^*_j \otimes \bigotimes_{i=j+1}^{n} Z_i(B) \\ &< \tilde{y}_j \otimes \bigotimes_{i=j+1}^{n} Z_i(B) \\ &= e.\end{aligned}$$

Which contradicts the statement that $\|y^*\|_{1_{max}} = e$, so $\tilde{y}$ and hence $\tilde{x}$ achieves the minimum. It is easy to see that $x = \tilde{x} \otimes b_{nn}$ also achieves the minimum. ■

We have a similar result for the max.

Proposition 4.2: Suppose $A \in \mathcal{M}^n$. Then

$$\max_{x \neq \epsilon} \left(\|A \otimes x\|_{1_{max}} \oslash \|x\|_{1_{max}} \right)$$

is satisfied with $x = \mathbf{e}$. Furthermore, if $B \in \mathcal{M}^n$, then

$$\max_{x \neq \epsilon} \left(\|A \otimes B \otimes x\|_{1_{max}} \oslash \|B \otimes x\|_{1_{max}} \right) \qquad (23)$$

is satisfied with the same choice of x.

The proof is very similar in structure to that of Proposition 4.1 and is supplied in [13].

B. Fixed Input Stability

Theorem 4.2: Let $A \in \mathcal{M}^n$. Then the linear autonomous system

$$x_{k+1} = A \otimes x_k$$

is stable in the sense of Definition 2.1.

The bulk of the proof is given in [13]. We will present two lemmata also proven in [13] to supply the proof of Theorem 4.2.

Lemma 4.4: Let $A \in \mathcal{M}^n$. Then A has cyclicity one.

Lemma 4.5: Let $A \in \mathcal{M}^n$. Then A is irreducible.

From [9] we get the following theorem which we will need in our proof of stability.

Theorem 4.3 ([9]): Let $A \in \mathbb{R}_{max}^{n\times n}$ be an irreducible matrix with eigenvalue λ and cyclicity $\sigma = \sigma(A)$. Then there is an N such that

$$A^{\otimes(\kappa+\sigma)} = \lambda^{\otimes\sigma} \otimes A^{\otimes\kappa}$$

for all $\kappa \geq N$.

These pieces combine to give us the proof for Theorem 4.2.

Proof: [Proof of Theorem 4.2] Let $A \in \mathcal{M}^n$ with eigenvalue λ. By Lemmata 4.5 and 4.4 and Theorem 4.3, we know that there is some N such that

$$^{\otimes(l+1)} = \lambda \otimes A^{\otimes l}$$

for all $l \geq N$. Let $x \in \mathbb{R}_{max}^n$, $l \geq N$ be given. Let $i \leq n$ be given. Consider $A^{\otimes(l+1)}x = \lambda \otimes A^{\otimes l} \otimes x$. From this we see that after l steps, $A^{\otimes l} \otimes x$ is an eigenvalue of A, therefore we see that for all $k \geq 1$, $[A^{\otimes(k+l)} \otimes x]_i \oslash [A^{\otimes(k+l)} \otimes x]_1$ is a constant value. ∎

V. Suboptimal Scheduling with Bounds

A. Model Reduction

Since any system in $\mathcal{M}^n$ is stable, clearly any system in our set $\mathcal{A} \subset \mathcal{M}^n$ is stable. If the input, u, is constant, the effect of the initial condition dies away. This motivates us to pose an approximation to the system that assumes only the most recent inputs effect the current state. To simplify notation, we will denote the sequence of inputs from time i to time j as $U_{i,j} = (u_i, u_{i+1}, \ldots, u_{j-1}, u_j)$ with $i \leq j$. Using this notation, we will also denote

$$A(U_{i,j}) = A_{u_j} \otimes \cdots \otimes A_{u_i}$$

and note that by Theorem 4.1 $A(U_{i,j}) \in \mathcal{M}$. For a sequence $U_{0,k} = \{u_0, \ldots, u_k\}$, we will write

$$\begin{aligned} x_{k+1} &= A(U_{0,k}) \otimes x_0 \\ y(U_{0,k}) &= x_{k+1} \oslash x_0 \\ &= \sum_{i=0}^{k} y(i). \end{aligned}$$

If the system is driven by a sequence $U_{0,k-1} = (u_0, \ldots, u_{k-1})$, then we have as a solution to (5)

$$x_k = A(U_{0,k-1}) \otimes x_0.$$

We want to build our approximation such that we lower bound the actual output, so we approximate the current state using what we call the *p-step approximation* for the subsequence $U_{k-p,k-1}$.

Definition 5.1: Given a system as in (5) and a sequence of inputs, $U_{0,k} = (u_0, \ldots, u_k)$, we define the p-step approximation of (x, y) to be

$$\begin{aligned} \hat{x}_k^p &= \begin{cases} A(U_{k-p,k-1}) \otimes x_\ell & \text{if } k > p, \\ A(U_{0,k-1}) \otimes x_0 & \text{otherwise} \end{cases} \\ \hat{y}_k^p &= \|A_{u_k} \otimes \hat{x}_k^p\|_{1_{max}} \oslash \|\hat{x}_k^p\|_{1_{max}} \end{aligned}$$

with $x_\ell = \begin{bmatrix} \epsilon & \ldots & \epsilon & e \end{bmatrix}$.

We use x_ℓ in order to arrive at the following proposition.

Proposition 5.1: Given a system as in (5) and a sequence $U_{0,k}$, the p-step approximation of (x, y) satisfies

$$\hat{y}_k^p \leq y_k$$

for all $k > p$, and

$$\hat{y}_k^p = y_k$$

for all $k \leq p$.

Proof: Let a system as in (5) and a sequence $U_{0,k}$ be given. By Proposition 4.1 and Theorem 4.1

$$\begin{aligned} \hat{y}_k^p =& \|A_{u_k} \otimes \hat{x}_k^p\|_{1_{max}} \oslash \|\hat{x}_k^p\|_{1_{max}} \\ =& \|A(u_k) \otimes A(U_{k-p,k-1}) \otimes x_\ell\|_{1_{max}} \\ & \oslash \|A(U_{k-p,k-1}) \otimes x_\ell\|_{1_{max}} \\ =& \min_x \|A_{u_k} \otimes A(U_{k-p,k-1}) \otimes x\|_{1_{max}} \\ & \oslash \|A(U_{k-p,k-1}) \otimes x\|_{1_{max}} \\ \leq& y_k \end{aligned}$$

if $k > p$, and

$$\hat{y}_k^p = y_k$$

when $p \leq k$ by definition of p-step approximation. ∎

Hence, our approximation gives a lower bound on y given the last p inputs. This differs from the approximation given in [12] as that approximation gave an upper bound on y.

This approximation leads to an approximation of (6):

$$\begin{aligned} \min_{U \text{ admissible}} \quad & \sum_{k=0}^{|\mathbf{q}|_1-1} \hat{y}_k^p \\ \text{subject to} \quad & \hat{x}_k^p = \begin{cases} A(U_{k-p,k-1}) \otimes x_\ell & \text{if } k > p, \\ A(U_{0,k-1}) \otimes x_0 & \text{otherwise} \end{cases} \\ & \hat{y}_k^p = \|A_{u_k} \otimes \hat{x}_k^p\|_{1_{max}} \oslash \|\hat{x}_k^p\|_{1_{max}}. \end{aligned} \tag{24}$$

This problem is easier to solve than (6) because we drastically reduce the search space of the optimal control problem.

B. Error Bounds

Let U^* be the solution to (6) and $\hat{U}^{p*}$ the solution to (24). We will now construct a bound for the difference between $y(U^*)$ and $y(\hat{U}^{p*})$.

Lemma 5.1: Suppose we have the set $\mathcal{A} \subset \mathcal{M}^n$ with a corresponding set of indices $\mathcal{U} = \{1, 2, \ldots, m\}$ and a quota, $\mathbf{q} \in \mathbb{N}^m$. Then we have the following relationship

$$y(\hat{U}^{p*}) - y(U^*) \leq y(\hat{U}^{p*}) - \hat{y}^p(\hat{U}^{p*}). \tag{25}$$

Proof: Because our approximation is a lower bound on y, we have the chain of inequalities:

$$\hat{y}^p(\hat{U}^{p*}) \leq \hat{y}^p(U^*) \leq y(U^*) \leq y(\hat{U}^{p*}).$$

This ensures that the left hand side of (25) is non-negative and leads easily to

$$y(\hat{U}^{p*}) - y(U^*) \leq y(\hat{U}^{p*}) - \hat{y}^p(\hat{U}^{p*}).$$

∎

This means that when we have a solution, we know the optimal solution is between the cost as calculated using the approximation and the actual cost of the best sequence of the approximated problem.

We will define the maximum error of the p-step approximation much like as in [12]. Given a set $\mathcal{A}$ and a sequence $U_{k-p,k}$, we say that

$$\gamma^p(U_{k-p,k}) = \max_x \left\{ \frac{\frac{\|A(U_{k-p,k}) \otimes x\|_{1_{max}}}{\|A(U_{k-p,k-1}) \otimes x\|_{1_{max}}}}{\frac{\|A(U_{k-p,k}) \otimes x_l\|_{1_{max}}}{\|A(U_{k-p,k-1}) \otimes x_l\|_{1_{max}}}} \right\}.$$

From this we define

$$\Gamma^p = \max_U \gamma^p(U).$$

Note that Γ^p specifies another combinatorial optimization problem. This problem is no more complex to solve than the p-step approximation problem. Thus if there is enough computational power to solve the p-step approximation for p, then it is possible to calculate Γ^p.

Lemma 5.2: Given a system as in (5) and a sequence $U_{0,k}$, for $k > p$, $y_k \oslash \hat{y}_k^p \leq \Gamma^p$. If $k \leq p$, then $y_k \oslash \hat{y}_k^p = e$.

Proof: Let a system as in (5) and a sequence $U_{0,k}$ be given. Consider the p-step approximation for $k > p$. Then

$$\begin{aligned} y_k \oslash \hat{y}_k^p =& \frac{\frac{\|A(U_{k-p,k}) \otimes x_{k-p}\|_{1_{max}}}{\|A(U_{k-p,k-1}) \otimes x_{k-p}\|_{1_{max}}}}{\frac{\|A(U_{k-p,k}) \otimes x_l\|_{1_{max}}}{\|A(U_{k-p,k-1}) \otimes x_l\|_{1_{max}}}} \\ \leq& \gamma^p(U_{k-p,k}) \\ \leq& \Gamma^p. \end{aligned}$$

Now consider the p-step approximation for $k \leq p$. Then by Proposition 5.1

$$y_k = \hat{y}_k^p$$

for all k. ■

These Lemmata lead to the main result of the paper. A bound on the error of the approximated best solution when compared to the true best solution exists and this bound improves as p increases until it reaches 0 at $p = |q|_1$.

Theorem 5.1: If $\hat{U}^{p*}$ is the optimal solution to problem (24), then

$$y(\hat{U}^{p*}) \oslash \hat{y}^p(\hat{U}^{p*}) \leq \bigotimes_{i=p+1}^{|q|_1} \Gamma^p$$

where

$$\Gamma^{p+1} \leq \Gamma^p.$$

Furthermore,

$$y(\hat{U}^*_{|q|_1}) \oslash \hat{y}_{|q|_1}(\hat{U}^*_{|q|_1}) = 0.$$

Proof: Suppose $\hat{U}^{p*}$ is the optimal solution to problem (24). Then by Lemma 5.2, $y_k \oslash \hat{y}_k \leq \Gamma^p$ for $k > p$, so

$$\begin{aligned} y(\hat{U}^{p*}) \oslash \hat{y}(\hat{U}^{p*}) =& \bigotimes_{k=0}^{|q|_1} (y_k \oslash \hat{y}_k) \\ \leq& \bigotimes_{k=p+1}^{|q|_1} \Gamma^p. \end{aligned}$$

Let p be given. We will let $\tilde{U}_{0,p+1}$ be such that $\gamma^{p+1}(\tilde{U}_{0,p+1}) = \Gamma^{p+1}$. It was shown in [12] that for any sequence, U, $\gamma^{p+1}(U_{0,p+1}) \leq \gamma^p(U_{0,p})$, so

$$\begin{aligned} \gamma^{p+1}(\tilde{U}_{0,p+1}) \leq& \gamma_p(\tilde{U}_{0,p}) \\ \leq& \Gamma^p. \end{aligned}$$

The final statement of the theorem follows since for $k \leq p$, the approximation is exact. So if $p = |q|_1$, then the approximation is exact up to $k = |q|_1$. ■

VI. Conclusion

We have studied a scheduling problem posed over a class of systems in the max-plus algebra. This problem is motivated by makespan minimization in batch manufacturing systems. We have shown that this problem is $\mathcal{NP}$-complete and hence there is a need to derive an approximation method. We modify the model reduction method presented in [12] so the solution of the approximated problem gives a lower bound on the true optimum solution. Furthermore, we showed that the error created by approximating the problem using this method is bounded and that the bound improves as the approximation is refined and eventually reaches zero.

References

[1] C. Beck, J. Doyle, and K. Glover. Model reduction of multidimensional and uncertain systems. *IEEE Transactions on Automatic Control*, 41(10):1466–1477, October 1996.

[2] A. Bemporad. Multiparametric nonlinear integer programming and explicit quantized optimal control. *Proc. of the 42nd IEEE Conference on Decision and Control*, pages 3167–3172, 2003.

[3] D. Corona, A. Giua, and C. Seatzu. Quantized optimal control of discrete-time systems. *10th IEEE Conference on ETFA*, 1, 2005.

[4] X. Deng, H. Feng, G. Li, and G. Lui. A PTAS for minimizing total completion time of bounded batch scheduling. *Internation Journal of Foundations of Computer Science*, 13(6):817–827, 2002.

[5] G. Dullerud and F. Paganini. *A Course in Robust Control Theory: A Convex Approach*. Texts in Applied Mathematics. Springer, 2000.

[6] L. Dupont and C. Dhaenens-Flipo. Minimizing the makespan on a batch machine with non-identical job sizes: an exact procedure. *Computers and Operations Research*, 29(7):807–819, 2002.

[7] C. Glass, C. Potts, and V. Strusevich. Scheduling batches with sequential job processing for two-machine flow and open shops. *INFORMS Journal on Computing*, 13(2):120–137, Spring 2001.

[8] N. Hall and C. Sriskandarajah. A survey of machine scheduling problems with blocking and no-wait in process. *Operations Research*, 44(3):510–525, May/June 1996.

[9] B. Heidergott, G. J. Olsder, and J. van der Woude. *Max Plus at Work: Modeling and Analysis of Synchronized Systems: A Course on Max-Plus Algebra and its Applications*. Princeton Series in Applied Mathematics. Princeton University Press, 2006.

[10] E. Kondili, C. C. Pantelides, and R. W. H. Sargent. A general algorithm for short-term scheduling of batch operations - I. MILP formulation. *Computers and Chemical Engineering*, 17(2):211–227, 1993.

[11] E. Sanmartí, L. Puigjaner, T. Holczinger, and F. Friedler. Combinatorial framework for effective scheduling of multipurpose batch plants. *AIChE Journal*, 48(11):2557–2570, Nov 2002.

[12] W. Weyerman and S. Warnick. Monotonically improving error bounds for a sequence of approximations for makespan minimization of batch manufacturing systems. *Proc. IEEE Conference on Decisions and Control*, 2007.

[13] W. Weyerman and S. Warnick. Model reduction for batch flow shop scheduling with fixed batch sizes. Working Paper, Information and Decision Algorithms Laboratories, September 2008.