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Adaptive Learning Control for Nonlinear Systems with Extended Matching Unstructured Uncertainties

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Abstract— **The output tracking control problem via state feedback is addressed for a class of single input-single output nonlinear systems which are affected by extended matching unstructured uncertainties. Under the assumption that the output reference signal is sufficiently smooth and periodic with known period, a robust adaptive learning control is designed, which learns the unstructured unknown periodic disturbance signals due to system uncertainties by identifying the Fourier coefficients of any truncated approximation while guaranteeing $\mathcal{L}_2$ and $\mathcal{L}_\infty$ transient performances. For any initial condition of the system in an arbitrary given compact set, by properly setting the control parameters: i) the output tracking error exponentially converges to a residual set which may be arbitrarily reduced by increasing the number of terms in the truncated Fourier series expansions; ii) when the unknown periodic disturbances can be represented by finite Fourier series expansions, the output tracking error exponentially converges to zero.**

I. Introduction

Output tracking control of nonlinear systems under various types of uncertainty has attracted the interest in the control community in the last decade. If the uncertainties satisfy structural conditions (such as matching, extended matching or triangularity) and are linearly parameterized, adaptive state feedback controls can be designed to achieve asymptotic tracking of smooth bounded reference signals. However, in the presence of unstructured uncertainties (no parameterization available), asymptotic tracking may be still guaranteed in the case of periodic output reference signal (with known periodicity) by following the learning control approach: a control input is generated to achieve output tracking over a finite or infinite time interval for systems performing repetitive tasks.

When iterative or repetitive (see [6]) learning controls are designed ([3]-[4], [7], [9], [13], [17]-[21], [23]-[26]) on the basis of either the contraction mapping approach or a Lyapunov-like theory (to overcome limitations such as resetting of initial conditions, derivative measurements or global Lipschitz conditions), only asymptotic (and not exponential) convergence may be in general proved. On the other hand, even when the tracking problem with unstructured uncertainties can be reduced (by using suitable approximation methods) to a tracking problem with linearly parameterized uncertainties and disturbances (see [27]-[28], [22], [8], [2], [15]), satisfying persistency of excitation conditions in a general feedback closed loop and achieving exponential tracking (guaranteeing certain closed loop robustness properties) constitute rather difficult problems to be solved. Robust adaptive controls are designed in [27] and [28] for nonlinear systems with parametric uncertainties and uncertain nonlinearities: arbitrary output tracking transient performances are guaranteed by adjusting the control parameters while asymptotic output tracking is achieved in the presence of parametric uncertainties only. A neural network control design approach (see also [22] for an adaptive neural control of a nonlinear Brunovsky system) is proposed in [8] for a class of nonlinear systems in semi-strict feedback form with unknown nonlinearities: the output tracking error can be made arbitrarily small by increasing the feedback gains while asymptotic output tracking is guaranteed when the unknown nonlinear functions are in the functional range of the corresponding neural networks and the ideal network weights lie within the chosen fictitious bounds. Robust controllers are designed in [2] for a class of single input-single output nonlinear systems in strict feedback form with structurally unknown dynamics and uncertain virtual coefficients: under the assumption that the parameters characterizing the neural network approximator lie in known compact sets, arbitrarily small output tracking error can be obtained by increasing the control effort.

This paper deals with the design of a state feedback control achieving exponential output tracking of a sufficiently smooth and periodic (with known period) reference signal for a class of single input-single output nonlinear systems with unstructured uncertainties. Systems in strict feedback form with extended matching unstructured uncertainties satisfying a certain growth condition are allowed. The unstructured unknown periodic disturbances due to system uncertainties are developed in Fourier series and the estimates of their coefficients are continuously adapted by using robust adaptive techniques including parameter estimate projections. Persistency of excitation conditions are guaranteed to hold. For any initial condition of the system in an arbitrary given compact set, by properly setting the control parameters, the guaranteed output tracking error is reduced as the number of Fourier coefficients is increased while exponential output tracking is obtained when the unknown periodic disturbances are represented by finite Fourier series expansions. $\mathcal{L}_2$ and $\mathcal{L}_\infty$ transient performances are guaranteed during the learning phase. While in [5] a local output tracking problem has been solved in the case of unstructured matching uncertainties and maximal relative degree, the class of nonlinear systems condidered here is a wider class of systems, possibly includ-

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ing systems in strict feedback form with extended matching unstructured uncertainties.

II. Problem Statement

Consider the class of single input-single output nonlinear systems

$$\begin{aligned} \dot{x}_i &= x_{i+1},\ 1 \le i \le n-1 \text{ (for } n \ge 2) \\ \theta \dot{x}_n &= f(x) + u \\ \dot{u} &= q(x,u) + v \\ y &= h(x) \end{aligned}$$

in which: $x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$; $u \in \mathbb{R}$; $v \in \mathbb{R}$; $y \in \mathbb{R}$; h is a known smooth function; θ is an uncertain constant parameter; f and q are uncertain smooth functions. Let $\xi = [x^T, u]^T$ so that we can use the more compact notation

$$\begin{aligned} \dot{x}_i &= x_{i+1},\ 1 \le i \le n-1 \text{ (for } n \ge 2) \\ \theta \dot{x}_n &= f(x) + u \\ \dot{u} &= q(\xi) + v \\ y &= h(x). \end{aligned} \tag{1}$$

Remark 1: Let $\Theta \in \Omega \subset \mathbb{R}^p$ be a vector of unknown parameters belonging to the compact set Ω and let f_z, q_z, g_z be known smooth vector fields and $h_z(\cdot)$ be a known smooth function. The $(n+1)$ dimensional single input-single output nonlinear system of relative degree $r_* \ge 2$ $[g_z(\cdot) \ne 0]$

$$\begin{aligned} \dot{z} &= f_z(z) + q_z(z,\Theta) + g_z(z)v \\ y &= h_z(z) \end{aligned}$$

such that:

i) the nominal system (f_z, g_z) is globally feedback linearizable;
ii) the extended matching condition

$$q_z \in \mathcal{G}_1 = \text{span}\{g_z, ad_{f_z} g_z\}$$

is satisfied in $\mathbb{R}^{n+1}$;
iii) the strict triangularity assumption

$$ad_{q_z} \mathcal{G}_i \subset \mathcal{G}_i,\ 0 \le i \le n-1$$

is satisfied in $\mathbb{R}^{n+1}$ in terms of $\mathcal{G}_i = \text{span}\{g_z, ad^i_{f_z} g_z\}$,

is globally feedback equivalent to a nonlinear system belonging to the class (1) with $\theta = 1$ and f, q uncertain smooth functions (see [12] and [14]). In this case, the variable u constitutes the last component of the system state vector.

Remark 2: The variable u in (1) may be considered as the control variable for the x-subsystem whose uncertain dynamics (forced by the input v) can be taken into account: systems with uncertain actuator dynamics comply with this interpretation.

We address the problem of designing a state feedback control [the state variable ξ is available from measurements] in order: i) to track an output reference signal $y_r(t)$ (for the output y) belonging to the following class [p_y is a positive integer]:

A$_y$) $y_r(t)$ is periodic of known period T_r, of class $\mathcal{C}^{n+p_y}$, with bounded time derivatives up to order n;

ii) to guarantee closed loop boundedness along with $\mathcal{L}_2$ and $\mathcal{L}_\infty$ output tracking transient performances.
Assume that:

A.1) θ is of known sign (positive without loss of generality) and satisfies

$$\theta_m \le \theta \le \theta_M$$

with θ_m, θ_M known positive reals;
A.2) there exist known positive reals γ_f, γ_q, p_u and known smooth functions α_f, α_0, α_1 such that

$$\begin{aligned} &\text{a)} \quad |f(0)| \le \gamma_f \\ &\text{b)} \quad |f(x) - f(x_*)| \le \alpha_f(x,x_*)\|x - x_*\| \\ &\text{c)} \quad |q(0)| \le \gamma_q \\ &\text{d)} \quad |q(\xi) - q(\xi_*)| \le \Big[|u|^{p_u}\alpha_0(x,\xi_*) \\ &\qquad\qquad + \alpha_1(x,\xi_*)\Big]\|\xi - \xi_*\| \\ &\qquad\qquad \doteq \alpha_q(\xi,\xi_*)\|\xi - \xi_*\| \end{aligned}$$

for all $x, x_* \in \mathbb{R}^n$ and for all $\xi, \xi_* \in \mathbb{R}^{n+1}$;
A.3) there exists a known periodic vector signal $x_*(t) = [x_{1*}(t), \ldots, x_{n*}(t)]$ of period T_r satisfying

$$\begin{aligned} \dot{x}_{i*} &= x_{i+1*},\ 1 \le i \le n-1 \text{ (for } n \ge 2) \\ y_r &= h(x_*) \end{aligned}$$

with $x_{1*}(\cdot)$ of class $\mathcal{C}^{n+p_y}$, with bounded time derivatives satisfying

$$|x^{(i)}_{1*}(t)| \le \varrho_{ri}, \quad \forall t \in [0, T_r)$$

in terms of known positive reals ϱ_{ri}, $1 \le i \le n$.

Remark 3: In [5], it has been studied the case in which: i) no dynamics of u are considered and u constitutes the control input to be designed; ii) $h(x) = x_1$; iii) f is locally bounded by a known positive real and locally Lipschitz with known Lipschitz constant.

Remark 4: While in [5] a class of nonlinear systems with maximal relative degree has been studied, in this paper nonlinear systems which may not have a well-defined global relative degree are allowed provided that assumption A.3) holds.

III. Nonlinear Control Design with Stability Analysis

Define as in [5]

$$\begin{aligned} e_i &= x_i - x^{(i-1)}_{1*} - x^r_i,\ 1 \le i \le n \\ x^r_{i+1} &= -e_{i-1} - \lambda e_i + \dot{x}^r_i,\ 2 \le i \le n-1 \text{ (for } n \ge 3) \\ x^r_1 &= 0 \\ x^r_2 &= -\lambda e_1 \end{aligned} \tag{2}$$

where λ is a positive control parameter, so that we can write $[c(0) = c(1) = 0,\ c(s) = 1 \text{ for } s \geq 2]$

$$\begin{aligned}\dot{e}_i &= -\lambda e_i + e_{i+1} - c(i)e_{i-1},\ 1 \leq i \leq n-1 \\ &\quad (\text{for } n \geq 2) \\ \theta \dot{e}_n &= f(x) + u - \theta(x_{1*}^{(n)} + \dot{x}_n^r). \end{aligned} \tag{3}$$

Let $x^r = [x_1^r, \ldots, x_n^r]^T \in \mathbb{R}^n$ and $e = [e_1, \ldots, e_n]^T \in \mathbb{R}^n$ so that we have [I is the identity matrix]

$$\begin{aligned} x^r &= \Lambda_0(x - x_*) \\ e &= (I - \Lambda_0)(x - x_*) \doteq \Lambda(x - x_*) \\ \dot{x}_n^r &= c(n)\left[a_{n,1}(x_2 - x_{1*}^{(1)}) + \ldots + a_{n,n-1}(x_n - x_{1*}^{(n-1)})\right] \\ &\doteq c(n)a^T(x - x_*) \end{aligned}$$

with

$$\Lambda_0 = \begin{bmatrix} 0 & 0 & \ldots & \ldots & 0 \\ a_{2,1} & 0 & \ldots & \ldots & 0 \\ a_{3,1} & a_{3,2} & 0 & \ldots & 0 \\ \vdots & \ldots & \ldots & \ddots & 0 \\ a_{n,1} & \cdots & \cdots & a_{n,n-1} & 0 \end{bmatrix}$$

and the known coefficients a_{ij}, $2 \leq i \leq n$, $1 \leq j \leq i-1$, depending on λ. The tracking error dynamics (3) may be rewritten as

$$\begin{aligned}\dot{e}_i &= -\lambda e_i + e_{i+1} - c(i)e_{i-1},\ 1 \leq i \leq n-1 \\ &\quad (\text{for } n \geq 2) \\ \theta \dot{e}_n &= \eta_0(x_*, x_{1*}^{(n)}) + u + f(x) - f(x_*) \\ &\quad -c(n)\theta a^T(x - x_*) \\ &\doteq \eta_0(x_*, x_{1*}^{(n)}) + u + \sigma(x, x_*) \end{aligned} \tag{4}$$

with $\eta_0(x_*, x_{1*}^{(n)}) = f(x_*) - \theta x_{1*}^{(n)}$ and the uncertain function $\sigma(x, x_*)$ satisfying, according to assumptions A.1) and A.2), the following inequality

$$\begin{aligned} |\sigma(x, x_*)| &\leq [\alpha_f(x, x_*) + c(n)\theta_M\|a\|]\|x - x_*\| \\ &\leq [\alpha_f(x, x_*) + c(n)\theta_M\|a\|]\|\Lambda^{-1}\|\|e\|. \end{aligned}$$

Define $\nu_0(t) = \eta_0(x_*(t), x_{1*}^{(n)}(t))$ which, by virtue of assumption A$_y$), is a class $\mathcal{C}^{p_y}$ periodic function of known period T_r and can be approximated according to Fourier approximation theory (see for instance [10]). Let $\rho[N] = [\rho_0, \ldots, \rho_{N-1}]^T \in \mathbb{R}^N$ be the vector of the first N Fourier coefficients of function $\nu_0(t)$: for any $N > 1$ (N is an odd number) we can write

$$\nu_0(t) = \sum_{l=0}^{N-1} \rho_l \varphi_l(t) + \varepsilon(t), \quad \text{with } |\varepsilon(t)| \leq \varepsilon_N$$

where $(l = 1, \ldots, (N-1)/2,\ 2 \leq p \leq p_y)$

$$\begin{aligned} \varphi_0(t) &= 1 \\ \varphi_{2l}(t) &= \sqrt{2}\cos\Big(lt\frac{2\pi}{T_r}\Big) \\ \varphi_{2l-1}(t) &= \sqrt{2}\sin\Big(lt\frac{2\pi}{T_r}\Big) \end{aligned} \tag{5}$$

$$\varepsilon_N = \Big[\Big(\frac{2\pi}{T_r}\Big)^p (N-1)^{\frac{p-1}{2}}\Big]^{-1} 2^{\frac{p-1}{2}} B_{\nu p} \tag{6}$$

with $B_{\nu p}$ an upper bound on $\left|\frac{d^p \nu_0(t)}{dt^p}\right|$. According to assumptions A.1) and A.2), a known bound for $\nu_0(t)$ is given by

$$\begin{aligned} |\nu_0(t)| \leq \max_{0 \leq \tau \leq T_r} \{&\gamma_f + \alpha_f(x_*(\tau), 0)\|(x_*(\tau))\| \\ &+ \theta_M \varrho_{rn}\} \doteq B_\nu \end{aligned} \tag{7}$$

so that, by virtue of Parseval identity, we obtain

$$\sum_{l=0}^{N-1} \rho_l^2 \equiv \|\rho[N]\|^2 \leq \frac{1}{T_r}\int_0^{T_r} \nu_0^2(\tau)d\tau \leq B_\nu^2.$$

Hence, for any choice of N, we have

$$\|\rho[N]\| \leq B_\nu. \tag{8}$$

We then define an estimate $\hat{\nu_0}[N](t)$ of the Fourier approximation for $\nu_0(t)$ as $\hat{\nu_0}[N](t) = \sum_{l=0}^{N-1} \hat{\rho}_l(t)\varphi_l(t) \equiv \hat{\rho}[N](t)^T \Phi_N(t)$ in which $\Phi_N(t) = [\varphi_0(t), \ldots, \varphi_{N-1}(t)]^T$ and $\hat{\rho}[N](t) = [\hat{\rho}_0(t), \ldots, \hat{\rho}_{N-1}(t)]^T$ with $\hat{\rho}_l(t)$ being the estimate of the l-th Fourier coefficient ρ_l in $\nu_0(t)$, $0 \leq l \leq N-1$. Define the tracking and the estimation errors

$$\begin{aligned} \tilde{u} &= u - u^r \\ \tilde{\rho} &= \rho[N] - \hat{\rho}[N] \end{aligned} \tag{9}$$

where the reference signal u^r for the variable u and the adaptation law $\dot{\hat{\rho}}[N](t)$ for the parameter estimate $\hat{\rho}[N](t)$ are given by [k, ν are positive control parameters with $\nu \geq 1$, $N > 1$]

$$\begin{aligned} u^r &= -\lambda e_n - c(n)e_{n-1} - \hat{\rho}[N]^T\Phi_N - \Big(\frac{k}{4} + \frac{1}{2} \\ &\quad + \frac{s_m(x, x_*)}{2\lambda}\|\Lambda^{-1}\|^2\Big)e_n \\ s_m(x, x_*) &= 2[\alpha_f^2(x, x_*) + c(n)\theta_M^2\|a\|^2] \\ \dot{\hat{\rho}}[N] &= \text{Proj}\Big[\mu_\rho \Phi_N e_n, \hat{\rho}[N], d, B_\nu\Big] \\ \|\hat{\rho}[N](0)\| &\leq \frac{B_\nu}{\sqrt{N}}, \quad \mu_\rho = \frac{\nu k}{2} \end{aligned} \tag{10}$$

where the projection operator (see [16]) is the Lipschitz continuous function $\mathrm{Proj}\Big[\chi, \hat{\rho}[N], \cdot, \cdot\Big]$ given by

$$\mathrm{Proj}\Big[\chi, \hat{\rho}[N], d, B_\nu\Big] = \mathcal{M}\chi$$

$$\mathcal{M} = \begin{cases} I & \text{if } \mathcal{C}_{*1} \\ I & \text{if } \mathcal{C}_{*2} \\ I - \frac{s(\hat{\rho}[N])\mathrm{grad}[s(\hat{\rho}[N])]\mathrm{grad}[s(\hat{\rho}[N])]^T}{\|\mathrm{grad}[s(\hat{\rho}[N])]\|^2} & \text{if } \mathcal{C}_{*3} \end{cases}$$

$$s(\hat{\rho}[N]) = \frac{\|\hat{\rho}[N]\|^2 - B_\nu^2}{d^2 + 2dB_\nu}$$

$$\mathcal{C}_{*1} : s(\hat{\rho}[N]) \leq 0$$

$$\mathcal{C}_{*2} : s(\hat{\rho}[N]) > 0 \text{ and } \langle \mathrm{grad}[s(\hat{\rho}[N])], \chi\rangle \leq 0$$

$$\mathcal{C}_{*3} : s(\hat{\rho}[N]) > 0 \text{ and } \langle \mathrm{grad}[s(\hat{\rho}[N])], \chi\rangle > 0$$

in which d is an arbitrary positive real and B_ν is the radius of the closed ball $\overline{\mathcal{B}(0, B_\nu)}$ in $\mathbb{R}^N$ (with center the origin) in which $\rho[N]$ is constrained to be. If $\hat{\rho}[N](0) \in \overline{\mathcal{B}(0, B_\nu)}$, then the following properties hold: 1) $\|\hat{\rho}[N](t)\| \leq B_\nu + d, \ \forall t \geq 0$; 2) $\|\mathrm{Proj}[\chi, \hat{\rho}[N], d, B_\nu]\| \leq \|\chi\|$; 3) $(\rho[N] - \hat{\rho}[N])^T \mathrm{Proj}[\chi, \hat{\rho}[N], d, B_\nu] \geq (\rho[N] - \hat{\rho}[N])^T \chi$. Accordingly, the quadratic function

$$\mathcal{V} = \frac{1}{2}c(n)\sum_{i=1}^{n-1} e_i^2 + \frac{1}{2}\theta e_n^2 + \frac{1}{2\mu_\rho}\tilde{\rho}^T\tilde{\rho}$$

admits time derivative along the trajectories of the closed loop system satisfying the following inequalities

$$\dot{\mathcal{V}} \leq -\frac{\lambda}{2}\|e\|^2 + \frac{\varepsilon^2}{k} + \tilde{u}e_n$$

$$\dot{\mathcal{V}} \leq -\frac{\lambda}{2}\|e\|^2 + \frac{\varepsilon^2}{k} + \frac{\tilde{u}^2}{2}.$$

Let us write

$$\dot{x} = \dot{x}_* + \Lambda^{-1}\dot{e} = \dot{x}_* + \Big[\Lambda_*^{-1} \ \Big| \ \Gamma_2\Big]\dot{e} = \dot{x}_* + \Gamma_1 + \Gamma_2\dot{e}_n$$

in which Γ_1 is a known vector depending on e and Γ_2 is the n-th column vector of the square matrix Λ^{-1}. By direct computation we have

$$\dot{\tilde{u}} = q(\xi) + v - \phi_{0u} - \phi_{1u}\Big[\frac{\tilde{u}}{\theta} + \frac{\mathcal{K}_c e_n}{\theta} - \frac{c(n)e_{n-1}}{\theta} + \frac{\Phi_N^T\tilde{\rho}}{\theta} + \frac{\sigma(x, x_*)}{\theta} + \frac{\varepsilon}{\theta}\Big]$$

in which the known terms

$$\begin{aligned} \phi_{0u} &= c(n)\lambda e_{n-1} - c(n)e_n + c(n-1)e_{n-2} - \dot{\hat{\rho}}[N]^T\Phi_N \\ &\quad -\hat{\rho}[N]^T\dot{\Phi}_N - \frac{\nabla_x s_m(x, x_*)}{2\lambda}(\dot{x}_* + \Gamma_1)\|\Lambda^{-1}\|^2 e_n \\ &\quad -\frac{\nabla_{x_*} s_m(x, x_*)}{2\lambda}\dot{x}_*\|\Lambda^{-1}\|^2 e_n \qquad (11) \\ \phi_{1u} &= \mathcal{K}_c - \frac{\|\Lambda^{-1}\|^2}{2\lambda}e_n\nabla_x s_m(x, x_*)\Gamma_2 \\ \mathcal{K}_c &= -\Big[\lambda + \frac{k}{4} + \frac{1}{2} + \frac{s_m(x, x_*)}{2\lambda}\|\Lambda^{-1}\|^2\Big] \end{aligned}$$

appear with the notation $\nabla_\vartheta\Psi(\vartheta) = [\frac{\partial\Psi(\vartheta)}{\partial\vartheta_1}, \ldots, \frac{\partial\Psi(\vartheta)}{\partial\vartheta_{n_\vartheta}}]$. Define the uncertain variable $u_* = -\eta_0(x_*, x_{1*}^{(n)})$ such that $u_*(t)$, by virtue of assumption A_y), is a class $\mathcal{C}^{p_y}$ periodic function of known period T_r with $|u_*(t)| \leq B_\nu$. Let $\xi_* = [x_*^T, u_*]^T$ and define $\mu_0(t) = q(\xi_*(t))$ which is a class $\mathcal{C}^{p_y}$ periodic function of known period T_r and can be approximated according to Fourier approximation theory. Let $\delta[M] = [\delta_0, \ldots, \delta_{M-1}]^T \in \mathbb{R}^M$ be the vector of the first M Fourier coefficients of function $\mu_0(t)$: for any $M > 1$ (M is an odd number) we can write

$$\mu_0(t) = \sum_{l=0}^{M-1}\delta_l\varphi_l(t) + \bar{\varepsilon}(t), \quad \text{with } |\bar{\varepsilon}(t)| \leq \bar{\varepsilon}_M$$

where ($2 \leq p \leq p_y$)

$$\bar{\varepsilon}_M = \Big[\Big(\frac{2\pi}{T_r}\Big)^p (M-1)^{\frac{p-1}{2}}\Big]^{-1} 2^{\frac{p-1}{2}} B_{\mu p} \qquad (12)$$

with $B_{\mu p}$ an upper bound on $\Big|\frac{\mathrm{d}^p\mu_0(t)}{\mathrm{d}t^p}\Big|$. According to assumption A.2), a known bound for $\mu_0(t)$ is given by

$$|\mu_0(t)| \leq \max_{\substack{0\leq\tau\leq T_r \\ |u_*|\leq B_\nu}} \{\gamma_q + \alpha_q([x_*(\tau)^T, u_*]^T, 0)\|[x_*(\tau)^T, u_*]^T\|\} \doteq B_\mu$$

so that, by virtue of Parseval identity, we obtain

$$\sum_{l=0}^{M-1}\delta_l^2 \equiv \|\delta[M]\|^2 \leq \frac{1}{T_r}\int_0^{T_r}\mu_0^2(\tau)d\tau \leq B_\mu^2.$$

Hence, for any choice of M, we have

$$\|\delta[M]\| \leq B_\mu. \qquad (13)$$

We then define an estimate $\hat{\mu_0}[M](t)$ of the Fourier approximation for $\mu_0(t)$ as $\hat{\mu_0}[M](t) = \sum_{l=0}^{M-1}\hat{\delta}_l(t)\varphi_l(t) \equiv \hat{\delta}^T[M](t)\Phi_M(t)$ in which $\Phi_M(t) = [\varphi_0(t), \ldots, \varphi_{M-1}(t)]^T$ and $\hat{\delta}[M](t) = [\hat{\delta}_0(t), \ldots, \hat{\delta}_{M-1}(t)]^T$ with $\hat{\delta}_l(t)$ being the estimate of the l-th Fourier coefficient δ_l in $\mu_0(t)$, $0 \leq l \leq M-1$. Let

$$\begin{aligned} e_u &= u - \hat{u} \qquad (14) \\ \tilde{\delta} &= \delta[M] - \hat{\delta}[M]. \end{aligned}$$

We design the control input and the adaptive observer [μ_δ, d^* are positive control parameters]

$$\begin{aligned} v &= \phi_{0u} - e_n - \hat{\delta}[M]^T\Phi_M - \mathcal{K}_u\tilde{u} \\ \dot{\hat{u}} &= \hat{\delta}[M]^T\Phi_M + v + \mathcal{K}_e e_u, \quad \hat{u}(0) = \hat{u}_0 \\ \dot{\hat{\delta}}[M] &= \mathrm{Proj}\Big[\mu_\delta\Phi_M e_u, \hat{\delta}[M], d^*, B_\mu\Big], \ \|\hat{\delta}[M](0)\| \leq B_\mu \end{aligned}$$

with the non-negative feedback terms $\mathcal{K}_u, \mathcal{K}_e$ yet to be chosen, so that the time derivative of the quadratic function

$$V = \mathcal{V} + \frac{1}{2}(\tilde{u}^2 + e_u^2)$$

is forced to satisfy along the trajectories of the closed loop system

$$\begin{aligned} \dot{V} \leq & -\frac{\lambda}{2}\|e\|^2 - [\mathcal{K}_u - m_2(\alpha_q)]\tilde{u}^2 - \mathcal{K}_e e_u^2 + \frac{\varepsilon^2}{k} \\ & +\|\tilde{\delta}\|\|\Phi_M\||\tilde{u}| + |\bar{\varepsilon}||\tilde{u}| + \|\tilde{\delta}\|\|\Phi_M\||e_u| + |\bar{\varepsilon}||e_u| \\ & +m_0(\alpha_q)|\varepsilon||\tilde{u}| + m_1(\alpha_q)\|e\||\tilde{u}| + m_3(\alpha_q)\|\tilde{\rho}\||\tilde{u}| \\ & +m_4(\alpha_q)\|e\||e_u| + m_5(\alpha_q)|\tilde{u}||e_u| \\ & +m_6(\alpha_q)|\varepsilon||e_u| + m_7(\alpha_q)\|\tilde{\rho}\||e_u| \end{aligned}$$

in terms of the functions

$$\begin{aligned}
m_0(\alpha_q) &= m_2(\alpha_q) = \frac{|\phi_{1u}|}{\theta_m} + \alpha_q(\xi,\xi_*) \\
m_1(\alpha_q) &= \alpha_q(\xi,\xi_*)[\|\Lambda^{-1}\| + 1 + |\mathcal{K}_c|] + \frac{|\phi_{1u}||\mathcal{K}_c|}{\theta_m} \\
&\quad + \frac{c(n)|\phi_{1u}|}{\theta_m} + \frac{|\phi_{1u}|}{\theta_m}[\alpha_f(x,x_*) \\
&\quad + c(n)\theta_M\|a\|]\|\Lambda^{-1}\| \\
m_2(\alpha_q) &= \alpha_q(\xi,\xi_*) + \frac{|\phi_{1u}|}{\theta_m} \\
m_3(\alpha_q) &= \alpha_q(\xi,\xi_*)\|\Phi_N\| + \frac{|\phi_{1u}|\|\Phi_N\|}{\theta_m} \qquad (15) \\
m_4(\alpha_q) &= \alpha_q(\xi,\xi_*)[\|\Lambda^{-1}\| + 1 + |\mathcal{K}_c|] \\
m_5(\alpha_q) &= m_6(\alpha_q) = \alpha_q(\xi,\xi_*) \\
m_7(\alpha_q) &= \alpha_q(\xi,\xi_*)\|\Phi_N\|.
\end{aligned}$$

According to assumption A.2) and Lemma 2.1 in [11], the known continuous real-valued function $\alpha_q(\xi,\xi_*)$ satisfies

$$\begin{aligned}
\alpha_q(\xi,\xi_*) &\le |u|^{p_u}(\eta_x(x,x_*) + \eta_u(u_*)) \\
&\quad + (\tilde{\eta}_x(x,x_*) + \tilde{\eta}_u(u_*)) \\
&\le |u|^{p_u}\eta_x(x,x_*) + \tilde{\eta}_x(x,x_*) \\
&\quad + |u|^{p_u}\max_{|u_*|\le B_\nu}\{\eta_u(u_*)\} + \max_{|u_*|\le B_\nu}\{\tilde{\eta}_u(u_*)\} \\
&\doteq \tilde{\alpha}_q(\xi,x_*)
\end{aligned}$$

in terms of the (known) smooth non-negative functions η_x, η_u, $\tilde{\eta}_x$, $\tilde{\eta}_u$, so that we can write $[1 \le i \le 7]$

$$m_i^2(\alpha_q) \quad \le \quad m_i^2(\tilde{\alpha}_q) \doteq \tilde{m}_i(\xi,x_*)$$

with $\tilde{m}_i$ known continuous non-negative real-valued functions. Let us choose the yet undefined feedback terms $\mathcal{K}_u$ and $\mathcal{K}_e$ as (k_u and k_e are positive control parameters)

$$\begin{aligned}
\mathcal{K}_u &= k_u + \frac{1}{2} + \frac{k}{4}\left(1 + \|\Phi_M\|^2\right) + \frac{k}{4}\tilde{m}_0(\xi,x_*) \\
&\quad + \frac{4}{\lambda}\tilde{m}_1(\xi,x_*) + \frac{\tilde{m}_2(\xi,x_*)}{2} + \frac{k}{4}\tilde{m}_3(\xi,x_*) \\
&\quad + \frac{\tilde{m}_5(\xi,x_*)}{2} \qquad (16) \\
\mathcal{K}_e &= k_e + \frac{1}{2} + \frac{k}{4}\left(1 + \|\Phi_M\|^2\right) + \frac{4}{\lambda}\tilde{m}_4(\xi,x_*) \\
&\quad + \frac{k}{4}\tilde{m}_6(\xi,x_*) + \frac{k}{4}\tilde{m}_7(\xi,x_*)
\end{aligned}$$

so that

$$\begin{aligned}
\dot{V} &\le -\frac{3\lambda}{8}\|e\|^2 - k_u\tilde{u}^2 - k_e e_u^2 + 3\frac{\varepsilon^2}{k} + 2\frac{\bar{\varepsilon}^2}{k} \\
&\quad + 2\frac{\|\tilde{\rho}\|^2}{k} + 2\frac{\|\tilde{\delta}\|^2}{k}
\end{aligned}$$

and therefore the following $\mathcal{L}_\infty$ and $\mathcal{L}_2$ inequalities [ε_N, $\bar{\varepsilon}_M$ are the approximation bounds (6) and (12) while $c_* = \min\{\frac{3\lambda}{4}, \frac{3\lambda}{4\theta_M}, 2k_u, 2k_e, 1\}$]

$$\begin{aligned}
\|\zeta(t)\|^2 &\le \frac{\max\{1,\theta_M\}}{\min\{1,\theta_m\}}\|\zeta(0)\|^2 e^{-c_* t} \\
&\quad + \frac{6}{kc_*\min\{1,\theta_m\}}\left(\varepsilon_N^2 + \bar{\varepsilon}_M^2\right) \\
&\quad + \frac{4\max\{1,3c_*^{-1}\}}{k\min\{1,\theta_m\}}\max_{0\le\tau\le t}\{\|\tilde{\rho}(\tau)\|^2 \\
&\quad + \|\tilde{\delta}(\tau)\|^2\} \\
\int_0^t \|\zeta(\tau)\|^2 d\tau &\le \frac{\max\{1,\theta_M\}}{c_*\min\{1,\theta_m\}}\|\zeta(0)\|^2 \\
&\quad + \frac{2}{kc_*\min\{1,\theta_m\}}\|\tilde{\rho}(0)\|^2 \\
&\quad + \frac{6}{kc_*\min\{1,\theta_m\}}\int_0^t\left(\varepsilon_N^2 + \bar{\varepsilon}_M^2\right. \\
&\quad \left. + \|\tilde{\rho}(\tau)\|^2 + \|\tilde{\delta}(\tau)\|^2\right)d\tau \qquad (17)
\end{aligned}$$

are satisfied for any $t > 0$ in terms of $\zeta = [e^T, \tilde{u}, e_u]^T$, where, according to (8), (13) and property (1) of the projection algorithm, for all $t \ge 0$

$$\begin{aligned}
\|\tilde{\rho}(t)\| &\le \sqrt{2B_\nu^2 + 2(B_\nu + d)^2} \qquad (18) \\
\|\tilde{\delta}(t)\| &\le \sqrt{2B_\mu^2 + 2(B_\mu + d^*)^2}.
\end{aligned}$$

By virtue of the quadratic function [a_p, s_*, a_{*p} are sufficiently small positive reals]

$$\begin{aligned}
W &= \mathcal{V} + \frac{1}{2}a_p\|Q(t)\tilde{\rho} - \Omega(t)e\|^2 + \frac{1}{2}\tilde{u}^2 \\
&\quad + \frac{s_*}{2}\left(e_u^2 + \frac{\|\tilde{\delta}\|^2}{\mu_\delta} + a_{*p}\|Q_*(t)\tilde{\delta} - \Phi_M(t)e_u\|^2\right)
\end{aligned}$$

in which

$$\Omega^T = \begin{bmatrix} 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\varphi_0}{\theta} & \dots & \dots & \frac{\varphi_{N-1}}{\theta} \end{bmatrix}$$

and $Q(t), Q_*(t)$ are generated by the filters

$$\begin{aligned}
\dot{Q}(t) &= -Q(t) + \Omega(t)\Omega^T(t), \quad Q(0) = \frac{T_r}{\theta_M^2}I \\
\dot{Q}_*(t) &= -Q_*(t) + \Phi_M(t)\Phi_M^T(t), \quad Q_*(0) = T_r I
\end{aligned}$$

and by using arguments similar to those used in the proof of Lemma 3.2 in [5] we can establish that, for any positive integer q_* and for any initial condition $[x(0)^T, u(0)]^T$ in the closed ball $\overline{\mathcal{B}(0;m_\mathcal{A})} \subset \mathbb{R}^{n+1}$ with center the origin and arbitrary radius $m_\mathcal{A} > 0$, there exist a positive integer $p_{y*}(q_*, p_u)$ and a positive real $k_u^*(N,M,m_\mathcal{A})$ such that for any $p_y > p_{y*}(q_*, p_u)$ and for any control parameter $k_u > k_u^*(N,M,m_\mathcal{A})$, $\zeta_m = [\zeta^T, \tilde{\rho}^T, \tilde{\delta}^T]^T$ satisfies the following properties: $\mathcal{P}_1$) $\zeta_m(t)$ is exponentially attracted, as $t \to +\infty$, into a closed ball of radius $r(N,M)$, with $r(N,M) = O\big(\min\{N,M\}^{-q_*}\big)$ for $N, M \to +\infty$; $\mathcal{P}_2$) if there exist odd integers N^*, M^* such that $\nu_0(t) = \sum_{l=0}^{N^*-1}\rho_l\varphi_l(t)$ and $\mu_0(t) = \sum_{l=0}^{M^*-1}\delta_l\varphi_l(t)$, then the control algorithm with

$N \geq N^*$ and $M \geq M^*$ guarantees exponential convergence of $\|\zeta_m(t)\|$ to zero. Since, according to Hadamard's Lemma (see [1]) and (17), there exists a positive constant c_y (not increasing when N and M increase) such that

$$|\tilde{y}| \doteq |y - y_r| \leq c_y \|x - x_*\| \leq c_y \|\Lambda^{-1}\| \|\zeta_m\|$$

we have proved the following:

Theorem 1: Consider system (1) under assumptions A1)-A3) and let the output reference signal $y_r(t)$ satisfy assumption A$_y$). Let $\tilde{y} = y - y_r$ be the output tracking error, N, M be arbitrary odd integers ($N, M > 1$), k_u be a positive control parameter and p_u, p_y be the positive scalars in assumptions A.2) and A$_y$), respectively. A dynamic state feedback control $v(k_u)$ exists such that: i) closed loop boundedness along with $\mathcal{L}_\infty$ and $\mathcal{L}_2$ output tracking transient performances are guaranteed for any initial condition $[x(0)^T, u(0)]^T$; ii) for any positive integer q_* and for any initial condition $[x(0)^T, u(0)]^T$ in the closed ball $\overline{\mathcal{B}(0; m_\mathcal{A})} \subset \mathbb{R}^{n+1}$ with center the origin and arbitrary radius $m_\mathcal{A} > 0$, there exist a positive integer $p_{y*}(q_*, p_u)$ and a positive real $k_u^*(N, M, m_\mathcal{A})$ such that, for any $p_y > p_{y*}(q_*, p_u)$ and for any control parameter $k_u > k_u^*(N, M, m_\mathcal{A})$: $\mathcal{Q}_1$) $\tilde{y}(t)$ is exponentially attracted, as $t \to +\infty$, into a closed ball of radius $r(N, M)$, with $r(N, M) = O\big(\min\{N, M\}^{-q_*}\big)$ for $N, M \to +\infty$; $\mathcal{Q}_2$) if there exist odd integers N^*, M^* such that $\nu_0(t) = \sum_{l=0}^{N^*-1} \rho_l \varphi_l(t)$ and $\mu_0(t) = \sum_{l=0}^{M^*-1} \delta_l \varphi_l(t)$, then the control algorithm with $N \geq N^*$ and $M \geq M^*$ guarantees exponential convergence of $|\tilde{y}(t)|$ to zero.

IV. Conclusions

Under the assumption that the output reference signal is sufficiently smooth and periodic with known period, a state feedback output tracking control has been designed for a class of single input-single output nonlinear systems (1) which are affected by unstructured uncertainties satisfying assumptions A.1)-A.2). $\mathcal{L}_2$ and $\mathcal{L}_\infty$ transient performances are guaranteed in the learning phase, while for any initial condition in an arbitrary given compact set, by properly setting the control parameters, the following properties hold: i) the guaranteed output tracking error is reduced by increasing the numbers N, M of the truncated series expansions; ii) when the uncertain functions $\nu_0(t)$ and $\mu_0(t)$ are represented by finite Fourier series expansions, they are exponentially reconstructed while the output tracking error exponentially converges to zero.

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