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Robust Constraint Satisfaction for Continuous-Time Nonlinear Systems

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Abstract— **A method to design feedback control laws that guarantee state-constraint satisfaction for a class of uncertain nonlinear systems is proposed. A recursive procedure is used to construct robustly controlled invariant sets for input affine nonlinear systems with bounded disturbances or parametric uncertainties. Constraint satisfaction is achieved by a modification of the control input on the boundaries of the constructed sets. The results are illustrated on a design example.**

I. INTRODUCTION

Every physical control system must deal with some limits or constraints in the operation space. Such limits can arise from physical constraints imposed on the system by its surroundings. They can also arise from safety constraints. In process control, for example, it is often economically desirable to operate close to limits of the feasible region. Unfortunately, such operating practices are not sustainable due to the potential damage and safety problems associated with the violations of the operation bounds. To avoid such problems, it is imperative that one incorporates all process constraints, and especially safety constraints, as an essential performance requirement in the design of a control system.
While the problem of saturated actuators has already gained much interest in recent years [5], structural results for state-constrained problems remain rare. A good insight in the problem is given in [12] and the references therein where state-constrained linear systems are considered. In the study of state-constrained nonlinear systems, different approaches have been proposed over the last few years. Basically two types of approaches have been considered. The first category consists of controller design methods in which state constraints are taken into account explicitly during the controller design procedure. Such approaches result in control schemes that guarantee closed-loop stability, constraint satisfaction and other specifications. In [10], an approach based on backstepping is proposed. A second category of strategies seeks to modify the command signal on-line to prevent constraint violations. A very successful example of such methods is the well-known Nonlinear Model Predictive Control [3], where optimization methods are used to determine a control signal that avoids constraint violation. Other approaches include so-called override schemes [7], [14] and reference governors [1], [6]. Of particular interest in this study is the invariance control approach proposed in [16]. In this method, an invariance control law is designed *a priori* and a subset of the state space that can be rendered invariant by this control law is defined. Using this method, it is possible to apply a nominal controller inside the invariant set, while switching to the invariance controller on the boundaries guarantees constraint satisfaction. The approach presented in the following is based on the ideas of invariance control. The advantage of this approach is that the controller design method is greatly simplified by considering the stabilization and the constraint satisfaction problem separately. The key point of the proposed approach is the construction of a robustly controlled invariant set. An interesting feature is that finally only a first order constraint has to be considered to guarantee the invariance of the resulting set. This leads directly to conditions on the control input that can be integrated in the controller design. The proposed design procedure is flexible and can handle bounded uncertainties in the system.
The remainder of the paper is organized as follows. In section II, the relevant class of systems is identified and a general problem formulation is given. The main result of this work is stated in section III where a method to construct robustly controlled invariant sets is presented. The controller synthesis is explained in section IV. In section V, the proposed design procedure is illustrated on a simulation example. A summary and an outlook on future work are given in section VI.

II. PROBLEM FORMULATION

Throughout this paper we consider the following class of uncertain input affine control systems

$$\dot{x} = f(x) + g(x)u + q(x, \theta(t)) \tag{1}$$

where $x \in \mathbb{R}^n$ are the state variables and $u \in \mathbb{R}$ is the input variable. The vector fields $f(x)$ and $g(x)$ are assumed to be sufficiently smooth on a domain $\mathcal{D} \subset \mathbb{R}^n$. The uncertainty vector $\theta(t) = [\theta_1, \ldots, \theta_p] \in \Omega \subset \mathbb{R}^p$ belongs to a known compact set $\Omega \subset \mathbb{R}^p$. Furthermore the uncertainty is assumed to be linearly parametrized, i.e. $q(x, \theta(t)) = \sum_{l=1}^{p} q_l(x)\theta_l(t)$. State constraints of the system are given by a set of output functions

$$y_i = h_i(x) \leq 0, \quad i = 1, \ldots, m \tag{2}$$

where $h_i(x) : \mathbb{R}^n \to \mathbb{R}$ are sufficiently smooth functions. Each output is assumed to have a well defined relative degree ρ_i, that is

$$L_g h_i(x) = \cdots = L_g L_f^{\rho_i - 2} h_i(x) = 0, \quad L_g L_f^{\rho_i - 1} h_i(x) \neq 0$$

for all $x \in \mathcal{D}$, where $L_g h_i(x)$ denotes the standard Lie (directional) derivative of $h_i(x)$ along the vector field $g(x)$. The disturbances are assumed to satisfy the triagularity condition [9, Theorem 3.1.1]. Therefore the following assumption is made.

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Assumption 1. For each output function $h_i(x)$, there exists a change of coordinates

$$T(x) = \left[\begin{array}{c} \Phi_1(x) \\ \vdots \\ \Phi_{n-\rho}(x) \\ \hline h_i(x) \\ \vdots \\ L_f^{\rho-1} h_i(x) \end{array}\right] =: \left[\begin{array}{c} \xi \\ \hline z \end{array}\right], \quad (3)$$

where Φ_1 to $\Phi_{n-\rho}$ are chosen such that $T(x)$ is a diffeomorphism on the domain $\mathcal{D} \subset \mathbb{R}^n$, that transforms the system into the normal form

$$\dot{\xi} = \Phi(\xi, z, \theta(t)) \quad (4)$$

$$\dot{z}_j = z_{j+1} + \sum_{l=1}^{p} \phi_{jl}(z_1, \ldots, z_j)\theta_l(t), \quad 1 \le j \le \rho_i - 1$$

$$\dot{z}_{\rho_i} = a(z,\xi) + b(z,\xi)u + \sum_{l=1}^{p} \phi_{\rho_i l}(z_1, \ldots, z_{\rho_l})\theta_l(t) \quad (5)$$

$$y_i = z_1.$$

Note that no further assumption on the ξ-dynamics needs to be made at this point. It will become obvious in the following that for the purpose of constraint satisfaction no other requirement, i.e. no minimum-phase assumption is needed since only the z-dynamics will be considered for the construction of robustly controlled invariant sets. Thus, a transformation $T(x)$ can be found in a straight forward manner for many control problems. The most restrictive assumption concerns the existence of the $b(z,\xi)$ term, that is the well defined relative degree. This term has to exist and be bounded in specific regions of the state space yet to be determined.

A feedback controller for the system (1) can be designed using conventional design techniques while ignoring the constraints. Well known robust controller design techniques can be found in [4] or [9]. In the following, this robust feedback controller is termed nominal controller and is denoted by u_{nom}. It is assumed that the nominal controller stabilizes the system (1) with respect to an equilibrium x_d or a time-varying trajectory $x_d(t)$. Furthermore, it is assumed that a suitable Lyapunov function $V(e)$, $e = x - x_d$, for the closed-loop system is known.

III. Design Procedure

In this section, we state the main contribution of this study. A design method,based on the idea of backstepping, is proposed for the construction of robustly invariant sets that are constraint admissible. First, a well known result is reviewed for the sake of clarity.

Definition [[2], Def. 2.3] The set $\mathcal{S} \subset \mathbb{R}^n$ is said to be robustly controlled invariant for the system

$$\dot{x} = f(x(t), u(t), \theta(t)), \quad \theta(t) \in \Omega,$$

if there exists a feedback control law $u(t) = k(x(t))$ which assures the existence and uniqueness of the solution on $\mathbb{R}^+$ and is s.t. for all $x(0) \in \mathcal{S}$ the solution $x(t) \in \mathcal{S}$ for $t > 0$.

Clearly not every subset of the state space is robustly controlled invariant. In the following, a procedure is proposed to design suitable sets and to derive corresponding conditions on the control inputs for systems of the form (1),(2). The basic idea is to restrict the set of safe initial conditions step by step by constructing new constraints, whereas satisfaction of each constraint guarantees satisfaction of the previous constraint. The procedure is repeated until one obtains a constraint whose time derivative depends explicitly on the control input. Starting inside the constructed set, the control input can be used to guarantee constraint satisfaction for the last and therefore for all previous constraints.

The construction algorithm is first introduced for a single constraint. However, it is possible to repeat the algorithm for each constraint $h_i(x)$ and under some additional conditions on the constraint relation is it possible to guarantee satisfaction for the whole set of constraints (2). This topic will be discussed later on. Under the previous assumptions, the main result can be stated as follows.

Theorem 1: Consider the uncertain nonlinear system (1) subject to one of the constraints (2). Suppose furthermore that assumption 1 holds, then a constraint admissible safe set $\mathcal{S}$ can be constructed and there exists a control law u_{inv} that guarantees robust invariance of this set.

Remark The proof of this result is constructive and introduces an algorithm for the construction of robustly controlled invariant sets. For simplicity, ρ_i will be denoted by ρ and the ξ-dynamics is neglected in the following. This does not impose a problem as clarified later.

Proof. By assumption 1, there exists a change of coordinates that transforms the system into the normal form

$$\dot{z}_j = z_{j+1} + \sum_{i=1}^{p} \phi_{ji}(z_1, \ldots, z_j)\theta_i(t), \quad 1 \le j \le \rho - 1$$

$$\dot{z}_\rho = a(z) + b(z)u + \sum_{i=1}^{p} \phi_\rho(z_1, \ldots, z_\rho)\theta_i(t),$$

where the constraint is now imposed on the coordinate z_1, i.e. $y = z_1 \le 0$. To construct a set of safe initial conditions, one starts with the first subsystem

$$\dot{z}_1 = z_2 + \sum_{i=1}^{p} \phi_{1i}(z_1)\theta_i(t). \quad (6)$$

Since the uncertainties belong to a known and compact set Ω one can find a bound such that $|\theta_i(t)| \le \bar{\theta}_i$. To guarantee safety in the system, one has to consider the worst case disturbance and find an upper bound for the right hand side of (6). This can be done in different ways, but it is advantageous to find a smoothly differentiable bounding function. In general, it is possible to find some smooth functions Ξ and Ψ that guarantee

$$\sum_{i=1}^{p} \phi_{1i}(z_1)\theta_i(t) \leq \Xi_1(z_1) + \Psi_1(\bar{\theta}) \tag{7}$$

for $t > 0$ in the subset of interest in the state-space, i.e. where $z_1 \leq 0$. Now, the right hand side of (6) can be overestimated and rewritten as

$$\begin{aligned}\dot{z}_1 &= z_2 + \sum_{i=1}^{p} \phi_{1i}(z_1)\theta_i(t) \leq z_2 + \Xi_1(z_1) + \Psi_1(\bar{\theta}) \\ &= -k_1 z_1 + \left[z_2 + k_1 z_1 + \Xi_1(z_1) + \Psi_1(\bar{\theta})\right],\end{aligned}$$

where k_1 is a positive constant. The new variable

$$v_1 := z_2 + k_1 z_1 + \Xi_1(z_1) + \Psi_1(\bar{\theta})$$

can be introduced and one can rewrite the upper system as

$$\dot{z}_1 \leq -k_1 z_1 + v_1.$$

The comparison principle [8, Lemma 3.4] can be used to provide an upper bound for the trajectories of the differential inequality above by those of the comparison system $\dot{\tilde{z}}_1 = -k_1\tilde{z} + v_1$ (*i.e.* whenever it holds that $z_1(t_0) \leq \tilde{z}_1(t_0)$ then $z_1(t) \leq \tilde{z}_1(t),\ (\forall t > 0)$). The main objective here is to guarantee that $z_1(t) \leq 0$ which implies that $z_1(0) \leq 0$. Considering now the comparison system, one can use the Lyapunov function $V_1 = \frac{1}{2}\tilde{z}_1^2$ that satisfies $\dot{V} \leq -\tilde{z}_1^2 + \tilde{z}_1 v_1$ to show that if $\tilde{z}_1(0) \leq 0$ and $v_1(t) \leq 0$ then it is guaranteed that $\tilde{z}_1(t) \leq 0$ and therefore that $z_1(t) \leq 0$. The second requirement for constraint satisfaction is that $\sup_t v_1(t) \leq 0$ which yields a new constraint given by

$$v_1 := z_2 + k_1 z_1 + \Xi_1(z_1) + \Psi_1(\bar{\theta}) \leq 0.$$

If the initial condition $x(0)$ satisfies $z_1 \leq 0$ and $v_1 \leq 0$ and if it can be guaranteed furthermore that $v_1(t) \leq 0,\ (\forall t > 0)$ than it holds that $z_1(t) \leq 0,\ (\forall t > 0)$.

The next steps are to apply the procedure inductively to determine $\rho - 2$ more constraints written as

$$v_l := z_{l+1} + \Xi_l(z_1, \ldots, z_l) + \Psi_l(\bar{\theta}), \quad l = 1 \ldots n-2.$$

Taking their time derivatives, one obtains the following expressions

$$\dot{v}_l = z_{l+2} + \Xi_{l+1}^{'}(z_1, \ldots, z_{l+1}) + \sum_{i=1}^{p} \bar{\phi}_{l+1,i}\theta_i(t),$$

where all terms without uncertainty resulting from the derivatives of $k_l v_{l-1}$ and Ξ_l are collected in the new expression Ξ_{l+1}'. Using the inequality conditions, this derivative can be bounded by a smoothly differentiable function, i.e.

$$\dot{v}_l \leq z_{l+2} + \Xi_{l+1}(z_1, \ldots, z_{l+1}) + \Psi_{l+1}(\bar{\theta}_i)$$

with Ξ_{l+1} resulting from $\Xi_{l+1}^{'}$ and the bounding of $\sum_{i=1}^{p} \bar{\phi}_{l+1,i}\theta_i(t)$. Again, applying the same reasoning as above, the next constraint can be defined as

$$v_{l+1} = z_{l+2} + k_{l+1}v_l + \Xi_{l+1}(z_1, \ldots, z_{l+1}) + \Psi_{l+1}(\bar{\theta}_i) \leq 0$$

The last constraint that has to be defined is the constraint $v_{\rho-1}$. As above, this constraint takes the form

$$v_{\rho-1} := z_\rho + \Xi_{\rho-1}(z_1, \ldots, z_{\rho-1}) + \Psi_{\rho-1}(\bar{\theta}),$$

but its derivative can be controlled directly by the input, i.e.

$$\dot{v}_{\rho-1} \leq a(z) + b(z)u + \Xi_\rho(z_1, \ldots, z_\rho) + \Psi_\rho(\bar{\theta}).$$

By the input-output linearizability of the nominal system, it follows that $b(z) \neq 0$. Therefore the control input u can be used to guarantee $\dot{v}_{\rho-1} \leq 0$ using an appropriate control law $u_{inv}(x)$. More precisely, any control law that guarantees

$$a(z) + b(z)u + \Xi_\rho(z_1, \ldots, z_n) + \Psi_\rho(\bar{\theta}) \leq 0, \tag{8}$$

guarantees constraint satisfaction. There are different ways to construct such a controller. One possible choice will be proposed later in this paper. The constraint admissible set of safe initial conditions is given by

$$\mathcal{S} = \{x \in \mathbb{R}^n | z_1 \leq 0, v_i \leq 0, i = 1, \ldots, \rho - 1\} \tag{9}$$

This completes the proof. ∎

At this point, the results can be shortly summarized as follows. Starting with the given state constraint $y = h(x) \leq 0$, $\rho - 1$ new constraints $v_1(x), \ldots, v_{\rho-1}$ are designed recursively. The satisfaction of each constraint $v_i \leq 0$ guarantees satisfaction of the previous constraint $v_{i-1} \leq 0$ directly. Therefore, if the initial state is such that $z_1(x(0)) \leq 0, v_i(x(0)) \leq 0,\ i = 1 \ldots \rho - 1$, it is sufficient to verify satisfaction of the last constraint $v_{\rho-1}(x(t)) \leq 0,\ (\forall t > 0)$, which can be achieved with the control input. The function $v_{\rho-1}(x)$ is negative inside the safe set that should be rendered invariant, it is zero on its boundary and positive in the unsafe region. Due to the special importance of this last constraint the function $B(x) := v_{\rho-1}$ referred to as a *barrier certificate* [11], [15] for the constraint admissible set $\mathcal{S}$. Control inputs that guarantee the invariance of the set $\mathcal{S}$ are described by the condition (8). It remains to select a suitable control input in accordance to this condition.

The advantage of the backstepping-like procedure proposed here is that, like the original backstepping methods, it allows one to tailor the approach to address many problem dependent issues often encountered in the study of nonlinear control systems. Note that it is not necessary to transform each constraint's dynamics to $\dot{v}_i \leq -k_i v_i + v_{i+1}$. Alternatively, one could easily assign another function with similar properties e.g. $\dot{v}_i \leq -\alpha(v_i) + v_{i+1}$, where α is a Lipschitz function in $(0, \infty)$. For most cases however, the proposed approach yields good results.

Remark When the system (1) is not subject to disturbances, all inequalities become equalities and the system takes the form

$$\begin{aligned}\dot{z}_1 &= -k_1 z_1 + v_1\\ \dot{v}_1 &= -k_2 v_1 + v_2\\ &\vdots\\ \dot{v}_{\rho-1} &= a(z) + b(z)u.\end{aligned}$$

If furthermore the remaining ξ-dynamics are input-to-state stable, then the controller $u = (-a(z) - v_{\rho-1})/b(z)$ stabilizes the system on the original constraint $y = h(x) = 0$.

If the system (1) is subject to more than one constraint, the proposed algorithm has to be repeated for each constraint $h_i(x)$ to construct the different sets $\mathcal{S}_i$. The overall safe set is then given by $\mathcal{S} = \bigcap \mathcal{S}_i$ and is characterized by the barrier certificate $B(x) = \max_i [B_i(x)]$. In this case it might happen that two or more constraints are active at the same time. It is necessary to guarantee that there is no contradiction in the constraint satisfaction. In general, the question is whether there exists at least one control input u_{inv} that decreases simultaneously the barrier certificates of all active constraints. This very general question of existence is hard to answer and therefore a simplifying assumption on the intersecting constraints is made here.

Assumption 2 For constraint satisfaction of multiple intersecting constraints, it is assumed that the $L_g B(x)$-terms have the same sign at the intersection points. That is,

$$\text{sign} L_g B_i(x) = \text{sign} L_g B_j(x), \text{ if } B_{i,j}(x) = 0, x \in \mathcal{S}. \quad (10)$$

If this requirement is not met, it might still be possible to find an inner approximation of one constraint or to introduce an additional constraint such that the condition is satisfied. The reasoning for this additional assumption is the following. When the given condition is satisfied, both invariance control laws have the same sign and therefore $u_{inv} = u_{inv_i} + u_{inv_j}$ renders $\dot{B}_i(x)$ as well as $\dot{B}_j(x)$ negative. However, even if the given condition on the constraint relation seems to be satisfied quite often, situations where the condition is not met require clearly further investigation.

IV. Controller Synthesis

In this section, some basic concepts associated with the controller design procedure and the controller implementation are presented. The basic premise of the controller design and implementation is to use the nominal controller inside the safe set and to design a complementary controller that renders the safe set robustly positively invariant. The nominal controller u_{nom} is assumed to be known. The complementary controller implementation seeks to minimize its effects on the nominal controller. The strategy is to use the function $B(x)$ as a decision variable for the activation of the complementary controller.

The first step is to design controllers, u_{inv_i} that guarantee the invariance of the sets $\mathcal{S}_i$. A good choice, derived from (8), could be the following

$$u_{inv_i} = \begin{cases} u_{nom}, & \text{if } u_{nom}, \text{ sat. (8)}\\ b(z)^{-1}(-a(z) - \Xi_\rho - \Psi_\rho(\bar{\theta})), & \text{o.w.}\end{cases}$$

This control law u_{inv_i} guarantees that

$$\dot{B}_i(x) = L_f B_i(x) + L_g B_i(x) u_{inv_i}(x) + L_p B_i(x) \le 0,$$

for all disturbances $\theta \in \Omega$. Note that the function u_{inv_i} is continuous since there exists a unique u that makes the condition (8) hold with an equality sign. To avoid discontinuities in the control law, the switching procedure is implemented using a sigmoidal switching surface as proposed in [15]. The applied control input is defined by

$$u = k(x) = \prod_{i=1}^{m}(1 - \sigma_i(x)) \cdot u_{nom} + \sum_{i=1}^{m} \sigma_i(x) u_{inv_i} \quad (11)$$

with

$$\sigma_i(x) = \begin{cases} 1, & -B_i(x) < 0\\ 1 - 2\left(\frac{B_i(x)}{\epsilon}\right)^3 - 3\left(\frac{B_i(x)}{\epsilon}\right)^2, & 0 \le -B_i(x) \le \epsilon\\ 0, & -B_i(x) > \epsilon\end{cases}$$

for some small constant $\epsilon > 0$. The control law u acts like a feed-through term for the nominal controller inside the set $\mathcal{S}$. If the systems' states come close to a given constraint, the proposed controller structure guarantees that the corresponding invariance controller is activated, ensuring that $\dot{B} \le 0$. Hence the switching controller affects the nominal controller as little as possible while guaranteeing constraint satisfaction for the system. As mentioned in the previous section, the sum of two invariance control laws guarantees constraint satisfaction for both constraints. The proposed structure is a suitable choice for a continuous controller that guarantees constraint satisfaction.

A crucial problem in systems with switching controls is the question of stability. One way to ensure complete integration of the constraints into a Lyapunov based control approach is to further impose the restriction that

$$\text{sign}\,\{L_g V(x)\} = \text{sign}\,\{L_g B_i(x)\}, \quad (12)$$

whenever $-\epsilon \le B(x) \le 0$. In this case the control actions of both controllers act in the same direction. More precisely, the invariance controller decreases the Lyapunov function and stability is guaranteed since $V(x)$ can be seen as a common Lyapunov function for the switching control law. This case is very convenient, since stabilization of the desired equilibrium is possible without violation of the constraint. This property may not be satisfied everywhere on the constraint or at all times, e.g. it may be violated when a reference signal to be tracked lays outside of the set $\mathcal{S}$. However, many situations may arise where the geometry of the constraint *vis-a-vis* the system dynamics is such that the sign-condition is violated (even in situations where the reference point lies inside the set $\mathcal{S}$). Such cases remain highly problematic because they single out areas on the boundary of the set $\mathcal{S}$ where constraint satisfaction and minimization of the Lyapunov function are in contradiction. Since this situation can lead to convergence of the closed-loop system on the boundary of $\mathcal{S}$, it is of interest to change the overall control objective to stabilization on or close to the constraint. Note that, as mentioned in the previous section, this is always possible for minimum phase constraints and undisturbed systems. In this case stability can be concluded using simple dwell-time considerations. Unfortunately, the quest for Lyapunov functions of the switch controller remains an open problem at this and is beyond the scope of the current study. It will be considered further in future work.

V. Simulation Example

In this section, the design procedure is illustrated on an Active Magnetic Bearing (AMB). An AMB is an apparatus that is used to bear a rotating mass between two electromagnets. It is advantageous for various reasons. It avoids friction losses due to the lack of contact between the rotor and the stator. It also provides active disturbance rejection.

A simplified one-dimensional AMB system, shown in figure 1 is considered in the following. A nominal model for the system is proposed in [13]. In this study, a time-varying disturbance is added to the nominal system yielding the following uncertain system representation

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= \epsilon x_3 + x_3|x_3| + \theta x_1 \sin(bt)\\ \dot{x}_3 &= u,\end{aligned} \tag{13}$$

where x_1 represents the position and x_2 the velocity of the mass. The third state x_3 is proportional to the magnetic flux. The system parameter $\epsilon \geq 0$ is in general smaller than one. In the following it is assumed that $\epsilon = 0.1$. The voltage $V_1 = V$ and $V_2 = -V$ is used as the control input u. A stabilizing controller for the nominal system is proposed in [13], i.e. $u_{nom} = -1.7538x_1 - 6.6957x_2 - 2.588x_3 - 3.1582x_3|x_3|$. The additional uncertainty influences the acceleration, i.e. it can be seen as an external force. In the disturbance term, the constant θ is known, but b is unknown. To avoid collisions with the magnet, the position x_1 has to be constrained such that $|x_1| \leq a$. The nominal controller cannot stabilize the disturbed problem adequately. Nevertheless one needs to guarantee that the mass does not come too close to the magnets. This should be done via a robust invariance control. The design starts with the upper constraint $x_1 < a$. For the nominal system and with respect to the fictive output $y = x_1 - a$ one can define a coordinate transformation $z_1 := x_1 - a,\ z_2 := x_2,\ z_3 := \epsilon x_3 + x_3|x_3|$. In the new coordinates the system can be written as

$$\begin{aligned}\dot{z}_1 &= z_2\\ \dot{z}_2 &= z_3 + \theta(z_1 + a)\sin bt\\ \dot{z}_3 &= (\epsilon + 2|x_3|)u.\end{aligned} \tag{14}$$

Note that the disturbance does not satisfy the matching condition but he triangularity condition is met. The backstepping procedure can now be applied to design a robustly invariant set and the corresponding controller. One starts with the z_1 subsystem and rewrites it as

$$\dot{z}_1 = -k_1 z_1 + [z_2 + k_1 z_1] = -k_1 z_1 + v_1.$$

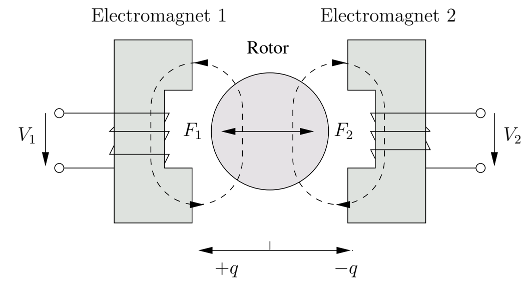

Fig. 1. Active Magnetic Bearing.

The second constraint is now given by $v_1 := z_2 + k_1 z_1 \leq 0$. The dynamics of the constraint is given by

$$\dot{v}_1 = k_1 z_2 + \theta(z_1 + a)\sin bt + z_3.$$

Therefore a third constraint can be defined by considering

$$\begin{aligned}&z_3 + k_2(z_2 + k_1 z_1) + k_1 z_2 + \theta(z_1 + a)\sin bt\\ &\leq z_3 + (k_2 + k_1)z_2 + k_2 k_1 z_1 + \theta|z_1| + \theta|a|\\ &\leq z_3 + (k_2 + k_1)z_2 + k_2 k_1 z_1 - \theta z_1 + \theta a,\end{aligned}$$

since $z_1 \leq 0$. Hence, the third relevant constraint is set as

$$v_2 := z_3 + (k_2 + k_1)z_2 + (k_2 k_1 - \theta)z_1 + \theta a \leq 0. \tag{15}$$

This is the ρ-th constraint and, by construction, provides a barrier certificate for the constrained system. Its derivative depends directly on the input, i.e.

$$\begin{aligned}\dot{v}_2 = &(\epsilon + 2|x_3|)u + (k_2 + k_1)z_3 + (k_2 k_1 - \theta)z_2\\ &+ (k_2 + k_1)\theta(z_1 + a)\sin bt.\end{aligned}$$

Since $(k_2 + k_1)\theta(z_1 + a)\sin bt \leq (k_2 + k_1)\theta|z_1 + a|$, any control law that satisfies the inequality

$$\begin{aligned}u \leq &\frac{1}{(\epsilon + 2|x_3|)}(-(k_2 + k_1)z_3 - (k_2 k_1 - \theta)z_2\\ &- (k_2 + k_1)\theta|z_1 + a|)\end{aligned}$$

is an invariance controller for the designed safe set $\mathcal{S} = \{x \in \mathbb{R}^n | z_1(x) \leq 0, v_1(x) \leq 0, v_2(x) \leq 0\}$.

The same design procedure can be applied to design a invariance controller that guarantees safety for the lower constraint $x_1 \geq -a$. Proceeding as above, the following new states are defined, i.e. $\xi_1 = -x_1 - a,\ \xi_2 = -x_2\ \xi_3 = -\epsilon x_3 - x_3|x_3|$. In these new coordinates a barrier certificate can be defined as $B_2 = (k_1 + k_2)\xi_2 + (k_1 k_2 - \theta)\xi_1 + \xi_3 + \theta a$, and a corresponding invariance control law has to satisfy

$$\begin{aligned}u \geq &-\frac{1}{(\epsilon + 2|x_3|)}(-(k_1 k_2 - \theta)\xi_2\\ &+ (k_1 + k_2)\xi_3 + (k_1 + k_2)\theta| - \xi_1 - a|).\end{aligned}$$

The control law is implemented as proposed in section IV. For both constraints an invariance control law u_{inv} and a switching surface σ are designed as proposed previously. The control input u is synthesized in the form (11).

Simulation results for the given problem with $a = 1$, $b = 0.2$

and $\theta = 0.3$ are shown In figure 2. The nominal controller fails to stabilize the disturbed system in this case. It has to be redesigned. Nevertheless the invariance control guarantees that the mass does not hit the magnet, while this would clearly happen without the invariance control. In figure 3 the time trajectories of the three constraint variables z_1, v_1, v_2 are shown. All three constraints are kept below zero at all times, hence the overall constraint is satisfied. The constraint v_2 (solid line) would be the first to become positive, which is avoided by the invariance controller.

VI. Conclusion

An algorithm to construct control laws for uncertain nonlinear systems under state constraints is proposed. In a recursive design procedure, a set of safe initial conditions is constructed that is robustly controlled invariant. The design leaves some degree of freedom that can be used to adapt the procedure to specific requirements. A switching control is designed to guarantee positive invariance of the set and to satisfy a nominal control objective whenever possible. The proposed method allows one to consider constraints on single states as well as on combinations of states and can handle multiple constraints. The applicability of the proposed procedure is illustrated on a design example. It is shown that the ideas are easily applicable to constraints with a high relative degree.

Even if the current work focuses primarily on single input systems, a generalization to multi-input systems can be easily establshed at this point. An open problem that remains is with regards to the simultaneous satisfaction of state and input constraints. This question is connected to the problem of maximizing the set $\mathcal{S}$. Furthermore it is of interest to focus on robust stabilization of the system on the constraint if the nominal control objective and constraint satisfaction contradict each other.

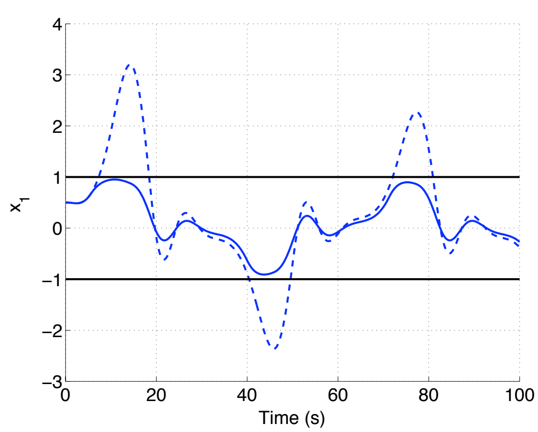

Fig. 2. Position x_1 of the uncertain Active Magnetic Bearing System for the initial condition $x(0) = [0.5, 0, 0]$ with (solid) and without (dotted) the invariance control. The systems constraints are indicated by the solid horizontal lines.

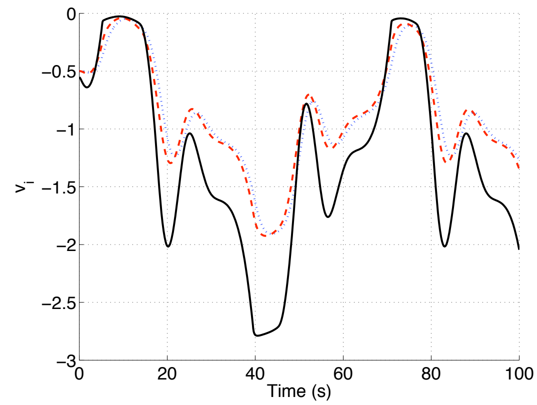

Fig. 3. Constraint variables z_1 (dotted), v_1 (dashed), and v_2 (solid) for the upper constraint of the uncertain Active Magnetic Bearing System for the initial condition $x(0) = [0.5, 0, 0]$ and the safety control law.

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