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# Finite-time Control of Cross-chained Nonholomic Systems by Switched State Feedback

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*Abstract*— **This paper is concerned with the control problem of a class of nonholonomic systems having *cross-chained* structure. Such systems are structurally incompatible with the chained systems, so the conventional methods proposed for chained systems are not valid any more and an entirely new control approach is required. In this paper, we propose a switched state feedback law which delivers the initial state to the origin *in finite time using bounded control inputs*, without infinitely high gain and frequent switchings in spite of its discontinuity. The effectiveness of the proposed method is shown by numerical simulations. Possible mechanical applications of this study include snake robots, rolling sphere problem and attitude control of free-flying robots.**

## I. INTRODUCTION

Nonholonomic systems have been providing challenging topics to nonlinear control theory since early 90's. One of the most important class of nonholonomic systems is *driftless systems* described by nonlinear state equation without drift vector-fields, which represents kinematic mechanical systems with non-integrable velocity constraints. Owing to the well-known necessary condition for asymptotic stabilizability indicated by Brockett[3], it is impossible to asymptotically stabilize a driftless system by any continuous state feedback, even though its reachability is guaranteed by Chow's rank condition[6]. Now there remain two ways to get rid of this restriction: one is to abandon pure state feedback, by adopting feedforward or time-dependent components with the aid of off-line path generation technique. The other is to give up continuity by using discontinuous or switched terms in the feedback law.

Among several subclasses of driftless systems, *chained systems* is the one that has been most actively investigated, followed by many successful results such as sinusoidal path generation[12], discontinuous state feedback ([1], [11]), time-state control[14], time-varying feedback control[15] and mixed-up approaches([13]). The clue for this success was the simplicity of its structure, i.e., existence of the single *generator* vector-field, along which the controllability is ensured in the sense of linear approximation. Through these intensive and thorough studies, the key concept for controlling chained systems has been well established. Typical time-dependent feedback approach consists of periodic excitation of the generator and continuous feedback[15], while the discontinuous feedback approach consists of monotonic decreasing of the generator combined with high-gain feedback[1].

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However, there are many stimulating examples of nonholonomic systems that are structurally incompatible with chained systems. *Multi-generator systems*contains two or more generator vector-fields in the basis of its controllability Lie algebra, so the conventional control approaches, based on single generator structure, are not sufficient any more. A relatively easy subclass in this category is the *first-order controllable systems*, for which some satisfactory feedback control results have been proposed ([2], [10]).

In this paper, we step further to deal with *higher-order* multi-generator systems. The simplest example of such systems is found in the case of 2 inputs and 5 states with second-order controllability structure, which we call the *cross-chained* system in this paper. This system is not only stimulating from theoretical viewpoint, but also includes interesting physical applications, such as rolling sphere control[7], 3-link snake robot[9], double trailer system with off-axle hitch[17], attitude control of free-flying robot with two actuators[16]. So far, the quest for definitive control method for this system is still on the way; the author proposed a control algorithm based on generator-switching and extended time-state control form [8], but its convergence analysis was not sufficient. Casagrande et al. ([4], [5]) proposed a switching control algorithm with precise Lyapunov-based stability analysis, though it was not sufficiently free from frequent switchings.

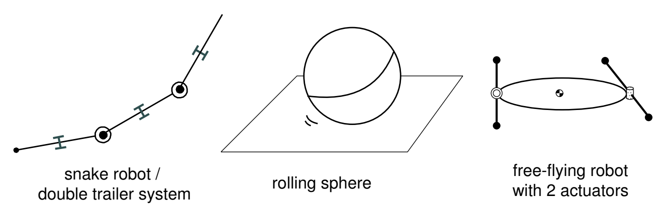


Fig. 1. Applications of cross-chained systems control

This paper presents an alternative method to this problem, which delivers the initial state to the origin *in finite time using bounded control inputs*, without infinitely high gain and frequent switchings in spite of its discontinuity. The key technique includes (i) a careful choice of coordinate transformation, which is parameterized by twisted loops filling up $\mathbb{R}^3$, and (ii) sliding-mode type switching rule which makes the predetermined subsets attractive and invariant.

The rest of this paper is organized as follows. We begin with the definition of the cross-chained system and fundamental analysis of its structure and controllability in section II. The construction of the proposed method is described

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in section III. The effectiveness of the proposed method is examined by numerical simulation in section IV. Section V concludes the results.

## II. System with Cross-Chained Structure

### A. State equation

Consider a driftless system of the following form:

$$\dot{\boldsymbol{x}} = h_1(\boldsymbol{x})u_1 + h_2(\boldsymbol{x})u_2 \tag{1}$$

$$h_1(\boldsymbol{x}) := \begin{pmatrix} 1 \\ 0 \\ -x_2 \\ -x_3 - x_1x_2 \\ -x_2^2 \end{pmatrix}, \; h_2(\boldsymbol{x}) := \begin{pmatrix} 0 \\ 1 \\ x_1 \\ x_1^2 \\ -x_3 + x_1x_2 \end{pmatrix}$$

In this paper, we call this a *cross-chained system*.

Controllability of this system is systematically checked as follows. Let $[\cdot,\cdot]$ denote the Lie bracket of smooth vector-fields, then

$$h_3(\boldsymbol{x}) := [h_1,h_2](\boldsymbol{x}) = \left(\, 0,\, 0,\, 2,\, 4x_1,\, 4x_2 \,\right)^T \tag{2}$$

$$h_4(\boldsymbol{x}) := [h_1,h_3](\boldsymbol{x}) = \left(\, 0,\, 0,\, 0,\, 6,\, 0 \,\right)^T \tag{3}$$

$$h_5(\boldsymbol{x}) := [h_2,h_3](\boldsymbol{x}) = \left(\, 0,\, 0,\, 0,\, 0,\, 6 \,\right)^T \tag{4}$$

and the system satisfies the Lie Algebra Rank Condition (LARC) by considering

$$C^\infty \operatorname{span}\{h_1,h_2,h_3,h_4,h_5\}(\boldsymbol{x}) = \mathbb{R}^n \tag{5}$$

Note that this system is the *simplest* example of the non-chained and higher-order structure. This is the reason why it is worth exploiting as a basic platform for this class of systems.

*Remark 1 (Virtual first-order form):* Suppose the right hand side of $\dot{x}_3 = -x_2u_1 + x_1u_2$ is regarded as a *virtual input*, say $u_3$. Then the system (1) can be written as

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -x_3 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -x_3 \end{pmatrix} u_2 + \begin{pmatrix} 0 \\ 0 \\ 1 \\ x_1 \\ x_2 \end{pmatrix} u_3 \tag{6}$$

$$u_3 = -x_2u_1 + x_1u_2 \tag{7}$$

which resembles a *first-order controllable* system with three inputs[2]. It is interesting to see the derivative equations

$$dx_3 = -x_2dx_1 + x_1dx_2 = \int dx_1 \wedge dx_2 \tag{8}$$

$$dx_4 = -x_3dx_1 + x_1dx_3 = \int dx_1 \wedge dx_3 \tag{9}$$

$$dx_5 = -x_3dx_2 + x_2dx_3 = \int dx_2 \wedge dx_3 \tag{10}$$

which simply exhibit the principle of *holonomy* (or area rule). For instance, the displacement of $x_4$ is proportional to the area encircled by the closed trajectory projected on $x_1$-$x_3$ plane. This observation gives us a clear view of the symmetry of its controllability structure, as well as a motivation the coordinate transformation given in the next section. •

## III. Switched State Feedback Controller

In this section, we propose a switched state feedback controller for system (1) which delivers the initial state to the origin in finite time using bounded control inputs. An underlying idea of the proposed method is a discontinuous coordinate transformation motivated by the last remark. The principle of holonomy tells us, from eq. (8), that a circular loop on $x_1$-$x_2$ plane yields displacement in $x_3$ which is parallel to $[h_1,h_2]$ indeed. Similarly, from eq. (9), we may expect that a circular loop on $x_1$-$x_3$ plane yields displacement in $x_4$ which is parallel to the second-order Lie bracket $[h_1,[h_1,h_2]]$. In order that the system trajectory draws a circular loop indeed on $x_1$-$x_3$ plane, it should follow the corresponding figure-8-shaped loop on $x_1$-$x_2$ plane.

The following argument is basically a realization of this idea by means of coordinate transformation. Roughly speaking, the new coordinates are composed of the specification parameters for the family of figure-8 loops $(r,d)$, the path parameter $(\phi)$ and a pair of integral invariants along the loop $(z_1,z_2)$.

### A. Discontinuous coordinate transformation

In the rest of the paper, we adopt the following compact notations for $k \in \mathbb{N}$ and $\phi \in \mathbb{S}$:

$$C := \cos\phi, \quad C_k := \cos(k\phi) \tag{11}$$

$$S := \sin\phi, \quad S_k := \sin(k\phi) \tag{12}$$

where $\mathbb{S}$ denotes the unit circle, which is isomorphic to $\mathbb{R}/2\pi\mathbb{Z}$ (i.e., the interval $[-\pi,\pi]$ with $\pm\pi$ identified).

Suppose a family of continuous closed curves on the $x_1-x_2$ plane characterized by: Thus the parameterization above is simply rewritten as

$$x_1 = rC \tag{13}$$

$$x_2 = \alpha r S_2 + d. \tag{14}$$

where $r \in \mathbb{R}_+, d \in \mathbb{R}, \phi \in \mathbb{S}$ are all scalar parameters and $\alpha \in \mathbb{R}_+$ is a constant. As shown in Fig. 2, $r$ implies the size, $d$ implies the vertical displacement, and $\phi$ implies the path parameter of the figure-8 shaped loop on $x_1$-$x_2$ plane.

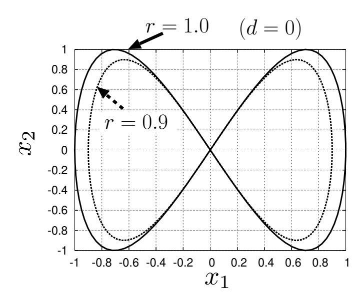


Fig. 2. Parameterization of $x_1$-$x_2$ plane by figure-8 loops ($\alpha = 1$)

Next, let us assign $x_3$ the following parameterization

$$\begin{aligned} x_3 &= \int_0^\phi \left( -x_2\frac{\partial x_1}{\partial \phi} + x_1\frac{\partial x_2}{\partial \phi} \right) d\phi \\ &= \alpha r^2 \left( \frac{1}{6}S_3 + \frac{3}{2}S \right) - rdC, \end{aligned} \tag{15}$$

then (13)-(15) represent a family of closed spatial curves parameterized by $(r, d, \phi)$ as illustrated in Fig. 3. Now let us

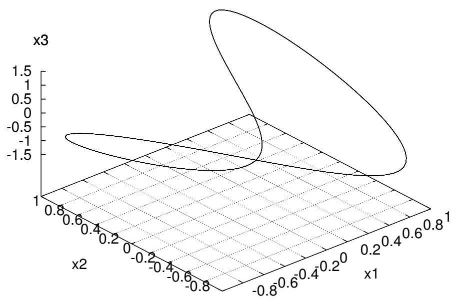


Fig. 3. Closed spatial curve in $x_1$-$x_2$-$x_3$ space ($r = \alpha = 1, d = 0$)

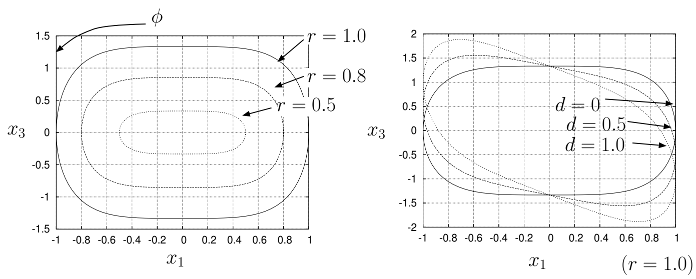


Fig. 4. Projection of the loops onto $x_1$-$x_3$ plane

introduce a discontinuous coordinate transformation defined by (13)-(15) together with the following relation

$$\begin{aligned} x_4 &= z_1 + \int_0^{\phi} \left( -x_3 \frac{\partial x_1}{\partial \phi} + x_1 \frac{\partial x_3}{\partial \phi} \right) d\phi \\ &= z_1 + \alpha r^3 \left( \frac{1}{24} S_4 + \frac{1}{6} S_2 + \frac{3}{2} \phi \right) \end{aligned} \tag{16}$$

$$\begin{aligned} x_5 &= z_2 + \int_0^{\phi} \left( -x_3 \frac{\partial x_2}{\partial \phi} + x_2 \frac{\partial x_3}{\partial \phi} \right) d\phi \\ &= z_2 + \alpha^2 r^3 \left( -\frac{1}{60} C_5 + \frac{1}{4} C_3 - \frac{11}{6} C + \frac{8}{5} \right) \\ &\qquad + \alpha r^2 d \left( \frac{1}{3} S_3 + 3S \right) - r d^2 C \end{aligned} \tag{17}$$

Then

$$\boldsymbol{\xi} := (r, d, \phi, z_1, z_2)^T \in M \tag{18}$$

$$M := \mathbb{R}^3 \times \mathbb{R}_+ \times \mathbb{S}^1 \tag{19}$$

defines the new coordinate and the new state space instead of $\boldsymbol{x} \in \mathbb{R}^5$. For reference, we also denote $\boldsymbol{q} := (x_1, x_2)^T \in \mathbb{R}^2$ and $\boldsymbol{z} := (z_1, z_2)^T \in \mathbb{R}^2$.

*Lemma 1:* Let

$$D := \{\boldsymbol{x} \in \mathbb{R}^5 | x_1^2 + x_3^2 = 0,\ x_4^2 + x_5^2 \neq 0\}. \tag{20}$$

If $\boldsymbol{x} \notin D$, the inverse transformation ($\boldsymbol{x} \to \boldsymbol{\xi}$) of (13)-(17) is given as follows.

1) If $x_1^2 + x_3^2 \neq 0$, let $r$ be the only positive real root of the polynomial

$$P(s) := s^6 + 3x_1^2 s^4 - \zeta^2 s^2 - 4x_1^6 \tag{21}$$

where $\zeta := \dfrac{3(x_3 + x_1 x_2)}{4\alpha}$. With $r$ fixed, the rest of the coordinates are given by:

$$\phi = \arctan\left( \frac{x_1}{r},\ \frac{\zeta}{r^2 + 2x_1^2} \right) \tag{22}$$

$$d = x_2 - r\alpha S_2 \tag{23}$$

$$z_1 = x_4 - \alpha r^3 \left( \frac{1}{24} S_4 + \frac{1}{6} S_2 + \frac{3}{2} \phi \right) \tag{24}$$

$$\begin{aligned} z_2 &= x_5 - \alpha^2 r^3 \left( -\frac{1}{60} C_5 + \frac{1}{4} C_3 - \frac{11}{6} C + \frac{8}{5} \right) \\ &\quad + d(2x_3 + dx_1) \end{aligned} \tag{25}$$

where $\arctan(x, y)$ implies the unique solution $\theta$ for $y \cos\theta = x \sin\theta$.

2) If $x_1^2 + x_3^2 = 0$, let $r = 0, d = x_2, z_1 = z_2 = 0$. $\phi \in \mathbb{S}$ is indefinite (can be set arbitrary).

•

*Proof:* This is verified by intricate but straightforward computation. ■

### B. Calculus in the new coordinates.

We rewrite the state equation (1) in the new coordinates.

*Lemma 2:* System (1) is convertible with

$$\begin{cases} \dot{r} &= C\mu_1 \\ \dot{d} &= \alpha \left( \frac{1}{6} S_3 + \frac{3}{2} S \right) \mu_1 \\ \dot{\phi} &= \mu_2 \\ \dot{z_1} &= -\alpha r^3 C \left( \frac{3}{24} S_4 + \frac{1}{2} S_2 + \frac{3}{2} \right) \mu_1 + \frac{3}{2} \alpha r^3 \mu_2 \\ \dot{z_2} &= -\left( \frac{1}{6} S_3 + \frac{3}{2} S \right) \left( 3\alpha r^2 \left( \frac{1}{6} S_3 + \frac{3}{2} S \right) - rdC \right) \mu_1 \end{cases} \tag{26}$$

under the coordinate transformation (13)-(17) and a feedback transformation

$$\begin{cases} u_1 &= C^2 \mu_1 - rS\,\mu_2 \\ u_2 &= \alpha \left( \frac{2}{3} S_3 + 2S \right) \mu_1 + 2r\alpha C_2\, \mu_2 \end{cases} \tag{27}$$

•

*Proof:* The time derivatives of $x_1, x_2, x_3$ are given by

$$\dot{x_1} = u_1 = C\,\dot{r} - rS\,\dot{\phi} \tag{28}$$

$$\dot{x_2} = u_2 = \alpha S_2\,\dot{r} + 2r\alpha C_2\,\dot{\phi} + \dot{q}_2 \tag{29}$$

$$\begin{aligned} \dot{x_3} &= \left( \frac{2x_3}{r} + dC \right) \dot{r} + \left( \frac{\alpha r^2}{2} (C_3 + 3C) + rdS \right) \dot{\phi} \\ &\quad - rC\,\dot{d} \end{aligned} \tag{30}$$

Substituting (7) into (30), we have

$$r x_1 \dot{d} = (x_3 + dx_1)\dot{r}$$

or more simply,

$$C\,\dot{d} = \alpha \left( \frac{1}{6} S_3 + \frac{3}{2} S \right) \dot{r} \tag{31}$$

which implies that $r$ and $d$ are kinematically related by this equation. Now the time derivatives of $r, d, \phi$ are given in the form of

$$\begin{cases} \dot{r} = C\mu_1 \\ \dot{d} = \alpha\left(\frac{1}{6}S_3 + \frac{3}{2}S\right)\mu_1 \\ \dot{\phi} = \mu_2 \end{cases} \quad (32)$$

$\mu_1$ can be considered the *radial* velocity which intersects the closed curve in Fig. 4, while $\mu_2$ implies the *tangential* velocity which lies along the curve. Similarly, with (24)(25) and (27),

$$\begin{aligned} \dot{z_1} &= \left(-x_3C + \alpha r^2\left(-\frac{1}{24}S_4 + \frac{1}{3}S_2 - \frac{9}{2}\phi\right)\right)\dot{r} \\ &= -\alpha r^3 C\left(\frac{3}{24}S_4 + \frac{1}{2}S_2 + \frac{3}{2}\right)\mu_1 + \frac{3}{2}\alpha r^3\,\mu_2 \\ \dot{z_2} &= -(3x_3^2 + 5x_1dx_3 + 2x_1^2d^2)\frac{\mu_1}{r^2} \\ &= -\left(\frac{1}{6}S_3 + \frac{3}{2}S\right)\left(3\alpha r^2\left(\frac{1}{6}S_3 + \frac{3}{2}S\right) - rdC\right)\mu_1 \end{aligned} \quad (33)$$

■

### C. Singularity and pre-rotation of the coordinates

We should note that the proposed coordinate transformation is *discontinuous* when $x_1^2 = x_3^2 = 0$. However, it is still possible to avoid the discontinuity if $x_2$ is not equal to zero; let $\theta$ be a constant parameter initialized as

$$\theta := \arctan(x_1(0), x_2(0)) \quad (34)$$

and $\mathrm{Rot}_\theta$ be the rotation matrix

$$\mathrm{Rot}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad (35)$$

Perform the following invertible transformation for coordinates and inputs:

$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} := \mathrm{Rot}_\theta^{-1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (36)$$

$$\bar{x}_3 := x_3 \quad (37)$$

$$\begin{pmatrix} \bar{x}_4 \\ \bar{x}_5 \end{pmatrix} := \mathrm{Rot}_\theta^{-1}\begin{pmatrix} x_4 \\ x_5 \end{pmatrix} \quad (38)$$

$$\bar{u} := \mathrm{Rot}_\theta^{-1}u \quad (39)$$

The system dynamics (1) is invariant under this transformation, i.e.:

$$\frac{d}{dt}\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ \bar{x}_4 \\ \bar{x}_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -\bar{x}_2 \\ -\bar{x}_3 - \bar{x}_1\bar{x}_2 \\ -\bar{x}_2^2 \end{pmatrix}\bar{u}_1 + \begin{pmatrix} 0 \\ 1 \\ \bar{x}_1 \\ \bar{x}_1^2 \\ -\bar{x}_3 + \bar{x}_1\bar{x}_2 \end{pmatrix}\bar{u}_2$$

We see that $\bar{x}_2(0) = 0$ and $\bar{x}_1(0) = \sqrt{x_1(0)^2 + x_2(0)^2}$, therefore $\bar{x}_1(0) = 0$ if and only if $x_1(0) = x_2(0) = 0$. With the aid of this pre-transformation, in practice, the set of singularity reduces to $\{\boldsymbol{x} \in \mathbb{R}^5 \mid x_1^2 + x_2^2 + x_3^2 = 0\}$.

### D. Switched State Feedback Law

Now we are ready to consider to control the system (26) using the new inputs $\mu_1, \mu_2$. To begin with, we check the following fundamental properties. Let $D_1 := \{\phi = 0, \pi\}$, $D_2 := \{\phi = \pm\frac{1}{2}\pi\}$ be subsets of $\mathbb{S}$ (Fig. 5). Note that $S = S_2 = S_3 = 0$ on $D_1$ and $C = C_3 = 0$, $C_2 = \pm 1$ on $D_2$.

*Lemma 3:* The following facts hold.

- If $\phi \in D_1$,

$$\begin{cases} \dot{r} = C\mu_1 & \dot{z_1} = \frac{3}{2}\alpha r^2\mu_1 \\ \dot{d} = 0 & \dot{z_2} = 0 \\ \dot{\phi} = \mu_2 \end{cases} \quad (40)$$

- If $\phi \in D_2$,

$$\begin{cases} \dot{r} = 0 & \dot{z_1} = \frac{3}{2}\alpha r^2\mu_2 \\ \dot{d} = \frac{4\alpha}{3}\mu_1 & \dot{z_2} = -\frac{16}{3}\alpha r^2\mu_1 \\ \dot{\phi} = \mu_2 \end{cases} \quad (41)$$

- For $^\forall\phi \in \mathbb{S}$ and $\mu_1 = 0$,

$$\begin{cases} \dot{r} = 0 & \dot{z_1} = \frac{3}{2}\alpha r^2\mu_2 \\ \dot{d} = 0 & \dot{z_2} = 0 \\ \dot{\phi} = \mu_2 \end{cases} \quad (42)$$

*Proof:* Just evaluate (26) under the given conditions. ■

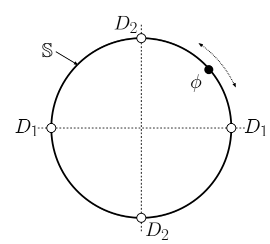


Fig. 5. Switching points on $\mathbb{S}$

*Remark 2:* In order to extend our free choice in controller design, let us introduce a binary design parameter

$$\gamma \in \{0, \pi\} \subset \mathbb{S}, \quad (43)$$

which specifies the *final reference of* $\phi$. $\gamma$ can be chosen arbitrarily from $\{0, \pi\}$. In case of $\gamma = \pi$, we have to slightly modify the coordinate transformation as follows:

$$z_1 := x_4 - \alpha r^3\left(\frac{1}{24}S_4 + \frac{1}{6}S_2 + \frac{3}{2}(\phi + \pi)\right) \quad (44)$$

•

Basic idea of the proposed method is to use the control $\mu_1 \neq 0$ only at $\phi \in D_1 \cup D_2$ to adjust $z_1, z_2$ to zero, while $\mu_1 = 0$ is kept otherwise. On the other hand, $\phi$ is moved towards the final value $\gamma \in D_1$ after passing once through the other end of $D_1$.

The feedback law will be described in a switch-case style using the following series of controlled-invariant sets:

$$\sigma_0 := \{\mathbf{0}\} \quad (45)$$

$$\sigma_1 := \{\boldsymbol{\xi} \in M | r = 0, \boldsymbol{z} = 0\} \quad (46)$$

$$\sigma_2 := \{\boldsymbol{\xi} \in M | \boldsymbol{z} = 0, \phi = \gamma\} \quad (47)$$

$$\sigma_3 := M \quad (48)$$

$\sigma_0$ is the origin to go for. On $\sigma_1$, only $d$ remains to be controlled. On $\sigma_2$, both $\phi$ and $\boldsymbol{z}$ are at their desired values simultaneously. $\sigma_3$ is the set of all $\boldsymbol{\xi}$, which corresponds to the original state space without the singularity set $D$. The following inclusion holds among these subsets.

$$\sigma_0 \subset \sigma_1 \subset \sigma_2 \subset \sigma_3 \quad (49)$$

**[SWITCHED STATE FEEDBACK CONTROL LAW]**

**Case 4:** If $\boldsymbol{x} \in D$ [Exit from the singularity]:
Apply a constant input $u_1 \neq 0, u_2 = 0$ to get out of $D$. Use the pre-rotation of the coordinates as in Sec. III-C if necessary.

**Case 3:** If $\boldsymbol{\xi} \in \sigma_3 \setminus \sigma_2$:
This is the main part of the control and divided into the following sub-cases.

- If $\phi \in D_1$, $z_1 \neq 0$:
  let $\mu_1 = -\operatorname{sgn}(z_1)$ and $\mu_2$ to ensure $z_1 \to 0$.
- If $\phi \in D_2$, $z_2 \neq 0$ :
  let $\mu_1 = \operatorname{sgn}(z_2)$ and $\mu_2$ to ensure $z_2 \to 0$.
- Otherwise:
  let $\mu_1 = 0$ and

$$\mu_2 = \begin{cases} -\operatorname{sgn}(\phi - \gamma) & \text{if } z_1 = 0 \\ \operatorname{sgn}(\phi - \gamma) & \text{if } z_1 \neq 0 \end{cases} \quad (50)$$

**Case 2:** If $\boldsymbol{\xi} \in \sigma_2 \setminus \sigma_1$:
let $\mu_1 = -\operatorname{sgn}(C)$ and $\mu_2 = 0$ to ensure $r \to 0$.

**Case 1:** If $\boldsymbol{\xi} \in \sigma_1 \setminus \sigma_0$:
let $\mu_1 = -\operatorname{sgn}(d)$, $\mu_2 = 0$ to ensure $d \to 0$.

**Case 0:** If $\boldsymbol{\xi} \in \sigma_0$: terminate. •

Whilst we omit the detailed convergence analysis due to lack of space, it is almost obvious that the number of switchings is finite, the control inputs are bounded and the reaching time is also finite since $\mu_1$ and $\mu_2$ belong to $\{0, \pm 1\}$.

## IV. SIMULATION RESULTS

Simulation results of the proposed method is shown in Fig. 6-9. The given initial state of (1) is

$$\boldsymbol{x}(0) = (-1.5, -0.1, -2.55, -1.58, 3.12)^T. \quad (51)$$

Aspect ratio of the figure-8 loop is $\alpha = 1.0$, and the final value of $\phi$ is set as $\gamma = \pi$.

Fig.6 shows the trajectory of $\boldsymbol{x}$ projected on $x_1$-$x_2$ and $x_1$-$x_3$ plane. The time response in terms of the new coordinates are shown in Fig. 7 and 8. The initial state starts from $\sigma_3$ (Case 3). Since $\gamma = \pi$, $\phi$ goes towards 0 at the beginning and gets to $D_1$ at around $t = 0.4$, followed by $z_1$ reaches 0 at around $t = 1.0$. Next, $\phi$ turns to head for $\gamma = -\pi (\equiv \pi \bmod 2\pi)$, passing through $D_2(\phi = -\frac{\pi}{2})$ at around $t = 2.5$. Then $z_2$ starts to decrease and reaches 0 at around $t = 4.4$. After that, $\phi$ goes on and reaches $\gamma = -\pi$ at around $t = 6.0$. Now the $z_1 = z_2 = 0$ and $\phi = \gamma$ are achieved, the Case 2 is selected; $r = 0$ is achieved at around $t = 6.9$. Finally, Case 1 is selected to make $d \to 0$, terminating at $t = 8.7$. The corresponding control inputs $\mu$ are shown in Fig. 9.

## V. CONCLUSION

In this paper, we dealt with the control problem of non-holonomic drift-free systems with cross chained structure. We proposed a switched feedback state controller which delivers the initial state to the origin in finite time using bounded control inputs, without (infinitely) high gain and frequent switchings. Here we re-emphasize that the proposed method is not a control *procedure*, but a *feedback law* in the sense that the control input is statically and uniquely assigned to each point in the state space.

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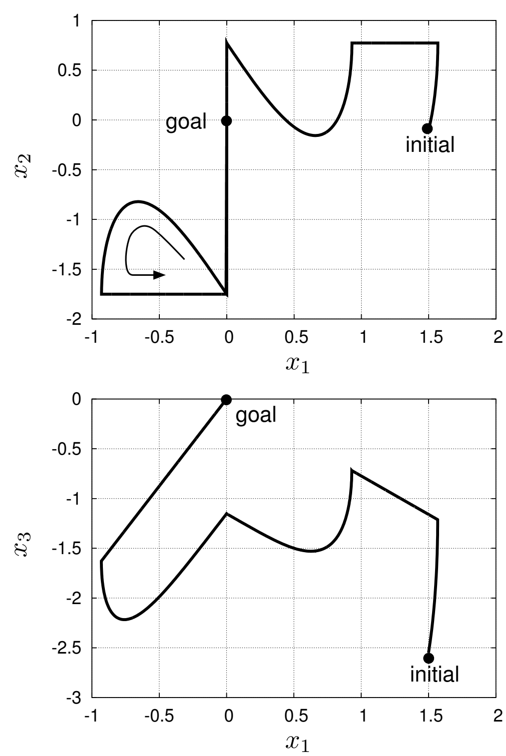


Fig. 6. Projections of the locus onto $x_1$-$x_2$ and $x_1$-$x_3$ planes

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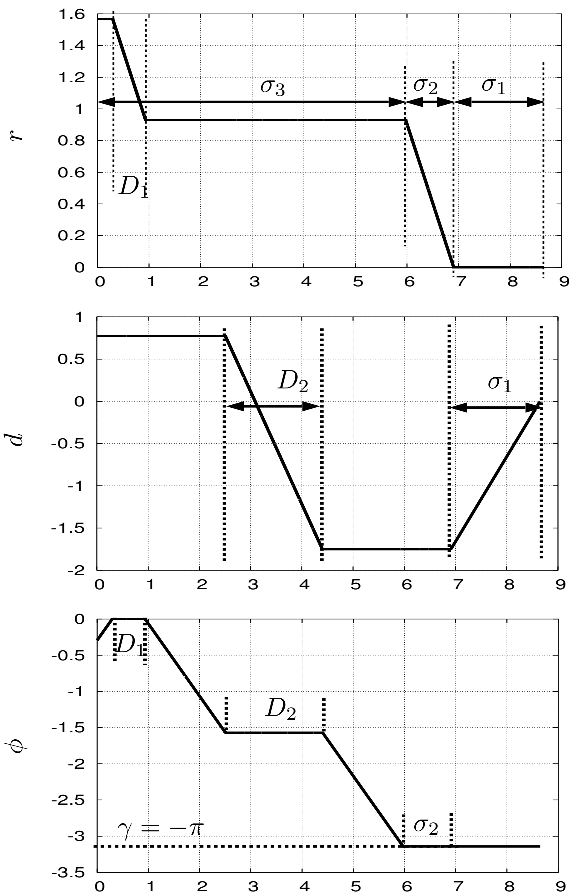


Fig. 7. Time response of $r$, $d$ and $\phi$

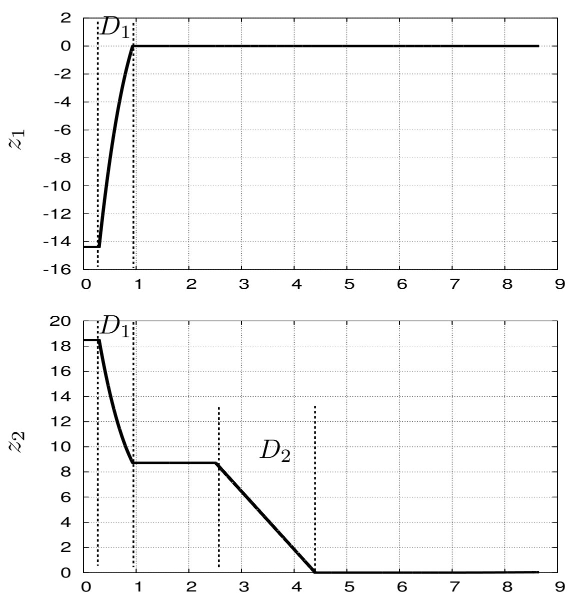


Fig. 8. Time response of $z_1$ and $z_2$

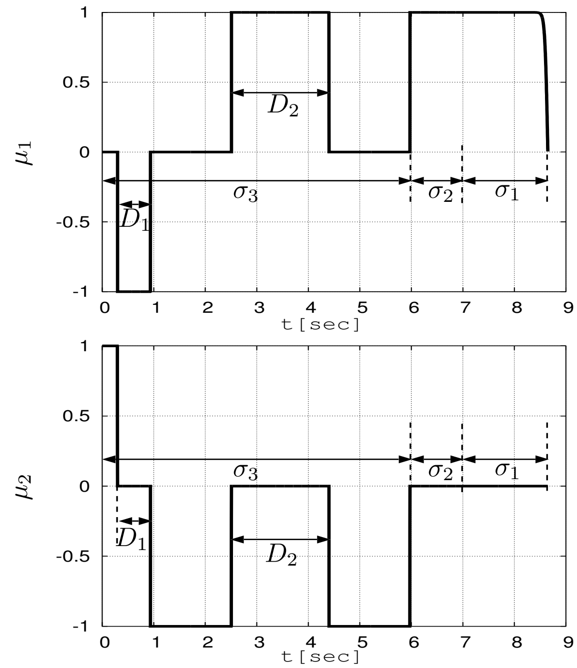


Fig. 9. New control inputs $\mu_1, \mu_2$

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