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Proportional-Integral observer design for nonlinear uncertain systems modelled by a multiple model approach

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Abstract— **In this paper, a *decoupled multiple model* approach is used in order to cope with the state estimation of uncertain nonlinear systems. The proposed decoupled multiple model provides flexibility in the modelling stage because the dimension of the submodels can be different and this constitutes the main difference with respect to the classically used multiple model scheme. The state estimation is performed using a Proportional Integral Observer (PIO) which is well known for its robustness properties with respect to uncertainties and perturbations. The Lyapunov second method is employed in order to provide sufficient existence conditions of the observer, in LMI terms, and to compute the optimal gains of the PIO. The effectiveness of the proposed methodology is illustrated by a simulation example.**

I. INTRODUCTION

In many real world engineering applications, the knowledge of the system state is often required not only for control purpose but also for monitoring and fault diagnosis. In practice however, the measurements of the system state can be very difficult or even impossible, for example when an appropriate sensor is not available or economically viable. Model-based state estimation is a largely adopted strategy used in order to cope with this important problem. Typically a state estimation is provided by means of an observer whose inputs are the inputs and the outputs of the system and the outputs are the estimated states. Note that the structure of an observer is based on the mathematical model of the considered system. Therefore, the accuracy of the state estimation depends on the accuracy of the used mathematical model and the quality of the employed measurements. However, a mathematical model is an abstract representation of the real world and it only provides an approximation to dynamic behaviours of the actual system. Consequently, modelling errors between the system and its model are unavoidable. Besides, the employed measurements are also affected by external disturbances due to the interactions between the system and its environment.

Hence, many efforts have been made in the past two decades to improve robustness of the state estimation of linear systems affected by disturbances and parametric uncertainties (e.g. in [1]–[3] norm-bounded uncertainties are considered). However, dynamic behaviour of most of real systems is nonlinear and consequently a linear model is not able to provide a good characterisation of the system in the whole operating range. On the other hand, the observer design problem for generic nonlinear models is delicate and so far this problem remains unsolved in a general way.

Multiple model approach is an appropriate tool for modelling complex systems using a mathematical model which can be used for analysis, controller and observer design. The basis of the multiple model approach is the decomposition of the operating space of the system into a finite number of *operating zones*. Hence, the dynamic behaviour of the system inside each operating zone can be modelled using a simple *submodel*, for example a linear model. The relative contribution of each submodel is quantified with the help of a *weighting function*. Finally, the approximation of the system behaviour is performed by associating the submodels and by taking into consideration their respective contributions. Note that a large class of nonlinear systems can accurately be modelled using multiple models.

The choice of the structure used to associate the submodels constitutes a key point in the multiple modelling framework. Indeed, the submodels can be aggregated using various structures [4]. Classically, the association of submodels is performed in the dynamic equation of the multiple model using a common state vector. This model, known as *Takagi-Sugeno multiple model*, has been initially proposed, in a fuzzy modelling framework, by Takagi and Sugeno [5] and in a multiple model modelling framework by Johansen and Foss [6]. This model has been largely considered for analysis, modelling, control and state estimation of nonlinear systems (see among others [7]–[9] and references therein).

In this paper, an other possible way for building a multiple model is employed. The used model, known as *decoupled multiple model*, has been suggested in [4] and results of the association of submodels only in the output equation of the multiple model. Note also that this multiple model has been successfully employed in modelling [10], [11], control [12]–[14] and state estimation [15], [16] of nonlinear systems. The main feature of the decoupled multiple model is that submodels of different dimensions (e.g. number of states) can be used. This fact introduces some flexibility degrees in the modelling stage in particular when the model is obtained using a black box modelling strategy. Indeed, the dimensions of the submodels can be well adapted to each operating zone and consequently the total number of parameters necessary for describing the system can be reduced.

This paper deals with the design of a Proportional-Integral Observer (PIO) for a class of nonlinear systems modelled by a decoupled multiple model with parameter uncertainties.

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The parameter uncertainties are assumed to be unknown, time-varying and norm bounded. With respect to the classic proportional observer, the PIO offers more additional degrees of freedom which can be used for improving its robustness properties with respect to perturbations and imperfections in the model (details are given in section III). The PIO design problem consists in finding the gains of the observer such that the state estimation error converges toward zero or at least remains globally bounded for all admissible uncertainties and perturbations. Furthermore, the PIO design based on the multiple model representation does not seem to be reported previously to the best authors' knowledge.

The outline of this paper is as follows. Discussion about decoupled multiple model is proposed in section II. In section III, the PIO design problem is investigated and the gains of the observer are obtained by LMI optimization. Finally, in section IV, a simulation example illustrates the state estimation of a decoupled multiple model.

II. On the decoupled multiple model representation

In this paper, an uncertain nonlinear system described by a decoupled multiple model is considered. The state space representation of this multiple model is given by:

$$\dot{x}_i(t) = (A_i + \Delta A_i)x_i(t) + (B_i + \Delta B_i)u(t) + D_i w(t) \quad , \text{(1a)}$$

$$y_i(t) = C_i x_i(t) \quad , \quad \text{(1b)}$$

$$y(t) = \sum_{i=1}^{L} \mu_i(\xi(t)) y_i(t) + W w(t) \quad , \quad \text{(1c)}$$

where $x_i \in \mathbb{R}^{n_i}$ and $y_i \in \mathbb{R}^p$ are respectively the state vector and the output of the i^{th} submodel; $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^p$ the output and $w \in \mathbb{R}^r$ the perturbation. The matrices $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times m}$, $D_i \in \mathbb{R}^{n_i \times r}$, $C_i \in \mathbb{R}^{p \times n_i}$ and $W \in \mathbb{R}^{p \times r}$ are known and appropriately dimensioned. The parametric uncertainties in the system are represented by matrices ΔA_i and ΔB_i (details are given in section II-A).

The complete partition of the operating space of the system is performed using a characteristic variable of the system called *decision variable* $\xi(t)$ that is assumed to be known and real-time available (e.g. the inputs and/or exogenous signals). Note that the contribution of the submodels are quantified by the *weighting functions* $\mu_i(\xi(t))$ which are associated with each operating zone. They satisfy the following convex sum constraints:

$$\sum_{i=1}^{L} \mu_i(\xi(t)) = 1 \text{ and } 0 \leq \mu_i(\xi(t)) \leq 1 \,, \forall i = 1 ... L, \forall t. \quad (2)$$

Thanks to the above properties, the contributions of several submodels can be taken into account simultaneously and therefore the dynamic behaviour of the multiple model can be truly nonlinear instead of a piecewise linear behaviour.

Note that the contributions of the submodels are taken into account via a weighted sum in the output equation of the multiple model. Consequently, dimensions of the submodels can be different and therefore this multiple model form is suitable for black box modelling of complex systems with variable structure and/or variable complexity in each operating zone. The model parameters can be obtained from a set of measured input and output data using appropriate black box identification tools proposed for instance in [10], [11], [17].

Remark 1: It should be mentioned that the outputs $y_i(t)$ of the submodels are intermediary modelling signals only used in order to provide a representation of the real system behaviour. The submodel outputs $y_i(t)$ are internal signals of the multiple model. They are not physically available and consequently no measurement is possible. Hence, they cannot be employed for driving an observer. Only the global output $y(t)$ of the multiple model can be used for this purpose.

A. Model uncertainties

The parametric uncertainties in the system are represented by the following norm-bounded matrices:

$$\Delta A_i = \mu_i(\xi(t)) M_i F_i(t) N_i \quad , \quad (3)$$

$$\Delta B_i = \mu_i(\xi(t)) H_i S_i(t) E_i \quad , \quad (4)$$

where M_i, N_i, H_i and E_i are known constant matrices of appropriate dimensions and $F_i(t)$ and $S_i(t)$ are unknown, real and possibly time-varying matrices with Lebesgue-measurable elements satisfying:

$$F_i^T(t) F_i(t) \leq \mathrm{I} \quad \text{and} \quad S_i^T(t) S_i(t) \leq \mathrm{I} \quad \forall t \ . \quad (5)$$

Note that the uncertainties of each submodel are taken into consideration according to the validity degree of each submodel via its associated weighting function $\mu_i(\xi(t))$. Indeed, the uncertainties of a submodel can be neglected when its respective contribution is not taken into consideration for providing the overall multiple model output.

Notations: the following notations will be used all along this paper. $P > 0$ ($P < 0$) denotes a positive (negative) definite matrix P; X^T denotes the transpose of matrix X, I is the identity matrix of appropriate dimension and $diag\{A_1, ..., A_n\}$ stands for a block-diagonal matrix with the matrices A_i on the main diagonal. The $\mathcal{L}_2-$norm of a signal, quantifying its energy is denoted and defined by $\|e(t)\|_2^2 = \int_0^\infty e^T(t) e(t) dt$. Finally, we shall simply write $\mu_i(\xi(t)) = \mu_i(t)$.

III. On the proportional-integral observer

The conventional Luenberger or proportional observer only uses a proportional correction injection term given by the output estimation error. In the PIO an additional injection term $z(t)$, given by the integral of the output estimation error, is included in the dynamic equation of the observer. Thanks to this additional degree of freedom some robustness degrees of the state estimation with respect to the system uncertainties and perturbation are introduced [1], [18], [19]. The PIO has also been successfully employed in the synchronization of a chaotic system by [20]. The extension of the PIO design, based on dissipativity framework, to a particular nonlinear system whose non linearity is assumed to satisfy a sector bounded constraint, has been recently proposed in [21].

In this section, sufficient conditions for ensuring convergence and optimal disturbance attenuation of the estimation error are established in LMI terms [22] using the Lyapunov method. Note that the classic observer design cannot be employed directly in the multiple model framework because the interaction between submodels must be taken into consideration in the observer design procedure for ensuring the observer stability for any blend between the submodels.

Firstly new notation of the decoupled multiple model needed to design a PIO is introduced. The suggested PIO is then presented and its design is proposed by introducing some $\mathcal{H}_\infty$ performances.

A. Augmented form of the decoupled multiple model

Consider the following augmented state vector:

$$x(t) = \left[x_1^T(t)\cdots x_i^T(t)\cdots x_L^T(t)\right]^T \in \mathbb{R}^n, n = \sum_{i=1}^{L} n_i \tag{6}$$

and the supplementary variable $z(t) = \int_0^t y(\xi)d\xi$ needed for the PIO design. Thus, the decoupled multiple model (1) may be rewritten in the following compact form:

$$\begin{aligned} \dot{x}(t) &= (\tilde{A}+\Delta\tilde{A})x(t) + (\tilde{B}+\Delta\tilde{B})u(t) + \tilde{D}w \ , && (7a)\\ \dot{z}(t) &= \tilde{C}(t)x(t) + Ww(t) \ , && (7b)\\ y(t) &= \tilde{C}(t)x(t) + Ww(t) \ , && (7c)\end{aligned}$$

where

$$\begin{aligned} \tilde{A} &= diag\{A_1\cdots A_i\cdots A_L\} \ , && (8)\\ \tilde{B} &= \left[{B_1}^T\cdots {B_i}^T\cdots {B_L}^T\right]^T \ , && (9)\\ \tilde{D} &= \left[{D_1}^T\cdots {D_i}^T\cdots {D_L}^T\right]^T \ , && (10)\\ \tilde{C}(t) &= \sum_{i=1}^{L}\mu_i(t)\tilde{C}_i \ , && (11)\\ \tilde{C}_i &= \begin{bmatrix}0 & \cdots & C_i & \cdots & 0\end{bmatrix} && (12)\end{aligned}$$

with the parametric uncertainties given by:

$$\begin{aligned} \Delta\tilde{A} &= \sum_{i=1}^{L}\mu_i(t)\tilde{M}_iF_i(t)\tilde{N}_i \ , && (13)\\ \Delta\tilde{B} &= \sum_{i=1}^{L}\mu_i(t)\tilde{H}_iS_i(t)E_i \ , && (14)\\ \tilde{M}_i &= \left[0\cdots {M_i}^T\cdots 0\right]^T \ , && (15)\\ \tilde{N}_i &= \left[0\cdots N_i\cdots 0\right] \ , && (16)\\ \tilde{H}_i &= \left[0\cdots {H_i}^T\cdots 0\right]^T \ . && (17)\end{aligned}$$

Finally, the equations (7) can be rewritten in the following augmented form:

$$\begin{aligned} \dot{x}_a(t) &= (\tilde{A}_a(t) + \overline{C}_1\Delta\tilde{A}\,\overline{C}_1^T)x_a(t) + \overline{C}_1(\tilde{B}+\Delta\tilde{B})u(t)\\ &+ \tilde{D}_aw(t) \ , && (18a)\\ y(t) &= \tilde{C}(t)\overline{C}_1^Tx_a(t) + Ww(t), && (18b)\\ z(t) &= \overline{C}_2^Tx_a(t) \ , && (18c)\end{aligned}$$

where

$$\begin{aligned} x_a(t) &= \begin{bmatrix}x(t)\\ z(t)\end{bmatrix}, \quad \tilde{A}_a(t) = \begin{bmatrix}\tilde{A} & 0\\ \tilde{C}(t) & 0\end{bmatrix}, \quad \tilde{D}_a = \begin{bmatrix}\tilde{D}\\ W\end{bmatrix} \ ,\\ \overline{C}_1 &= \begin{bmatrix}\mathrm{I} & 0\end{bmatrix}^T, \quad \overline{C}_2 = \begin{bmatrix}0 & \mathrm{I}\end{bmatrix}^T \ .\end{aligned}$$

Let us notice that, by using the convex properties of the weighting functions, the matrix $\tilde{A}_a(t)$ can be rewritten as:

$$\tilde{A}_a(t) = \sum_{i=1}^{L}\mu_i(t)\overline{A}_i \ , \tag{19}$$

where

$$\overline{A}_i = \begin{bmatrix}\tilde{A} & 0\\ \tilde{C}_i & 0\end{bmatrix}. \tag{20}$$

B. PIO structure

The state estimation of the decoupled multiple model (18) is achieved by using the following PIO:

$$\begin{aligned} \dot{\hat{x}}_a(t) &= \tilde{A}_a(t)\hat{x}_a(t) + \overline{C}_1\tilde{B}u(t) + K_P(y(t)-\hat{y}(t))\\ &\quad + K_I(z(t)-\hat{z}(t)) \ , && (21a)\\ \hat{y}(t) &= \tilde{C}(t)\overline{C}_1^T\hat{x}_a(t) \ , && (21b)\\ \hat{z}(t) &= \overline{C}_2^T\hat{x}_a(t) && (21c)\end{aligned}$$

which has a similar structure to the PIO used in [20]. Notice that the use of the auxiliary integral signal $z(t)$ in the dynamic equation is at the origin of the designation Proportional-Integral Observer. The matrix K_I introduces a freedom degree in the observer design.

C. Design of the PIO

Consider the state estimation error defined by:

$$e_a(t) = x_a(t) - \hat{x}_a(t) \tag{22}$$

and its dynamics by:

$$\begin{aligned} \dot{e}_a(t) &= (\tilde{A}_a(t) - K_PC(t)\overline{C}_1^T - K_I\overline{C}_2^T)e_a(t) + \overline{C}_1\Delta\tilde{A}x(t)\\ &+ \overline{C}_1\Delta\tilde{B}u(t) + (\tilde{D}_a - K_PW)w(t) \ . && (23)\end{aligned}$$

Finally, (7a) and (23) can be gathered as follows:

$$\dot{\varepsilon}(t) = A_{obs}(t)\varepsilon(t) + \Phi\bar{w}(t) \ , \tag{24}$$

where

$$\begin{aligned} \varepsilon(t) &= \begin{bmatrix}e_a^T(t) & x^T(t)\end{bmatrix}^T \ , && (25)\\ \bar{w}(t) &= \begin{bmatrix}w^T(t) & u^T(t)\end{bmatrix}^T \ , && (26)\\ A_{obs}(t) &= \begin{bmatrix}\tilde{A}_a(t) - K_PC(t)\overline{C}_1^T - K_I\overline{C}_2^T & \overline{C}_1\Delta\tilde{A}\\ 0 & \tilde{A}+\Delta\tilde{A}\end{bmatrix}, && (27)\\ \Phi &= \begin{bmatrix}\tilde{D}_a - K_PW & \overline{C}_1\Delta\tilde{B}\\ \tilde{D} & \tilde{B}+\Delta\tilde{B}\end{bmatrix} \ . && (28)\end{aligned}$$

Notice that the proportional gain K_P can be used to reduce the impact of the perturbation on the estimation error $e_a(t)$. On the other hand, the observer dynamics can be improved with the help of the integral gain K_I. Note also that, from equation (24), $\varepsilon(t)$ is stable if and only if the decoupled multiple model (7) with admissible uncertainties $\Delta\tilde{A}$ is stable and the observer gains K_P and K_I are chosen so that $\tilde{A}_a(t) - K_PC(t)\overline{C}_1^T - K_I\overline{C}_2^T$ is also stable. In the sequel, the two following assumptions will be considered:

Assumption 1: The decoupled multiple model (7) with admissible uncertainties $\Delta\tilde{A}$ is stable.

Assumption 2: The input and the perturbation are bounded energy signals, i.e. $\|u(t)\|_2^2 < \infty$ and $\|w(t)\|_2^2 < \infty$.

The robust PIO design problem can thus be formulated as finding the matrices K_P and K_I such that the influence of $\bar{w}(t)$ on the estimation error $e_a(t)$ is attenuated and the state estimation error remains globally bounded for any blend between the submodels. To this end, the following *objective signal* which only depends on the estimation error $e_a(t)$ is introduced:

$$\nu(t) = \begin{bmatrix} Y & 0 \end{bmatrix} \varepsilon(t) \ , \tag{29}$$

where Y is a matrix of appropriate dimension chosen by the designer. Finally, the expected performances of the PIO can be formulated by the following $\mathcal{H}_\infty$ performances:

$$\lim_{t\to\infty} e_a(t) = 0 \text{ for } w(t) = 0, F_i(t) = 0, S_i(t) = 0 \ , \tag{30a}$$

$$\|\nu(t)\|_2^2 \le \gamma^2 \|\overline{w}(t)\|_2^2 \text{ for } \overline{w}(t) \neq 0 \text{ and } \nu(0) = 0 \ , \tag{30b}$$

where γ is the $\mathcal{L}_2$ gain from $\bar{w}(t)$ to $\nu(t)$ to be minimized.

Theorem 1: Consider the uncertain model (18) and assumptions 1 and 2. There exists a PIO (21) ensuring the objectives (30) if there exists symmetric positive definite matrices $P_1 \in \mathbb{R}^{(n+p)\times(n+p)}$ and $P_2 \in \mathbb{R}^{n\times n}$, matrices $L_P \in \mathbb{R}^{(n+p)\times p}$ and $L_I \in \mathbb{R}^{(n+p)\times p}$ and positive scalars $\overline{\gamma}$, τ_1^i and τ_2^i such that the following condition holds for $i = 1 \ldots L$

$$\min \overline{\gamma} \quad \text{subject to}$$

$$\begin{bmatrix} \Gamma_i + \Gamma_i^T + Y^T Y & 0 & \Psi & 0 & P_1\overline{C}_1\tilde{M}_i & P_1\overline{C}_1\tilde{H}_i \\ 0 & \Lambda_i & P_2\tilde{D} & P_2\tilde{B} & P_2\tilde{M}_i & P_2\tilde{H}_i \\ (*) & (*) & -\overline{\gamma}\mathrm{I} & 0 & 0 & 0 \\ 0 & (*) & 0 & \phi_i & 0 & 0 \\ (*) & (*) & 0 & 0 & -\tau_1^i \mathrm{I} & 0 \\ (*) & (*) & 0 & 0 & 0 & -\tau_2^i \mathrm{I} \end{bmatrix} < 0 \ ,$$

where

$$\begin{aligned} \Gamma_i &= P_1\overline{A}_i - L_P\tilde{C}_i\overline{C}_1^T - L_I\overline{C}_2^T \ , \\ \Psi &= P_1\tilde{D}_a - L_P W \ , \\ \Lambda_i &= P_2\tilde{A} + \tilde{A}^T P_2 + \tau_1^i \tilde{N}_i^T \tilde{N}_i \ , \\ \phi_i &= -\overline{\gamma}\mathrm{I} + \tau_2^i E_i^T E_i \end{aligned}$$

for a prescribed matrix Y. The observer gains are given by $K_P = P_1^{-1} L_P$ and $K_I = P_1^{-1} L_I$; the $\mathcal{L}_2$ gain from $\bar{w}(t)$ to $\nu(t)$ is given by $\gamma = \sqrt{\overline{\gamma}}$.

Proof: The proof is deferred to the appendix. □

IV. A SIMULATION EXEMPLE

Consider the decoupled multiple model with $L = 2$ submodels with different dimensions ($n_1 = 3$ and $n_2 = 2$), given by:

$$A_1 = \begin{bmatrix} -0.1 & -0.3 & 0.6 \\ -0.5 & -0.4 & 0.1 \\ -0.3 & -0.2 & -0.6 \end{bmatrix} , \quad A_2 = \begin{bmatrix} -0.3 & -0.1 \\ 0.4 & -0.2 \end{bmatrix} ,$$

$$B_1 = \begin{bmatrix} 0.3 & 0.5 & 0.6 \end{bmatrix}^T , \quad B_2 = \begin{bmatrix} 0.4 & 0.3 \end{bmatrix}^T ,$$

$$D_1 = \begin{bmatrix} 0.1 & -0.1 & 0.1 \end{bmatrix}^T , \quad D_2 = \begin{bmatrix} -0.1 & -0.1 \end{bmatrix}^T ,$$

$$C_1 = \begin{bmatrix} -0.4 & 0.3 & 0.5 \\ 0.5 & 0.3 & 0.4 \end{bmatrix} , \quad C_2 = \begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.2 \end{bmatrix} ,$$

$$M_1 = \begin{bmatrix} -0.1 & 0.2 & -0.1 \end{bmatrix}^T , \quad M_2 = \begin{bmatrix} -0.2 & 0.1 \end{bmatrix}^T ,$$

$$N_1 = \begin{bmatrix} 0.1 & -0.2 & 0.3 \end{bmatrix} , \quad N_2 = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix} ,$$

$$H_1 = \begin{bmatrix} 0.3 & -0.1 & 0.2 \end{bmatrix}^T , \quad H_2 = \begin{bmatrix} -0.1 & -0.2 \end{bmatrix}^T ,$$

$$E_1 = -0.2 \ , \quad E_2 = -0.3 \ ,$$

$$W = \begin{bmatrix} 0.1 & -0.1 \end{bmatrix} , \quad Y = \mathrm{I}_{(7\times 7)} \ .$$

Here, the decision variable $\xi(t)$ is the input signal $u(t) \in [-1, 1]$. The weighting functions are obtained from normalised Gaussian functions:

$$\mu_i(\xi(t)) = \eta_i(\xi(t)) / \sum_{j=1}^{L} \eta_j(\xi(t)), \tag{31}$$

$$\eta_i(\xi(t)) = \exp\left(-(\xi(t) - c_i)^2 / \sigma^2\right), \tag{32}$$

with the standard deviation $\sigma = 0.6$ and the centres $c_1 = -0.3$ and $c_2 = 0.3$. The perturbation $w(t)$ is a normally distributed random signal with zero mean and standard deviation equal to one. The input, the weighting functions and the outputs are shown in figure 1. The time-varying signals $F_i(t)$, $S_i(t)$ and the perturbation $w(t)$ are plotted in figure 2. Notice that for $0 < t < 120$ no uncertainties in the multiple model are considered.

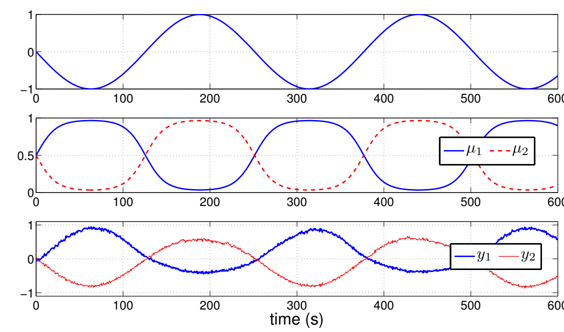

Fig. 1. Input, weighting functions and outputs

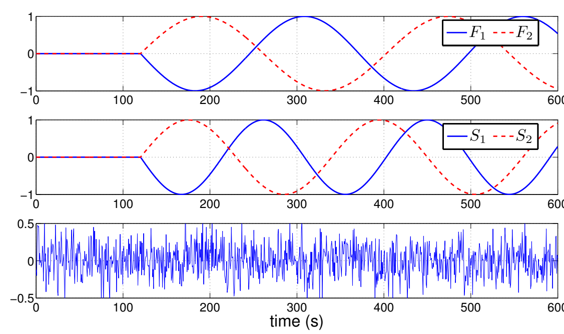

Fig. 2. $F_i(t)$, $S_i(t)$ and $w(t)$

A solution satisfying conditions of theorem 1 is obtained using YALMIP interface and SEDUMI solver. The gains of the PIO are:

$$K_P = \begin{bmatrix} -3.13 & 1.21 & -2.02 & 3.68 & -0.90 & 0.50 & 0.49 \\ 0.20 & 0.86 & 1.00 & -1.51 & 2.74 & -0.08 & 1.08 \end{bmatrix}^T ,$$

$$K_I = \begin{bmatrix} 0.16 & -0.12 & -0.41 & 0.31 & -0.57 & 0.65 & -0.04 \\ -0.13 & 0.12 & 0.41 & -0.32 & 0.56 & 0.02 & 0.65 \end{bmatrix}^T$$

the minimal attenuation level is $\gamma = 0.8654$ and $\tau_1^1 = 0.74$, $\tau_2^1 = 1.37$, $\tau_1^2 = 7.67$, $\tau_2^2 = 2.08$.

In the simulation the initial conditions of the multiple model are $x(0) = \begin{bmatrix} 0.1 & -0.1 & 0.1 & -0.1 & 0.1 \end{bmatrix}$ and the initial conditions of the observer are equal to zero. Figures 3 and 4 display the comparison between the states of the submodels and their estimates. Note that the interaction between submodels is at the origin of some compensation phenomenons in the state estimation. For example, if the output of submodel 1 is only taken into consideration (i.e. $\mu_1(t) \approx 1$) then naturally a bad state estimation of the submodel 2 is provided by the observer. However, the overall output estimation of the multiple model is not truly affected by this bad estimation. Finally, a comparison between the outputs of the multiple model and their estimates is shown in figure 5. Note that the output estimation errors remain globally bounded despite that model uncertainties and perturbations appear in the model.

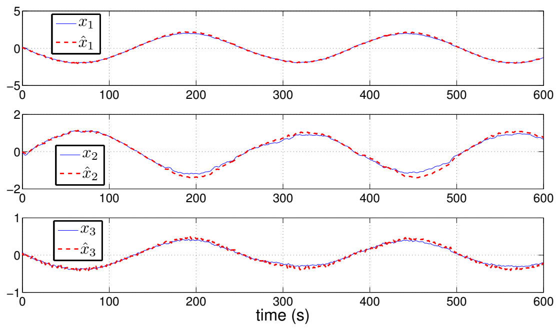

Fig. 3. States of submodel 1 and its estimates

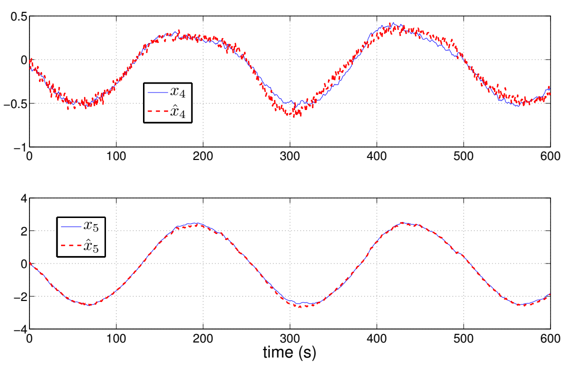

Fig. 4. States of submodel 2 and its estimates

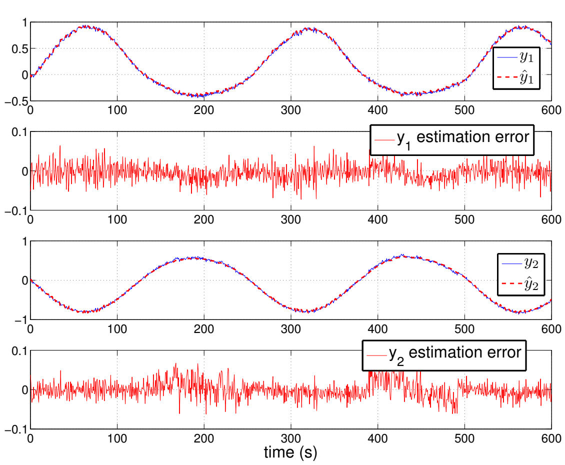

Fig. 5. Output, its estimates and the output estimation errors

V. Conclusions

In this paper a PIO design is presented for a class of uncertain nonlinear system which can be modelled with the help of a decoupled multiple model. This model is suitable for modelling variable structure systems because the dimension of the submodels can be different in each operating zone. Sufficient conditions, in LMI terms, for ensuring $\mathcal{H}_\infty$ performances of the estimation error are established using Lyapunov method. The effectiveness of the proposed approach is illustrated via a simulation example. Further research, in a fault diagnosis perspective, will be to investigate the sensitivity of the state estimation with respect to perturbations, model uncertainties and faults in order to establish the sensitivity of the fault symptoms of the system.

Appendix proof of the theorem

Lemma 1: For any constant real matrices X and Y with appropriate dimensions, a matrix function $F(t)$ bounded-norm, i.e. $F^T(t)F(t) \leq \mathrm{I}$, then the following property holds for any positive matrix Q

$$XF(t)Y + Y^T F^T(t) X^T \leq XQ^{-1}X^T + Y^T QY \ .$$

Consider the following quadratic Lyapunov function:

$$V(t) = e_a^T(t) P_1 e_a(t) + x^T(t) P_2 x(t) \ , \tag{33}$$

where $P_1 = P_1^T > 0$ et $P_2 = P_2^T > 0$. The objectives (30) are guaranteed if there exists a Lyapunov function (33) such that [22]:

$$\dot{V}(t) < -\nu^T(t)\nu(t) + \gamma^2 \overline{w}^T(t)\overline{w}(t) \ . \tag{34}$$

The time-derivative of (33) along the trajectories of (24) and (7a) is given by:

$$\begin{aligned}
\dot{V}(t) &= \Omega^T(t) \begin{bmatrix} PA_{obs}(t) + A_{obs}^T P(t) & P\Phi \\ (*) & 0 \end{bmatrix} \Omega(t) \ , && (35)\\
P &= diag\{P_1, \quad P_2\} \ , && (36)\\
\Omega(t) &= \begin{bmatrix} \varepsilon^T(t) & \overline{w}^T(t) \end{bmatrix}^T \ . && (37)
\end{aligned}$$

Now, by taking into consideration (35), the condition (34) becomes:

$$\Omega^T(t) \begin{bmatrix} PA_{obs}(t) + A_{obs}^T(t)P + \begin{bmatrix} Y^TY & 0 \\ 0 & 0 \end{bmatrix} & P\Phi \\ (*) & -\gamma^2 \mathrm{I} \end{bmatrix} \Omega(t) < 0 \ , \tag{38}$$

which is a quadratic form in $\Omega(t)$. By using the definitions of A_{obs} and Φ given respectively by (27) and (28), the inequality (38) is also guaranteed if:

$$\begin{bmatrix} \Gamma + \Gamma^T + Y^T Y & P_1 \overline{C}_1 \Delta\tilde{A} & \Psi & P_1 \overline{C}_1 \Delta\tilde{B} \\ (*) & X_1 + X_2 & P_2 \tilde{D} & P_2(\tilde{B} + \Delta\tilde{B}) \\ (*) & (*) & -\gamma^2 \mathrm{I} & 0 \\ (*) & (*) & 0 & -\gamma^2 \mathrm{I} \end{bmatrix} < 0, \tag{39}$$

where

$$\begin{aligned}
\Gamma &= P_1(\tilde{A}_a(t) - K_P C(t)\overline{C}_1^T - K_I \overline{C}_2^T) \ , && (40)\\
\Psi &= P_1(\tilde{D}_a - K_P W) \ , && (41)\\
X_1 &= P_2\tilde{A} + \tilde{A}^T P_2 \ , && (42)\\
X_2 &= P_2\Delta\tilde{A} + \Delta\tilde{A}^T P_2 \ , && (43)
\end{aligned}$$

Notice that by using the definition of $\tilde{A}_a(t)$ and $C(t)$ given respectively by (19) and (11), Γ can be rewritten as :

$$\Gamma = \sum_{i=1}^{L} \mu_i(t)\Gamma_i \ , \tag{44}$$

$$\Gamma_i = P_1(\overline{A}_i - K_P\tilde{C}_i\overline{C}_1^T - K_I\overline{C}_2^T) \ . \tag{45}$$

At this point, by considering (44) and (43), the nominal and the uncertain terms in (39) may be dissociated as follows:

$$\sum_{i=1}^{L}\mu_i(t)\begin{bmatrix}\Gamma_i+\Gamma_i^T+Y^TY & 0 & \Psi & 0\\ 0 & X_1 & P_2\tilde{D} & P_2\tilde{B}\\ (*) & (*) & -\gamma^2\mathrm{I} & 0\\ 0 & (*) & 0 & -\gamma^2\mathrm{I}\end{bmatrix}+Z+Z^T<0, \tag{46}$$

where

$$Z=\begin{bmatrix}0 & P_1\overline{C}_1\Delta\tilde{A} & 0 & P_1\overline{C}_1\Delta\tilde{B}\\ 0 & P_2\Delta\tilde{A} & 0 & P_2\Delta\tilde{B}\\ 0&0&0&0\\ 0&0&0&0\end{bmatrix} \ . \tag{47}$$

Now, by introducing the definitions of $\Delta\tilde{A}$ and $\Delta\tilde{B}$ given by (13) and (14) then $Z+Z^T$ becomes:

$$Z+Z^T=\sum_{i=1}^{L}\mu_i(t)\left\{\tilde{X}_i\tilde{Y}_i+\tilde{Y}_i^T\tilde{X}_i^T\right\} \ , \tag{48}$$

where

$$\tilde{X}_i = \begin{bmatrix}P_1\overline{C}_1\tilde{M}_i & P_1\overline{C}_1\tilde{H}_i\\ P_2\tilde{M}_i & P_2\tilde{H}_i\\ 0&0\\0&0\end{bmatrix} , \tag{49}$$

$$\tilde{Y}_i = \begin{bmatrix}F_i(t) & 0\\ 0 & S_i(t)\end{bmatrix}\begin{bmatrix}0 & \tilde{N}_i & 0 & 0\\ 0&0&0&E_i\end{bmatrix} \ . \tag{50}$$

Notice that the dependence of the unknown functions $F_i(t)$ and $S_i(t)$ upon (48) can be removed, by using the lemma 1 with $Q_i = diag\left\{\tau_1^i,\tau_2^i\right\}$, as follows:

$$Z+Z^T\leq\sum_{i=1}^{L}\mu_i(t)\left\{\tilde{X}_i\begin{bmatrix}\tau_1^i & 0\\ 0 & \tau_2^i\end{bmatrix}^{-1}\tilde{X}_i^T+\tilde{Y}_i\begin{bmatrix}\tau_1^i & 0\\ 0&\tau_2^i\end{bmatrix}\tilde{Y}_i^T\right\}. \tag{51}$$

Finally, using the definition (42) of X_1, the inequality (46) is guaranteed if for $i=1...L$ the following inequality holds:

$$\begin{bmatrix}\Gamma_i+\Gamma_i^T+Y^TY & 0 & \Psi & 0 & P_1\overline{C}_1\tilde{M}_i & P_1\overline{C}_1\tilde{H}_i\\ 0 & \Lambda_i & P_2\tilde{D} & P_2\tilde{B} & P_2\tilde{M}_i & P_2\tilde{H}_i\\ (*) & (*) & -\gamma^2\mathrm{I} & 0 & 0 & 0\\ 0 & (*) & 0 & \phi_i & 0 & 0\\ (*) & (*) & 0 & 0 & -\tau_1^i\mathrm{I} & 0\\ (*) & (*) & 0 & 0 & 0 & -\tau_2^i\mathrm{I}\end{bmatrix}<0, \tag{52}$$

where

$$\Lambda_i = P_2\tilde{A}+\tilde{A}^TP_2+\tau_1^i\tilde{N}_i^T\tilde{N}_i \ , \tag{53}$$

$$\phi_i = -\gamma^2\mathrm{I}+\tau_2^iE_i^TE_i \ . \tag{54}$$

This condition follows from the use of (51) in (46), the use of the well known Schur complement and the convex sum properties of $\mu_i(t)$. Note that asymptotic convergence towards zero of the estimation error, when no uncertainties and no perturbations affect the system, is guaranteed by the negativity of the block $(1,1)$ in (52).

Finally, let us notice that (52) is not a LMI in P_1, K_P, K_I and γ. However, it becomes a LMI by setting $L_P = P_1K_P$, $L_I = P_1K_I$ and $\overline{\gamma}=\gamma^2$. Now, standard convex optimization algorithms can be used to find matrices P_1, P_2 L_P and L_I minimising $\overline{\gamma}$. This completes the proof of theorem 1.

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