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Output Control of Spacecraft in Leader Follower Formation

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***Abstract*— In this paper we will consider the control of spacecraft in a leader-follower formation using position measurements only. To analyze the formation under non vanishing disturbances, new definitions of practical exponential stability are given together with sufficient conditions for systems to satisfy these properties.**

I. INTRODUCTION

A. Background

The formal study of spacecraft formation requires solid theoretical roots. In this paper, we therefore provide a theoretical framework that fits realistic challenges related to this problem. Indeed, in presence of uncertainties or disturbances, it is often the case that a nominally *asymptotically* stable formation turns out to present a steady-state error in reality. In the case when this error can be reducible at will by a convenient tuning of some gains, this stability property is referred to as *practical*. Practical stability has been treated in several papers, see [1] and [2] and references therein. We will here give a very simple introductory example:

Example 1: Consider the scalar system

$$\dot{x} = -\theta x + d \tag{1}$$

where θ is a constant parameter and $d = d(t)$ is a non vanishing, time-varying disturbance. In this case the solutions are bounded by

$$|x(t)| \le (|x(0) - \frac{\beta_d}{\theta}|)e^{-\theta t} + \frac{\beta_d}{\theta} \tag{2}$$

where $\beta_d = \sup_t d(t)$. We see that for any θ such that $\theta > \beta_d \delta$, the solutions converge exponentially to a ball around the origin of radius $\delta = \beta_d/\theta$.

Tools for a formal analysis of more involved parameterized time-varying systems will be given in Section II-B. We will stress that ultimate boundedness as defined in [3] is a *weaker* property than practical stability. For a system possessing the latter property, the vicinity of the origin to which the solutions converge may be made arbitrary small by convenient tuning of some parameters of the system, typically the control gains.

B. Previous work

Research on spacecraft formation control is vast, so we will focus on previous work done on spacecraft formations where a relative position model similar to the one in Section III-A are used. For a more thorough treatment of the topic of spacecraft formation control, the interested reader is instead referred to the survey paper [4]. In [5] a full state feedback adaptive learning control algorithm was developed to give global asymptotic convergence of position and velocity tracking errors, in the presence of periodic disturbances and unknown spacecraft masses. An internal model based approach was taken in [6] to design a controller that handles parametric uncertainties and unknown disturbances. The methodology was shown to be robust to persistent disturbances, such as gravitational perturbations. Assuming boundedness of orbital perturbations and the leader control force only, an adaptive controller was designed in [7] to prove that the closed-loop system is uniformly semiglobally practically asymptotically stable (USPAS). A velocity filter was used to provide sufficient knowledge about the relative velocity to solve the control problem. These results were extended in [8] to also include the case of uncertainty in spacecraft mass. In [9] two controller-observer schemes were proposed which render the origin uniformly globally exponentially stable (UGES) in the case of no disturbances, and uniformly globally practically asymptotically stable (UGPAS) if the disturbances are bounded.

Finally, we note that the results in this paper builds on results achieved for the control of robot manipulators, e.g. [10].

C. Contribution

The contribution of this paper is twofold. Firstly, we present a theoretical contribution consisting of new definitions and theorems of sufficient conditions for nonlinear time-varying systems to be exponentially stable with respect to balls that can be arbitrarily reduced by a convenient tuning. We denote a system satisfying these properties in the whole state-space uniformly globally practically exponentially stable (UGPES). For the sake of completeness, we will also discuss uniform semiglobal exponential stability (USES) and uniform semiglobal practical exponential stability (USPES), in which case the domain of attraction in not the whole state-space, but a compact set that can be arbitrarily enlarged.

Secondly, the stability of a leader/follower formation is analyzed using a controller-observer scheme originally designed for the control of robot manipulators. While, in the nominal case, the solutions of the system are proven to be exponentially convergent to zero, we will show that the steady-state error resulting from external disturbances and lack of measurement can be arbitrarily diminished by a convenient tuning of some controller gains. In fact, based on knowledge on the bounds of the disturbances and the acceptable steady state error, the presented theorems give information on how to pick the controller gains.

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II. Mathematical preliminaries

A. Notation

We use the notation $\dot{x}$ for the time derivative of a vector x, i.e. $\dot{x} = dx/dt$. Moreover $\ddot{x} = d^2x/dt^2$. The solutions of the differential equation $\dot{x} = f(t, x)$ with initial conditions (t_0, x_0) is denoted by $x(t, t_0, x_0)$. We use $|\cdot|$ for the Euclidean norm of vectors. We use $\lambda_m(A)$ and $\lambda_M(A)$ to denote the minimum and maximum eigenvalue, respectively, of a matrix A. A closed ball in $\mathbb{R}^n$ of radius δ centered at the origin is denoted by $\mathcal{B}_\delta$, i.e. $\mathcal{B}_\delta := \{x \in \mathbb{R}^n : |x| \leq \delta\}$.

B. Definitions

Semiglobal and practical exponential stability properties pertain to parameterized nonlinear time-varying systems of the form

$$\dot{x} = f(t, x, \theta), \tag{3}$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}_{\geq 0}$, $\theta \in \mathbb{R}^m$ is a constant parameter and $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz in x and satisfies Carathéodory conditions for any parameter θ under consideration. θ is a free tuning parameter, that can for instance be a control gain, see [1] for details.

Definition 1 (UGPES): Let $\Theta \subset \mathbb{R}^m$ be a set of parameters. The system (3) is said to be *uniformly globally practically exponentially stable on* Θ if, given any $\delta > 0$, there exists a parameter $\theta^\star(\delta) \in \Theta$, and positive constants $k(\delta)$ and $\gamma(\delta)$ such that, for any $x_0 \in \mathbb{R}^n$ and any $t_0 \in \mathbb{R}_{\geq 0}$ the solutions of (3) satisfies, for all $t \geq t_0$,

$$|x(t, t_0, x_0, \theta^\star)| \leq \delta + k(\delta)\,|x_0|\, e^{-\gamma(\delta)(t-t_0)}.$$

Definition 2 (USES): Let $\Theta \subset \mathbb{R}^m$ be a set of parameters. The system (3) is said to be *uniformly semiglobally exponentially stable on* Θ if, given any $\Delta > 0$, there exists a parameter $\theta^\star(\Delta) \in \Theta$ and positive constants $k(\Delta)$ and $\gamma(\Delta)$ such that, for any $x_0 \in \mathcal{B}_\Delta$ and any $t_0 \in \mathbb{R}_{\geq 0}$ the solutions of (3) satisfies, for all $t \geq t_0$,

$$|x(t, t_0, x_0, \theta^\star)| \leq k(\Delta)\,|x_0|\, e^{-\gamma(\Delta)(t-t_0)}$$

Definition 3 (USPES): Let $\Theta \subset \mathbb{R}^m$ be a set of parameters. The system (3) is said to be *uniformly semiglobally practically exponentially stable on* Θ if, given any $\Delta > \delta > 0$, there exists a parameter $\theta^\star(\delta, \Delta) \in \Theta$ and positive constants $k(\delta, \Delta)$ and $\gamma(\delta, \Delta)$ such that, for any $x_0 \in \mathcal{B}_\Delta$ and any $t_0 \in \mathbb{R}_{\geq 0}$ the solutions of (3) satisfies, for all $t \geq t_0$,

$$|x(t, t_0, x_0, \theta^\star)| \leq \delta + k(\delta, \Delta)\,|x_0|\, e^{-\gamma(\delta,\Delta)(t-t_0)}\,.$$

These properties are strongly related to their *asymptotic* counterpart (UGPAS, USAS and USPAS) introduced (and commented in detail) in [1], [11]. They are however stronger properties as they impose an *exponential* behavior of the solutions in the considered domain of the state-space and a linear dependency in the initial condition.

C. Lyapunov sufficient conditions

We here present sufficient conditions for the above properties to hold. They are expressed as a condition on the sign of a Lyapunov-like function's derivative, on a restricted region of the state space.

1) UGPES:

Theorem 1 (Sufficient condition for UGPES): Let Θ be a subset of $\mathbb{R}^m$ and suppose that, given any $\delta > 0$, there exist a parameter $\theta^\star(\delta) \in \Theta$, a continuously differentiable Lyapunov function $V_\delta : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and positive constants $\kappa(\delta)$, $\underline{\kappa}(\delta)$, $\overline{\kappa}(\delta)$ such that, for all $x \in \mathbb{R}^n \setminus \mathcal{B}_\delta$ and all $t \in \mathbb{R}_{\geq 0}$,

$$\underline{\kappa}(\delta)\,|x|^p \leq V_\delta(t, x) \leq \overline{\kappa}(\delta)\,|x|^p\,, \tag{4}$$

$$\frac{\partial V_\delta}{\partial t}(t, x) + \frac{\partial V_\delta}{\partial x}(t, x) f(t, x, \theta^\star) \leq -\kappa(\delta)\,|x|^p\,, \tag{5}$$

where p denotes a positive constant. Then, under the condition that

$$\lim_{\delta \to 0} \frac{\overline{\kappa}(\delta)\delta^p}{\underline{\kappa}(\delta)} = 0\,, \tag{6}$$

the system $\dot{x} = f(t, x, \theta)$ introduced in (3) is UGPES on the parameter set Θ.

Proof: Let δ be any given positive constant. Along the solutions of (3), we get from (4) and (5) that

$$\begin{aligned}&|x(t, t_0, x_0, \theta^\star)| \geq \delta \Rightarrow \\ &\dot{V}_\delta(t, x(t, t_0, x_0, \theta^\star)) \leq -\kappa'(\delta) V_\delta(t, x(t, t_0, x_0, \theta^\star))\,,\end{aligned}$$

where $\kappa'(\delta) := \kappa(\delta)/\overline{\kappa}(\delta)$. Invoking [11, Lemma 13], we then get that, for all $x_0 \in \mathbb{R}^n$, all $t_0 \in \mathbb{R}_{\geq 0}$ and all $t \geq t_0$,

$$\begin{aligned}&|x(t, t_0, x_0, \theta^\star)| \leq \\ &\left(\frac{\overline{\kappa}(\delta)\delta^p}{\underline{\kappa}(\delta)}\right)^{1/p} + \left(\frac{\overline{\kappa}(\delta)}{\underline{\kappa}(\delta)}\right)^{1/p} |x_0|\, e^{-\kappa'(\delta)(t-t_0)/p}.\end{aligned}$$

In view of (6), we see that the quantity $\overline{\kappa}(\delta)\delta^p/\underline{\kappa}(\delta)$ may be reduced at will by originally choosing δ small enough and the conclusion follows. ∎

Compared to classical results for Lyapunov stability, conditions (4) and (5) are natural (see [3, Theorem 4.10]). For perturbed systems, (4) is notably satisfied by the Lyapunov function associated to the UGES of the origin of the corresponding nominal systems. (5) is similar to the Lyapunov sufficient condition for global ultimate boundedness (*cf. e.g.* [3]). Intuitively, one may expect that these two requirements, when valid for any arbitrarily small δ, suffice to conclude UGPES. However, we see that an additional assumption (6) is required, establishing a relationship between the bounds on the Lyapunov function. Indeed, in the present framework, the Lyapunov function may here depend on the tuning parameter θ, and consequently on the radius δ. As clearly shown in [12], [13], this parametrization of the Lyapunov function may induce unexpected behaviors if (6) is not assumed.

2) USES:

Theorem 2 (Sufficient condition for USES): Let Θ be a subset of $\mathbb{R}^m$ and suppose that, given any $\Delta > 0$, there exist a parameter $\theta^\star(\Delta) \in \Theta$, a continuously differentiable Lyapunov function $V_\Delta : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and positive constants $\kappa(\Delta)$, $\underline{\kappa}(\Delta)$, $\overline{\kappa}(\Delta)$ such that, for all $x \in \mathcal{B}_\Delta$ and all $t \in \mathbb{R}_{\geq 0}$,

$$\underline{\kappa}(\Delta)\,|x|^p \leq V_\Delta(t, x) \leq \overline{\kappa}(\Delta)\,|x|^p \tag{7}$$

$$\frac{\partial V_\Delta}{\partial t}(t, x) + \frac{\partial V_\Delta}{\partial x}(t, x) f(t, x, \theta^\star) \leq -\kappa(\Delta)\,|x|^p\,, \tag{8}$$

where p denotes a positive constant. Then, under the condition that

$$\lim_{\Delta\to\infty}\frac{\underline{\kappa}(\Delta)\Delta^p}{\overline{\kappa}(\Delta)}=\infty\,, \tag{9}$$

the system $\dot{x}=f(t,x,\theta)$ introduced in (3) is USES on the parameter set Θ.

The proof is omitted, but follows along the same lines as Theorem 1.

3) USPES:

Theorem 3 (Sufficient condition for USPES): Let Θ be a subset of $\mathbb{R}^m$ and suppose that, given any $\Delta>\delta>0$, there exist a parameter $\theta^\star(\delta,\Delta)\in\Theta$, a continuously differentiable Lyapunov function $V_{\delta,\Delta}:\mathbb{R}_{\geq 0}\times\mathbb{R}^n\to\mathbb{R}_{\geq 0}$ and positive constants $\kappa(\delta,\Delta)$, $\underline{\kappa}(\delta,\Delta)$, $\overline{\kappa}(\delta,\Delta)$ such that, for all $x\in\mathcal{B}_\Delta\setminus\mathcal{B}_\delta$ and all $t\in\mathbb{R}_{\geq 0}$,

$$\underline{\kappa}(\delta,\Delta)\,|x|^p\leq V_{\delta,\Delta}(t,x)\leq\overline{\kappa}(\delta,\Delta)\,|x|^p \tag{10}$$

$$\frac{\partial V_{\delta,\Delta}}{\partial t}(t,x)+\frac{\partial V_{\delta,\Delta}}{\partial x}(t,x)f(t,x,\theta^\star)\leq-\kappa(\delta,\Delta)\,|x|^p\,, \tag{11}$$

where p denotes a positive constant. Assume also that, given any $\Delta^\star>\delta^\star>0$, there exist $\Delta>\delta>0$ such that

$$\frac{\overline{\kappa}(\delta,\Delta)\delta^p}{\underline{\kappa}(\delta,\Delta)}\leq\delta^\star\quad\text{and}\quad\frac{\underline{\kappa}(\delta,\Delta)\Delta^p}{\overline{\kappa}(\delta,\Delta)}\geq\Delta^\star\,.$$

Then the system $\dot{x}=f(t,x,\theta)$ introduced in (3) is USPES on the parameter set Θ.

The proof is omitted, but follows along the same lines as Theorem 1.

III. Model

The spacecraft model used in this paper is found in [14], and can be traced back to [15]. The starting point for this model is the fundamental differential equation for the two-body problem

$$\ddot{r}+\frac{\mu}{|r|^3}r=0, \tag{12}$$

where $r\in\mathbb{R}^3$ is the relative position of two point masses m_1,$m_2\in\mathbb{R}$, and $\mu=G(m_1+m_2)\in\mathbb{R}$, with $G\in\mathbb{R}$, being the universal constant of gravity.

A. Model of follower spacecraft

Equation (12) is generalized to include disturbance forces f_l, $f_f\in\mathbb{R}^3$ due to aerodynamic drag, third gravitating bodies, solar radiation, magnetic fields, etc., and actuator forces u_l, $u_f\in\mathbb{R}^3$ for the leader and follower spacecraft, respectively, such that

$$\ddot{r}_l=-\frac{\mu}{|r_l|^3}r_l+\frac{f_l}{m_l}+\frac{u_l}{m_l}, \tag{13}$$

$$\ddot{r}_f=-\frac{\mu}{|r_f|^3}r_f+\frac{f_f}{m_f}+\frac{u_f}{m_f}, \tag{14}$$

where m_l and m_f are the mass of the leader and follower spacecraft, respectively. By defining $p_f:=r_f-r_l\in\mathbb{R}^3$ as the relative position in the leader spacecraft reference frame, the relative position dynamics can be written in the following form (cf. [14])

$$M_f\ddot{p}_f+C_f(\dot{\nu}_l)\dot{p}_f+D_f(\ddot{\nu}_l,\dot{\nu}_l,r_f)p_f+n_f(r_l,r_f)=U_f+F_f, \tag{15}$$

where ν_l is the true anomaly of the leader spacecraft,

$$M_f=m_fI\in\mathbb{R}^{3\times 3} \tag{16}$$

is a diagonal matrix,

$$C_f(\dot{\nu}_l)=2m_f\dot{\nu}_l\bar{C}\in\mathbb{R}^{3\times 3} \tag{17}$$

is a skew-symmetric matrix, and

$$D_f(\ddot{\nu}_l,\dot{\nu}_l,r_f)=m_f\frac{\mu}{|r_f|^3}I+m_f\dot{\nu}_l^2\bar{D}+m_f\ddot{\nu}_l\bar{C}\in\mathbb{R}^{3\times 3} \tag{18}$$

with

$$\bar{C}:=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}\text{ and }\bar{D}:=\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&0\end{bmatrix}$$

as introduced in [7], and

$$n_f(r_l,r_f)=m_f\mu\begin{bmatrix}\frac{|r_l|}{|r_f|^3}-\frac{1}{|r_l|^2}&0&0\end{bmatrix}^\top\in\mathbb{R}^3. \tag{19}$$

The composite disturbance force $F_f\in\mathbb{R}^3$ and the relative control force $U_f\in\mathbb{R}^3$ are given by

$$F_f=f_f-\frac{m_f}{m_l}f_l\text{ and }U_f=u_f-\frac{m_f}{m_l}u_l. \tag{20}$$

B. Model and Desired Trajectory Assumptions

The true anomaly of the leader spacecraft is the angle between the eccentricity vector

$$e_l=\frac{\dot{r}_l\times h}{\mu}-\frac{r_l}{|r_l|}\in\mathbb{R}^3 \tag{21}$$

where $h=r_l\times\dot{r}_l\in\mathbb{R}^3$, and the orbital state vector r_l given by (13), so that:

$$\nu_l=\begin{cases}\arccos\frac{e_l^\top r_l}{|e_l||r_l|}&\text{if}\quad r_l^\top\dot{r}_l\geq 0\\2\pi-\arccos\frac{e_l^\top r_l}{|e_l||r_l|}&\text{if}\quad r_l^\top\dot{r}_l<0\end{cases} \tag{22}$$

The eccentricity vector is conserved under forces that obey the inverse-square law as in (12), but due to the control and disturbance forces in (13), the eccentricity vector will vary. We choose the reference trajectory of the leader spacecraft to satisfy the inverse square law such that the eccentricity is constant. Then, the desired true anomaly rate and true anomaly rate of change of the leader spacecraft, denoted $\dot{\nu}_d$ and $\ddot{\nu}_d$, are given by:

$$\dot{\nu}_d(t)=\frac{n_d(1+e_d\cos\nu_d(t))^2}{(1-e_d^2)^{\frac{3}{2}}} \tag{23}$$

and

$$\ddot{\nu}_d(t)=\frac{-2n_d^2e_d(1+e_d\cos\nu_d(t))^3\sin\nu_d(t)}{(1-e_d^2)^3}, \tag{24}$$

with $n_d=\sqrt{\mu/a_d^3}\in\mathbb{R}$ as the desired mean motion of the leader, and $a_d\in\mathbb{R}$ and $e_d\in\mathbb{R}$ as the semimajor axis and the eccentricity of the desired spacecraft orbit, respectively.

The desired trajectory is an elliptic orbit around the Earth, and hence $e_d \in (0,1)$. We will assume that the control of the leader spacecraft is sufficiently good, such that even under disturbances the following hold:

Assumption 1: Define $\tilde{\nu} := \nu_l - \nu_d$, where ν_l and ν_d are the actual and the desired true anomaly, respectively. We assume that the actuation system of the leader spacecraft keeps $\dot{\tilde{\nu}}$ and $\ddot{\tilde{\nu}}$ bounded, i.e. $\dot{\tilde{\nu}} \leq \beta_{\dot{\tilde{\nu}}}$ and $\ddot{\tilde{\nu}} \leq \beta_{\ddot{\tilde{\nu}}}$ for all $t \geq t_0 \geq 0$, where $\beta_{\dot{\tilde{\nu}}}, \beta_{\ddot{\tilde{\nu}}}$ are positive constants.

In addition we will make the following assumptions regarding the desired trajectories of the follower spacecraft:

Assumption 2: The desired relative position $p_{f,d}(t)$, desired relative velocity $\dot{p}_{f,d}(t)$ and desired relative acceleration $\ddot{p}_{f,d}(t)$ are all smooth and bounded functions, i.e. there exists positive constants $\beta_{p_{f,d}}$, $\beta_{\dot{p}_{f,d}}$, $\beta_{\ddot{p}_{f,d}}$ such that $|p_{f,d}(t)| \leq \beta_{p_{f,d}}$, $|\dot{p}_{f,d}(t)| \leq \beta_{\dot{p}_{f,d}}$ and $|\ddot{p}_{f,d}(t)| \leq \beta_{\ddot{p}_{f,d}}$ for all $t \geq t_0 \geq 0$.

Finally, we assume that the disturbances acting on the spacecraft are bounded.

Assumption 3: The disturbances acting on the follower spacecraft are bounded, i.e. there exist a positive constant β_{f_f} such that

$$|f_f(t)| \leq \beta_{f_f} \tag{25}$$

and that the difference between thrust and external disturbances acting on the leader spacecraft is bounded, that is:

$$|u_l(t) + f_l(t)| \leq \beta_{(u_l+f_l)} \tag{26}$$

for a positive constant $\beta_{(u_l+f_l)}$.

IV. Controller-Observer Design

In this section the controller scheme of [16] as redefined for output feedback in [10] will be used.

A. Without disturbance

Define $e_f := p_f - p_{f,d} \in \mathbb{R}^3$ as the position error and $\tilde{p}_f := p_f - \hat{p}_f$ as the observer estimation error. Let the controller of the follower spacecraft be:

$$\begin{aligned} u_f = &M_f\ddot{p}_{f,d} + C_f(\dot{\nu}_l)\dot{p}_{f,d} + D_f(\ddot{\nu}_l, \dot{\nu}_l, r_f)p_f \\ &+ n_f(r_l, r_f) - K_{f,d}(\dot{p}_{f,0} - \dot{p}_{f,r}) \end{aligned} \tag{27}$$

$$\dot{p}_{f,r} = \dot{p}_{f,d} - \Lambda_f e_f \tag{28}$$

$$\dot{p}_{f,0} = \dot{\hat{p}}_f - \Lambda_f \tilde{p}_f, \tag{29}$$

where $\Lambda_f = \Lambda_f^\top \in \mathbb{R}^{3\times3} > 0$, $K_{f,d} \in \mathbb{R}^{3\times3} := k_{f,d}I$ with $k_{f,d} \in \mathbb{R} > m_f\lambda_M(\Lambda_f) + \sqrt{12}m_f\beta_{\dot{\nu}}$. Let the observer be:

$$\dot{\hat{p}}_f = a_f + L_{f,d}\tilde{p}_f \tag{30}$$

$$\dot{a}_f = \ddot{p}_{f,d} + L_{f,p2}\tilde{p}_f, \tag{31}$$

where $L_{f,d} \in \mathbb{R}^{3\times3} := l_{f,d}I + \Lambda_f$ and $L_{f,p2} \in \mathbb{R}^{3\times3} := l_{f,d}\Lambda_f$, with $l_{f,d} \in \mathbb{R} > \frac{2}{m_f}k_{f,d}$ scalar.

The following Proposition was also given in [9, Proposition 2].

Proposition 1: Let Assumption 2 hold. Assume that $u_l + f_l = 0$, $f_f = 0$ and that $\dot{\nu}_l$ and $\ddot{\nu}_l$ are known and bounded. Then the origin of (15), in closed loop with the controller (27-29) and the observer (30-31) is uniformly globally exponentially stable.

Proof: All the calculations in the proof can be found in [9]. A shorter version is given here for later reference. By combining the dynamic equations of the formation with the equations for the proposed controller, the closed-loop tracking error dynamics are found to be

$$M_f\ddot{e}_f + C_f(\dot{\nu}_l)\dot{e}_f + K_{f,d}(\dot{p}_{f,0} - \dot{p}_{f,r}) = 0, \tag{32}$$

since $p_f - p_{f,d} = e_f$. Now, defining the sliding variables $s_{f,1}, s_{f,2} \in \mathbb{R}^3$ as

$$s_{f,1} := \dot{p}_f - \dot{p}_{f,r} = \dot{e}_f + \Lambda_f e_f \tag{33}$$

$$s_{f,2} := \dot{p}_f - \dot{p}_{f,0} = \dot{\tilde{p}}_f + \Lambda_f\tilde{p}_f, \tag{34}$$

we get the tracking error dynamics

$$M_f\dot{s}_{f,1} = M_f\Lambda_f\dot{e}_f - C_f(\dot{\nu}_l)\dot{e}_f - K_{f,d}(s_{f,1} - s_{f,2}), \tag{35}$$

since $\dot{p}_{f,0} - \dot{p}_{f,r} = s_{f,1} - s_{f,2}$. The observer error dynamics is

$$M_f\dot{s}_{f,2} = -C_f(\dot{\nu}_l)\dot{e}_f - K_{f,d}(s_{f,1} - s_{f,2}) - M_f l_{f,d}s_{f,2}. \tag{36}$$

Let the Lyapunov function candidate be given by (cf. [17] and [10])

$$V(x,t) := \frac{1}{2}x^\top W^\top RWx, \tag{37}$$

where $x := (\dot{e}_f^\top, (\Lambda_f e_f)^\top, \dot{\tilde{p}}_f^\top, (\Lambda_f\tilde{p}_f)^\top) \in \mathbb{R}^{12}$, $R := \mathrm{diag}(M_f, 2K_{f,d}\Lambda_f^{-1} - M_f, M_f, 2K_{f,d}\Lambda_f^{-1}) \in \mathbb{R}^{12\times12}$ and

$$W := \begin{bmatrix} I & I & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & I \\ 0 & 0 & 0 & I \end{bmatrix} \in \mathbb{R}^{12\times12}. \tag{38}$$

Note that for $K_{f,d} > M_f\Lambda_f$, we have that

$$k_1 |x|^2 \leq V \leq k_2 |x|^2$$

with $k_1 = \frac{1}{6}\lambda_m(R)$ and $k_2 = \frac{3}{2}\lambda_M(R)$, where $\lambda_m(R) = m_f$ and $\lambda_M(R) = 2k_{f,d}\lambda_m(\Lambda_f)^{-1}$. This can be verified using the fact that $\frac{1}{3} \leq \lambda_m(W^\top W)$ and that $\lambda_M(W^\top W) \leq 3$, where $\lambda_m(W^\top W)$ and $\lambda_M(W^\top W)$ denote the minimum and maximum eigenvalue of $W^\top W$, respectively. The time derivative of the Lyapunov function candidate along the error dynamics (35) and (36) is

$$\begin{aligned} \dot{V} = &-x^\top Qx - s_{f,2}^\top(l_{f,d}M_f - 2K_{f,d})s_{f,2} \\ &-(s_{f,1}^\top + s_{f,2}^\top)C_f(\dot{\nu}_l)\dot{e}_f, \end{aligned} \tag{39}$$

where $Q := \mathrm{diag}(K_{f,d} - M_f\Lambda_f, K_{f,d}, K_{f,d}, K_{f,d}) \in \mathbb{R}^{12\times12}$. By using that $l_{f,d} \geq \frac{2}{m_f}k_{f,d}$ and that the true anomaly rate is bounded, i.e. $\dot{\nu} \leq \beta_{\dot{\nu}}$, we get that

$$\begin{aligned} \dot{V} &\leq -(k_{f,d} - m_f\lambda_M(\Lambda_f) - \sqrt{12}m_f\beta_{\dot{\nu}})|x|^2 \\ &\leq -k_3|x|^2, \end{aligned} \tag{40}$$

where k_3 is a positive constant. It has also been used that $\left|(s_{f,1} + s_{f,2})^\top C_f(\dot{\nu}_l)\dot{e}_f\right| \leq |s_{f,1} + s_{f,2}|\,|C_f(\dot{\nu}_l)\dot{e}_f| \leq \sqrt{12}m_f\beta_{\dot{\nu}}|x|^2$ where we wrote $(s_{f,1} + s_{f,2})$ as $y^\top x$ ($y \in \mathbb{R}^{12}$ is a vector with all elements equal to 1) and used that $\left|y^\top x\right| \leq |y|\,|x| = \sqrt{12}\,|x|$. Hence, according to [3, Theorem 4.10], the origin of the system is UGES. ■

B. With disturbances

In the previous section it is assumed that the true anomaly, $\dot{\nu}_l$, and true anomaly rate of change, $\ddot{\nu}_l$, of the leader spacecraft are available. Since these parameters can be considered as velocity and acceleration parameters, we will now treat the case where the true values of $\dot{\nu}_l$ and $\ddot{\nu}_l$ are unknown.

Let the controller of the follower spacecraft be:

$$\begin{aligned} u_f =& M_f \ddot{p}_{f,d} + C_f(\dot{\nu}_d)\dot{p}_{f,d} + D_f(\ddot{\nu}_d, \dot{\nu}_d, r_f)p_f \\ &+ n_f(r_l, r_f) - K_{f,d}(\dot{p}_{f,0} - \dot{p}_{f,r}) \end{aligned} \quad (41)$$

$$\dot{p}_{f,r} = \dot{p}_{f,d} - \Lambda_f e_f \quad (42)$$

$$\dot{p}_{f,0} = \dot{\hat{p}}_f - \Lambda_f \tilde{p}_f, \quad (43)$$

with observer (30-31). Note the difference in equation (41) from that of equation (27) in that the parameters $\dot{\nu}_d, \ddot{\nu}_d$, of the desired trajectory of the leader spacecraft are used, instead of the actual parameters, $\dot{\nu}_l, \ddot{\nu}_l$, of the leader spacecraft orbit. By using the error of the true anomaly, $\tilde{\nu} = \nu_l - \nu_d$, we get that the tracking error dynamics are

$$\begin{aligned} M_f \ddot{e}_f =& - C_f(\dot{\nu}_l)\dot{e}_f - K_{f,d}(s_{f,1} - s_{f,2}) \\ &- \frac{m_f}{m_l}u_l + f_f - \frac{m_f}{m_l}f_l - 2m_f\bar{C}\dot{\tilde{\nu}}\dot{p}_{f,d} \\ &- m_f\bar{D}\dot{\tilde{\nu}}^2 p_{f,d} - m_f\bar{C}\ddot{\tilde{\nu}}p_{f,d}. \end{aligned} \quad (44)$$

Similarly the observer error dynamics, using the observer (30) and (31) become

$$\begin{aligned} M_f \ddot{\tilde{p}}_f =& - M_f\Lambda_f\dot{\tilde{p}}_f - C_f(\dot{\nu}_l)\dot{e}_f - K_{f,d}(s_{f,1} - s_{f,2}) \\ &- M_f l_{f,d} s_{f,2} + f_f - \frac{m_f}{m_l}(u_l + f_l) \\ &- 2m_f\bar{C}\dot{\tilde{\nu}}\dot{p}_{f,d} - m_f\bar{D}\dot{\tilde{\nu}}^2 p_{f,d} - m_f\bar{C}\ddot{\tilde{\nu}}p_{f,d}. \end{aligned} \quad (45)$$

Proposition 2: Let Assumption 1-3 hold. The controller given by (41)-(43) and observer (30)-(31) in closed loop with (15) is UGPES on the parameter set $\Theta = \mathbb{R}^2_{>0}$, with $\theta = (k_{f,d}, l_{f,d}) \in \Theta$ as tuning parameters.

Proof: The proof is done by applying Theorem 1. Using (37) as the Lyapunov function candidate, we get that its time derivative along (44) and (45) is:

$$\begin{aligned} \dot{V} \leq& - \left(k_{f,d} - m_f\lambda_M(\Lambda_f) - \sqrt{12}m_f\beta_{\dot{\tilde{\nu}}}\right)|x|^2 \\ &+ \sqrt{12}\Big(\beta_f + \frac{m_f}{m_l}\beta_{(u_l+f_l)} + 2m_f\beta_{\dot{\tilde{\nu}}}\beta_{\dot{p}_{f,d}} \\ &+ m_f(\beta_{\dot{\tilde{\nu}}^2} + \beta_{\ddot{\tilde{\nu}}})\beta_{p_{f,d}}\Big)|x|, \end{aligned}$$

by similar calculations as in the Proof of Proposition 1. Let δ be any positive constant. Pick $l_{f,d} \geq l^\star_{f,d} := \frac{2}{m_f}k_{f,d}$. Pick $k_{f,d} \geq k^\star_{f,d}$, where

$$\begin{aligned} k^\star_{f,d} :=& 2m_f\lambda_M(\Lambda_f) + 2\sqrt{12}m_f\beta_{\dot{\tilde{\nu}}} \\ &+ \frac{2\sqrt{12}}{\delta}\Big(\beta_f + \frac{m_f}{m_l}\beta_{(u_l+f_l)} + 2m_f\beta_{\dot{\tilde{\nu}}}\beta_{\dot{p}_{f,d}} \\ &+ m_f(\beta_{\dot{\tilde{\nu}}^2} + \beta_{\ddot{\tilde{\nu}}})\beta_{p_{f,d}}\Big) \end{aligned} \quad (46)$$

Then, for any $|x| \geq \delta$ we have that

$$\dot{V} \leq -\frac{1}{2}k^\star_{f,d}|x|^2 \quad (47)$$

and we can apply Theorem 1 with $p = 2$, $V_\delta = V$, $\underline{\kappa}(\delta) = \frac{1}{6}\lambda_m(R) = \frac{1}{6}m_f$, $\overline{\kappa} = \frac{3}{2}\lambda_M(R(\delta)) = 3k^\star_{f,d}(\delta)/\lambda_M(\Lambda_f)$ and $\kappa(\delta) = \frac{1}{2}k^\star_{f,d}(\delta)$. Hence (4) and (5) of Theorem 1 are fulfilled. Finally we have

$$\lim_{\delta\to 0}\frac{\overline{\kappa}(\delta)\delta^p}{\underline{\kappa}(\delta)} = \lim_{\delta\to 0}\frac{18k^\star_{f,d}(\delta)\delta^2}{\lambda_M(\Lambda_f)m_f} = 0, \quad (48)$$

thus (6) is also satisfied and we can conclude UGPES of the driving subsystem (15), in closed loop with the controller (27), (28), (29) and observer (30), (31). ■

V. Simulations

In this section the performance of the controller-observer scheme will be illustrated by simulations. The desired orbit of the leader spacecraft is of eccentricity $e_d = 0.5$, and with semimajor axis $a_d = 20000$ km. The true anomaly rate and true anomaly rate-of-change are generated by (23) and (24). We want to illustrate the robustness of our controller-observer scheme even under perturbed motion of the leader spacecraft. For that reason the leader spacecraft is simulated according to (13) with $u_l + f_l = (0.5\sin\frac{1}{10}t, 0.2\sin\frac{1}{100}t, 0.3\sin\frac{1}{1000}t)$ to illustrate a control system that is not able to handle the periodic forces that an orbiting spacecraft are exposed to. The true anomaly rate and rate-of-change of the leader spacecraft are achieved by differentiation of (22). The desired trajectory of the follower spacecraft is given by $p_d(t) = (-10\cos\nu, 20\sin\nu, 0)$, which means that the follower spacecraft evolves around the leader spacecraft in an ellipse during their orbit around the Earth. This is a fuel efficient orbit, as it is close to a natural orbit of the spacecraft. We assume that the follower spacecraft is exposed to similar perturbations as the leader spacecraft, and we have chosen that $f_f = (0.1\sin\frac{1}{100}t, 0.3\sin\frac{1}{10}t, 0.4\sin t)$. The initial position and velocity of the follower spacecraft is chosen as $p(0) = (-10, 5, 7)$ and $\dot{p}(0) = (1, 0, -1)$, where as the initial states of the observer are $\hat{p}(0) = (4, -4, 1)$ and $z(0) = (-1, 4, 2)$. The controller and observer gains are as follows: $l_d = 0.5$, $K_d = 20I_{3\times3}$, $\Lambda = 0.06I_{3\times3}$. Both spacecraft are of mass $m_l = m_f = 100$ kg. Furthermore, the thrust is assumed to be continuous and available in all directions of the leader spacecraft frame, but limited to $\max u_f = 10$ N. Figure 1 and 2 show the tracking and estimation errors, respectively. As seen from Figure 1 the tracking error is big, but, as proven in the previous section, this error can be arbitrarily diminished by an appropriate choice of control gains, e.g. by increasing K_d. The control history is shown in Figure 3. The actuation of the follower spacecraft would be greatly reduced by a better controlled leader spacecraft, as we use $\dot{\nu}_d$ and $\ddot{\nu}_d$, instead of the actual parameters $\dot{\nu}_l$ and $\ddot{\nu}_l$ for the true anomaly rate and rate-of-change. To further save fuel, one can imagine that control parameters are changed so as extensive actuation is used only when high accuracy formation control is needed, e.g. only during performance of measurement.

VI. Conclusion

We have stated definitions for UGPES, USES and USPES and provided Lyapunov-like sufficient conditions for them

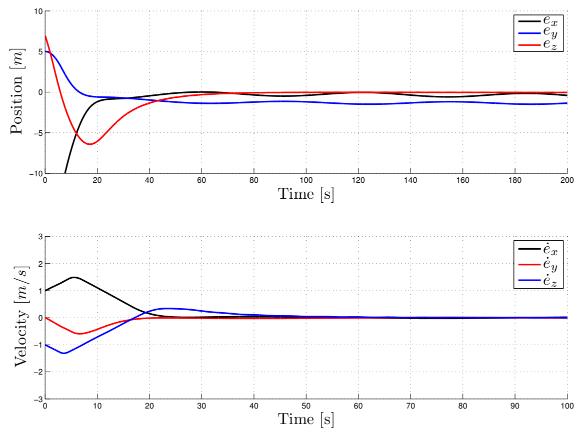

Fig. 1. Position and velocity tracking errors

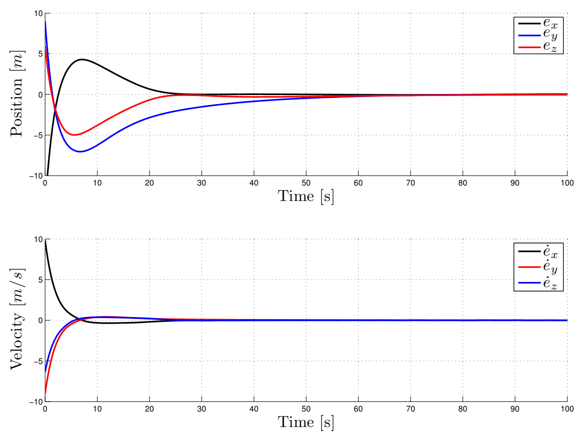

Fig. 2. Position and velocity estimation errors

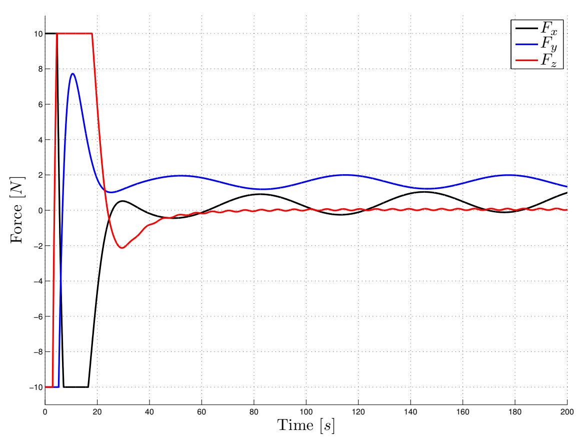

Fig. 3. Control history

to hold. Their utility was demonstrated through the stability analysis of a spacecraft formation under external disturbances.

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