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Controller synthesis for positive 2D systems described by the Roesser model

A. Hmamed, M. Ait Rami and M. Alfidi

Abstract— **This paper deals with the stability synthesis for a class of 2D linear systems described by the Roesser model. We provide necessary and sufficient conditions for stability, as well as stabilization for linear positive Roesser systems. This kind of systems have the property that the states take nonnegative values whenever the initial boundaries are nonnegative. The synthesis of state-feedback controllers, including the requirement of positivity of the controllers and the extension of the results to uncertain plants are solved in terms of Linear Programming. A numerical example is included to illustrate the proposed approach.**

Keywords: 2D systems, stability, stabilization, positive systems, positive control, positive Roesser model, linear programming.

I. INTRODUCTION

In last two decades, the two-dimensional (2D) system theory has been payed a considerable attention and developed by many researchers. The 2D linear models were introduced in the seventies [7], [10] and have found many applications, such as in digital data filtering, image processing [19], modeling of partial differential equations [18], etc. In connection with Roesser [19] and Fornasini-Marchesini [8] models, some important problems such as realization, controllability, minimum energy control, has been extensively investigated (see for example [13]). However, on the other hand, the stabilization problem is not fully investigated and still not completely solved.

The stability of 2D discrete linear systems can be reduced to checking the stability of 2D characteristic polynomial [22], [5]. This seems to be difficult task for the control synthesis problem. In the literature, various types of easily checkable but only sufficient conditions for asymptotic stability and stabilization problems for 2D discrete linear systems have been proposed [17], [16], [23], [9]. Recently, we observe a growing interest in the theory and application of positive 2D systems [12], [14], [6], [20], [21]. It seems that, originally, the positive 2D Roesser systems has been first studied in [11] (more detailed description can be found in [14]). In the present paper, we first analyze the stability of positive 2D Roesser model [11], [19] and obtain necessary and sufficient condition for its stability. On the other hand, we propose a simple numerical method for a complete treatment of the stabilization problem of positive 2D Roesser systems. This method is based on a previous approach for 1D positive systems initiated in [2], [3], [4], [1] where some synthesis problems are solved in term of Linear Programming (LP). In addition, based on this approach we provide LP necessary and sufficient conditions for the stabilization problem with positive controls. The robust stabilization problem in the presence of polytopic uncertainties is also investigated.

The remainder of the paper is structured as follows: In section 2 the problem under study is formulated and some preliminary results are given. Section 3 studies the stability analysis of positive 2D Roesser systems. Section 4 studies the synthesis problem and its extension to the robust case. In section 5 numerical example is given to illustrate the proposed results.

Notation: The following notation well be used throughout this paper. $\mathbf{N}$ denotes the set of integer numbers. R^n denote the n-dimensional Euclidean space and $R^{m\times n}$ denotes the set of $m \times n$ real matrices. The notation $M > 0$ (resp. $M \geq 0$), where M is a real matrix (or a vector), means that all the components of M are strictly positive (resp. nonnegative). For a complex number z, the quantity $|z|$ represents its modulus.

II. PROBLEM FORMULATION AND PRELIMINARIES

Consider the following 2D system described by Roesser model [19]:

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(i,j), \tag{1}$$

where $A_{11} \in R^{n_1\times n_1}$, $A_{12} \in R^{n_1\times n_2}$, $A_{21} \in R^{n_2\times n_1}$, $A_{22} \in R^{n_2\times n_2}$, $B_1 \in R^{n_1\times m}$, $B_2 \in R^{n_2\times m}$ are given constant real matrices. The vectors $x^h(i,j) \in R^{n_1}$ and $x^v(i,j) \in R^{n_2}$ are, respectively, the horizontal and vertical states at the point (i,j) and the vector $u(i,j) \in R^m$ is an input signal of System (1). Boundary initial conditions for System (1) are given by two sequences (x_0^h) and (x_0^v) such that:

$$\begin{cases} x^h(0,j) &= x_0^h(j) \quad \forall j \in \mathbf{N}, \\ x^v(i,0) &= x_0^v(i) \quad \forall i \in \mathbf{N}. \end{cases} \tag{2}$$

In the sequel, the following definition will be used.

Definition 2.1: System (1) with zeros input $u = 0$, is called positive if for any given nonnegative boundary conditions $x_0^h(j) \geq 0$ and $x_0^v(i) \geq 0$, the resulting states are also nonnegative $x(i,j) \geq 0, \ \forall i,j \in \mathbf{N}$.

The following result shows how one can check the positivity of System (1) (see [14]).

A. Hmamed and M. Alfidi are with LESSI, Dpartement de Physique Faculté des Sciences, Dhar El Mehraz B.P : 1796, 30000 Fes-Atlas Morocco.
M. Ait Rami is with the Departamento de Ingenieria de Sistemas y Automatica, University of Valladolid, 47005 Valladolid, Spain, e-mail:aitrami@autom.uva.es

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Proposition 2.1: System (1) with zeros input $u = 0$, is a positive system if and only if all the components of the matrix

$$A := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

are nonnegative (or by notation $A \geq 0$).

Asymptotic stability for general Roesser model [19] has been extensively studied in the literature. A well-known necessary and sufficient frequency condition for asymptotic stability is stated in the following.

Lemma 2.1: Let $A_{11} \in R^{n_1 \times n_1}$, $A_{12} \in R^{n_1 \times n_2}$, $A_{21} \in R^{n_2 \times n_1}$, $A_{22} \in R^{n_2 \times n_2}$ be given constant real matrices. Then, 2D system described by the Rosser model (1) with zeros input $u = 0$, is asymptotically stable if and only if the following condition holds

$$\begin{cases} \mathbf{det}(\begin{bmatrix} I_{n_1} - z_1 A_{11} & -z_1 A_{12} \\ -z_2 A_{21} & I_{n_2} - z_2 A_{22} \end{bmatrix}) \neq 0, \\ \forall (z_1, z_2) \in \{(z_1, z_2) : |z_1| \leq 1, |z_2| \leq 1\}. \end{cases} \quad (3)$$

In the sequel, our purpose is to investigate the existence of state-feedback control laws $u(i,j) = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}$ such that the resulting closed-loop system:

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} A_{11} + B_1 K_1 & A_{12} + B_1 K_2 \\ A_{21} + B_2 K_1 & A_{22} + B_2 K_2 \end{bmatrix} \times \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}, \quad (4)$$

is positive and asymptotically stable. Of course, if we utilize directly the results of Lemma 2.1 and Proposition 2.1, we have the following necessary and sufficient condition for the closed-loop system to be positive and asymptotically stable:

$$\begin{bmatrix} A_{11} + B_1 K_1 & A_{12} + B_1 K_2 \\ A_{21} + B_2 K_1 & A_{22} + B_2 K_2 \end{bmatrix} \geq 0,$$

$$det\{\begin{bmatrix} I_{n_1} - z_1(A_{11} + B_1 K_1) & -z_1(A_{12} + B_1 K_2) \\ -z_2(A_{21} + B_2 K_1) & I_{n_2} - z_2(A_{22} + B_2 K_2) \end{bmatrix}\} \neq 0 \quad (5)$$

$$\forall (z_1, z_2) \in \{(z_1, z_2) : |z_1| \leq 1, |z_2| \leq 1\}.$$

However, this is a very complicated formulation, since it leads to a very hard problem to solve numerically. As we can see we have a linear constraint (positivity constraint) but also mixed with the very highly nonlinear infinite dimensional constraint (asymptotic stabilizability constraint as stated above).

III. Stability Analysis

This section provides preliminary stability results for the free linear 2D system described by the Roesser model:

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}. \quad (6)$$

In fact, it will be shown that the asymptotic stability of System (6) (under the positivity constraint) is equivalent to the asymptotic stability of the following 1D discrete-time system:

$$x(k+1) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x(k). \quad (7)$$

We also use the following definition for 1D discrete-time system (7).

Definition 3.1: System (7) is called positive if for any given nonnegative initial conditions $x(0) \geq 0$, the resulting states are also nonnegative $x(i) \geq 0, \ \forall i \in \mathbf{N}$.

Obviously, the positiveness of System (7) is given by:

Proposition 3.1: System (7) is a positive system if and only if all the components of the matrix

$$A := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

are nonnegative (or by notation $A \geq 0$).

Next, recall that the spectral radius $\rho(M)$ of a matrix $M \in R^{n \times n}$ is defined as:

$$\rho(M) = \max\{|\lambda_1|, \ldots, |\lambda_n|\},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of M. Also, for a complex matrix $N = [n_{ij}]$ we define the real matrix $|N|$ as the matrix formed by the components $|n_{ij}|$.

Now, in order to establish our main stability result, we need some technical key role results which are provided by the following well-known lemmas.

Lemma 3.1: **[15]** Let M be a real matrix and N be a complex matrix such that $|N| \leq M$ ($M - |N|$ is a nonnegative matrix), then $\rho(N) \leq \rho(M)$.

Lemma 3.2: **[4]** Assume that the matrices A_{11}, A_{12}, A_{21} and A_{22} are constant nonnegative matrices (or equivalently that System (7) is positive). Then, the following statements are equivalent:

(i) 1D system described by (7) is asymptotically stable.

(ii) $\rho(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}) < 1$.

(iii) There exists a vector $d \in R^{n_1+n_2}$ such that

$$\begin{bmatrix} A_{11} - I_{n_1} & A_{12} \\ A_{21} & A_{22} - I_{n_2} \end{bmatrix} d < 0, \ \ d > 0. \quad (8)$$

In what follows we present new necessary and sufficient condition with regard to the asymptotic stability of 2D positive system described by the Roesser model (6).

Theorem 3.1: Assume that the system (6) is positive or equivalently that the matrices A_{11}, A_{12}, A_{21} and A_{22} are nonnegative. Then, the following statements are equivalent:

(i) $\begin{cases} \mathbf{det}(\begin{bmatrix} I_{n_1} - z_1 A_{11} & -z_1 A_{12} \\ -z_2 A_{21} & I_{n_2} - z_2 A_{22} \end{bmatrix}) \neq 0, \\ \forall (z_1, z_2) \in \{(z_1, z_2) : |z_1| \leq 1, |z_2| \leq 1\}. \end{cases}$

(ii) $\rho(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}) < 1.$

Proof: $(i) \Rightarrow (ii)$ by setting $z_1 = z_2 = z$ in condition (i), then we have obviously

$$det(I - z \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}) \neq 0, |z| \leq 1, \qquad (9)$$

which, in turn, is equivalent to the condition (ii).
$(ii) \Rightarrow (i)$ Let z_1 and z_2 be any arbitrary complex numbers such that $|z_1| \leq 1, |z_2| \leq 1$. Thus, we can easily see that

$$\begin{bmatrix} |z_1 A_{11}| & |z_1 A_{12}| \\ |z_2 A_{21}| & |z_2 A_{22}| \end{bmatrix} \leq \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

then, by using the spectral property given in Lemma 3.1, we obtain

$$\rho(\begin{bmatrix} z_1 A_{11} & z_1 A_{12} \\ z_2 A_{21} & z_2 A_{22} \end{bmatrix}) \leq \rho(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}) < 1.$$

Since z_1 and z_2 are arbitrary complex number with modulus less or equal to one, then the above inequality, in turn, implies condition (i) and the proof is complete. ■

Now, we are in position to state the main result of this section.

Corollary 3.1: The following statements are equivalent:

(i) The 2D system described by Roesser model (6) is positive and asymptotically stable.
(ii) The 1D system described by (7) is positive and asymptotically stable.
(iii) The matrices $A_{11}, A_{12}, A_{21}, A_{22}$ are nonnegative and there exists $d \in R^{n_1+n_2}$ such that:

$$\begin{bmatrix} A_{11} - I_{n_1} & A_{12} \\ A_{21} & A_{22} - I_{n_2} \end{bmatrix} d < 0, \quad d > 0.$$

Proof: Recall that the equivalence $(ii) \Leftrightarrow (iii)$ results from Lemma 3.2 and then the proof will be complete if we only show $(i) \Leftrightarrow (iii)$.
$(i) \Rightarrow (iii)$ First, using Proposition 2.1 we have that $A_{11}, A_{12}, A_{21}, A_{22}$ are nonnegative. Next, since by Lemma 2.1 the asymptotic stability of the 2D system (6) is equivalent to

$$\begin{cases} \mathbf{det}(\begin{bmatrix} I_{n_1} - z_1 A_{11} & -z_1 A_{12} \\ -z_2 A_{21} & I_{n_2} - z_2 A_{22} \end{bmatrix}) \neq 0, \\ \forall (z_1, z_2) \in \{(z_1, z_2) : |z_1| \leq 1, |z_2| \leq 1\}, \end{cases}$$

which by Theorem 3.1 is also equivalent to $\rho(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}) < 1,$. Finally by using Lemma 3.2 this implies (iii).
Reciprocally, to show that $(iii) \Rightarrow (i)$ it suffices to follow the same line of arguments by utilizing Proposition 2.1 combined with (in this order) Lemma 3.2, Theorem 3.1 and Lemma 2.1. ■

IV. Controller synthesis

This section studies the stabilization problem of linear 2D systems described by Roesser model for which the control law to be investigated has the state-feedback form $u(i,j) = K \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}$. This control law is designed in order to ensure the positivity and the asymptotic stability of the resulting closed-loop system.

Remarks 4.1: we point out that our proposed approach does not impose any restriction on the dynamics of the governed system. For instance, the free Roesser model can be possibly non positive. In this case, our synthesis design can be interpreted as enforcing the Roesser system to be positive.

Now, consider the following closed-loop Roesser model:

$$\begin{aligned} \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} &= (A + BK) \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}, \\ x^h(0,j) &= x_0^h(j), \ \forall j \in \mathbf{N}, \\ x^v(i,0) &= x_0^v(i), \ \forall i \in \mathbf{N}, \end{aligned} \qquad (10)$$

where $A := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $B := \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ are supposed to be any kind of given real matrices.

In what follows we provide the main result of this section.

Theorem 4.1: The closed-loop Roesser system (10) is positive and asymptotically stable for any nonnegative initial boundary conditions, if and only if there exist n+1 vectors $d = [d_1 \dots d_n]^T \in R^n$ and $y_1 \dots y_n \in R^m$ such that

$$\begin{cases} (A - I_n)d + B(\sum_{i=1}^{n} y_i) < 0, \\ d > 0, \\ a_{ij} d_j + b_i y_j \geq 0, \ \ 1 \leq i, j \leq n, \end{cases} \qquad (11)$$

with $A = [a_{ij}]$, $B^T = [b_1^T \dots b_n^T]$ *and* $n = n_1 + n_2$. Moreover, the gain matrix K is computed as:

$$K = [d_1^{-1} y_1 \dots d_n^{-1} y_n]. \qquad (12)$$

Proof: Assume that condition (11) is satisfied and define the appropriate matrix $K = [k_1, \dots, k_n]$ with the columns constructed as follows $k_i = d_i^{-1} y_i$ for $i = 1, \dots, n$. Now, by this construction, it is easy to see that $A + BK$ is nonnegative matrix. Effectively, from the last inequalities in condition (11) we have for $i, j = 1, \dots, n$:

$$0 \leq (a_{ij} d_j + b_i y_j) d_j^{-1} = a_{ij} + b_i k_j = (A + BK)_{ij}.$$

Next, we show the asymptotic stability under the feedback control $u(i,j) = K \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}$. Using the previous constructed gain, we obtain by calculation $BKd = B(\sum_{i=1}^{n} y_i)$, which is utilized in first inequality condition (11) and leads to $(A + BK - I_n)d < 0$. Now, since $d > 0$ and the matrix $(A + BK)$ is nonnegative, then by using Corollary 3.1, we conclude that the 2D system described by the closed-loop Roesser model (10) is positive and asymptotically stable.

The rest of the proof follows the same line of arguments and then is omitted. ∎

In the following, we show that the positivity of stabilizing controls can be also handled by using a similar LP approach and a similar proof as it is provided in the preceding result.

Theorem 4.2: The following statements are equivalent:

(i) There exists a positive state-feedback law $u = K\begin{bmatrix} x^h \\ x^v \end{bmatrix} \geq 0$ such that the closed-loop Roesser system (10) is positive and asymptotically stable for any initial boundary conditions.

(ii) There exists a matrix $K \in R^{m\times n}$ such that $K \geq 0$ and $A+BK$ is a nonnegative Schur matrix ($\rho(A+BK) < 1$).

(iii) The following LP problem in the variables $d = [d_1 \dots d_n]^T \in R^n$ and $y_1 \dots y_n \in R^m$ is feasible.

$$\begin{cases} (A-I_n)d + B(\sum_{i=1}^{n} y_i) < 0, \\ d > 0, \\ y_i \geq 0, \;\; 1 \leq i \leq n, \\ a_{ij}d_j + b_i y_j \geq 0, \;\; 1 \leq i,j \leq n, \end{cases} \tag{13}$$

with $A = [a_{ij}]$, $B^T = [b_1^T \dots b_n^T]$ *and* $n = n_1 + n_2$. Moreover, the gain matrix in conditions (i) and (ii) can be chosen as:

$$\text{K}=[d_1^{-1}y_1 \dots d_n^{-1}y_n],$$

where d and $y_1,\dots,y_n$ are given by any feasible solution to the above LP problem

Now, some significant remarks are provided.

Remarks 4.2: Note that if a negative state-feedback control law is to be considered it suffices to impose $y_i \leq 0$ instead of $y_i \geq 0$ in the previous LP formulation.

Remarks 4.3: We emphasize that our LP formulation in Theorem 4.1 and Theorem 4.2 does not impose any restriction on the dynamics of the governed system. In fact, the matrix A may not be nonnegative matrix, or equivalently, the autonomous system is not positive. Hence, our synthesis design can be viewed as enforcing the system to be positive.

Remarks 4.4: A positive unstable Roesser linear system cannot be stabilized by any positive state-feedback control law if the matrix B is nonnegative. Effectively, the existence of a stabilizing positive state-feedback control (by Theorem 4.2) necessarily implies that $(A-I_n)d < 0$, $d > 0$ which means that A must be a Schur matrix. This is impossible, since that, originally, the positive Roesser system is unstable.

V. Extension to uncertain plants

An important issue in the control design is the *robust stability*, that is, ensuring stability under uncertainty or against possible perturbations. Henceforth, it is of great interest to study the robust stabilization of Roesser systems for which the dynamics are not exactly known or subject to uncertainties that are captured in a polytopic domain.

Consider the following uncertain system:

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = A_\alpha \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + B_\alpha u(i,j), \tag{14}$$

where the matrices $A_\alpha \in R^{n\times n}$ and $B_\alpha \in R^{n\times m}$ are supposed to be not exactly known but they are assumed to belong to the following convex set:

$$\begin{bmatrix} A_\alpha & B_\alpha \end{bmatrix} \in \mathbf{D},$$

$$D := \Big\{ \sum_{i=1}^{l} \alpha_i \begin{bmatrix} A^i & B^i \end{bmatrix}, \sum_{i=1}^{l} \alpha_i = 1, \; \alpha_i \geq 0 \Big\},$$

where $[A^1 \;\; B^1], \dots, [A^l \;\; B^l]$ are known matrices.

The proposed robust synthesis design consists in finding a single constant gain matrix K for which the following closed-loop system is positive and asymptotically stable for every $[A_\alpha \;\; B_\alpha] \in \mathbf{D}$:

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = (A_\alpha + B_\alpha K) \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}. \tag{15}$$

This kind of uncertainties in the model (15) can be directly handled by the the following result.

Theorem 5.1: There exists a robust state-feedback law $u = K\begin{bmatrix} x^h \\ x^v \end{bmatrix}$ such that the resulting closed-loop system (15) is positive and asymptotically stable for any initial boundary conditions and for every $[A_\alpha \;\; B_\alpha] \in \mathbf{D}$, if the following LP problem in the variables $d = [d_1 \dots d_n]^T \in R^n$ and $y_1, \dots, y_n \in R^m$, is feasible.

$$\begin{cases} (A^k - I_n)d + B^k(\sum_{i=1}^{n} y_i) < 0, \; for\; k = 1,...,l \\ d > 0, \\ a_{ij}^k d_j + b_i^k y_j \geq 0, \;\; 1 \leq i,j \leq n, \; k = 1,...,l \end{cases} \tag{16}$$

with $A^k = [a_{ij}^k]$, $B^{k^T} = [b_1^{k^T} \dots b_n^{k^T}]$, *and* $n = n_1 + n_2$. Moreover, the gain matrix of a robust controller can be computed as

$$K = [d_1^{-1}y_1 \dots d_n^{-1}y_n]$$

where d and $y_1, \dots, y_n$ are any feasible solution to the above LP problem

Proof: By a simple convexity argument the proof is straightforward. ∎

A. Numerical Example

In this part, we illustrate the applicability of our approach by treating an uncertain Roesser system. Note that the proposed methodology is not restricted to positive systems (see remark 4.1).

We consider the following Roesser system (14) subject to a parametric perturbation as follows:

$$A_\alpha = \left[\begin{array}{cc|c} -1.5 & 0.1 & 0 \\ 0.2 & 0.5 & 0.3 \\ \hline 1 & 1 & 0.2 - 0.01\alpha \end{array}\right],$$

$$B_\alpha = \left[\begin{array}{c} 1 \\ 1 - 0.01\alpha \\ \hline 0 \end{array}\right],$$

where $0 \leq \alpha \leq 1$.

We look for a robust state-feedback control which stabilizes and enforces the positivity of all the plants between the two extreme plants ($\alpha = 0$ and $\alpha = 1$). By applying Theorem 5.1, the following conditions must be satisfied.

$$\begin{bmatrix} 1.5 & 0 & 0 & -1 & 0 & 0 \\ 0 & -0.1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ -0.2 & 0 & 0 & -1 & 0 & 0 \\ 0 & -0.5 & 0 & 0 & -1 & 0 \\ 0 & 0 & -0.3 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} < 0,$$

$$\begin{bmatrix} -2.5 & 0.1 & 0 & 1 & 1 & 1 \\ 0.2 & -0.5 & 0.3 & 1 & 1 & 1 \\ 1 & 1 & -0.8 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0.2 & -0.5 & 0.3 & 0.99 & 0.99 & 0.99 \\ 1 & 1 & -0.81 & 0 & 0 & 0 \\ -0.2 & 0 & 0 & -0.99 & 0 & 0 \\ 0 & -0.5 & 0 & 0 & -0.99 & 0 \\ 0 & 0 & -0.3 & 0 & 0 & -0.99 \\ 0 & 0 & -0.19 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} < 0.$$

One feasible solution to the above LP problem is:

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9.7490 \\ 192.8424 \\ 280.9313 \\ 15.922 \\ -14.7241 \\ 1.4626 \end{bmatrix},$$

from which we calculate the gain of the robust stabilizing controller (as $K = [y_1 d_1^{-1} \;\; y_2 d_2^{-1} \;\; y_3 d_3^{-1}]$):

$$K = [1.6332 \;\; -0.0764 \;\; 0.0052].$$

Hence, with this gain all the closed-loop systems between the two extreme plants ($\alpha = 0$ and $\alpha = 1$) are positive and asymptotically stable.

VI. Conclusion

In this paper, we have provided an approach for solving the stability synthesis problem for positive 2D systems described by the Roesser model. Necessary and sufficient conditions for the solvability of the stabilization problem have been proposed. Also, it has been shown how our method can take into account the positivity of the control laws and also the uncertainties in the model. It has been proved that all the proposed conditions are solvable in terms of Linear Programming.

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