

On the use of approximated predictive control laws for nonlinear systems

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Abstract—The problem of efficient nonlinear model predictive control (NMPC) implementation is investigated, using an approximating function $\hat{\kappa}$ to avoid on-line optimization. At first, sufficient conditions are given for $\hat{\kappa}$ to guarantee a finite computable bound on the approximation error (i.e. the difference between the exact and approximated control moves). Then, additional conditions are obtained to make such a bound arbitrary small. This result makes it possible to derive guaranteed closed loop stability properties. Finally, it is shown that set membership (SM) nonlinear function approximation theory can be employed to improve the performance of $\hat{\kappa}$. The resulting “fast” model predictive control law is given by the sum of $\hat{\kappa}$ with a SM approximated function and satisfies the above-mentioned conditions even if they are not met by $\hat{\kappa}$ alone. A nonlinear oscillator example shows the effectiveness of the proposed methodology.

I. INTRODUCTION

Nonlinear Model Predictive Control (NMPC) (see e.g. [1]) is a model based control technique where the control action is computed by means of a receding horizon (RH) strategy, which requires at each sampling time the solution of an optimization problem. For time invariant systems, the control move u_t at time t is a nonlinear static function of the system state x_t , i.e. $u_t = \kappa^0(x_t)$, evaluated implicitly by solving on-line the optimization. The application of predictive techniques has received an increasing attention in industrial world due to its efficiency in constraints handling. However, the RH strategy leads to strong limitations in using MPC techniques in the presence of fast plant dynamics, which require small sampling periods that do not allow to perform the optimization problem on-line. In order to allow the use of MPC to a larger range of applications, a significant research effort has been devoted in recent years to the problem of fast implementation of model predictive control laws. A viable solution to this problem is the use of an approximated control law $\hat{\kappa} \approx \kappa^0$, with low computational time, to be used for on-line implementation. In this context, a first contribution was given in [2], who considered the use of a neural approximation of κ^0 . However, computational problems may arise with such approach, related to the “curse of dimensionality” (causing an exponential dimension increase in the neural network parameter space) and to possible deteriorations in the approximation, due to trapping in local minima. Moreover,

no guaranteed approximation error and constraint satisfaction properties were obtained. Another methodology to approximate a NMPC controller has been proposed in [3], using an off-line algorithm for the construction of a piecewise affine (PWA) approximation of the nominal predictive control law and its implementation via a binary search tree. Stability properties can be obtained in this case. However, with this approach the computational times depend on the number of the state space partitions, which increases as the required error tolerance decreases. Moreover, the stability and constraint satisfaction properties rely on the assumption of the convexity of the optimal cost function. If such assumption is not met, ad-hoc solutions have to be used. Another technique to approximate a nonlinear MPC controller has been used in [4], by approximating the nonlinear system model with a set of PWA systems over the state space and computing for each one a PWA solution of the quadratic constrained finite-time optimal control (see e.g. [5] and [6]). Then, a set of off-line solutions of such PWA control laws is considered and a polynomial interpolation technique is employed to compute an approximation of the overall control law. However, the approximation of a given nonlinear model with a set of PWA systems is not a trivial task and model approximation errors are introduced. Moreover, no guarantees are given on the stabilizing properties of the computed polynomial law. A further technique has been considered in [7], [8] and [9], where approximated MPC laws, with guaranteed performance and stability properties, have been derived using set membership (SM) function approximation theory. Such techniques have been also applied to problems like control of power kites for energy generation ([10]) and control of semi-active suspension systems ([11]).

In the described scenario, the aim of this paper is to investigate the properties of generic approximated NMPC laws, with particular regard to their approximation accuracy. The paper contribution is twofold. At first, sufficient conditions are given for a generic approximating function $\hat{\kappa}$, derived with any technique, to obtain a finite approximation error and to make such error arbitrary small. This makes it possible to guarantee closed loop stability and constraint satisfaction properties (see e.g. [8]). Then, it is shown how SM function approximation theory can be employed to improve the performance of $\hat{\kappa}$. The resulting “fast” model predictive control law is given by the sum of $\hat{\kappa}$ and of a SM approximated function and satisfies the above-mentioned conditions even if they are not met by $\hat{\kappa}$ alone. A nonlinear oscillator example shows the effectiveness of the proposed methodology.

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II. NONLINEAR MODEL PREDICTIVE CONTROL

Consider the following nonlinear state space model:

$$x_{t+1} = f(x_t, u_t) \quad (1)$$

where $x_t \in \mathbb{R}^n$ and $u_t \in \mathbb{R}^m$ are the system state and control input respectively. In this paper, it is assumed that function f in (1) is continuous over $\mathbb{R}^n \times \mathbb{R}^m$. Assume that the control objective is to regulate the system state to the origin under some input and state constraints represented by a compact set $\mathbb{U} \subseteq \mathbb{R}^m$ and a convex set $\mathbb{X} \subseteq \mathbb{R}^n$ respectively, both containing the origin in their interiors.

Denoting by N_p and $N_c \leq N_p$ the prediction horizon and the control horizon respectively, the following objective function J can be defined:

$$J(U, x_{t|t}, N_p) = \Phi(x_{t+N_p|t}) + \sum_{j=0}^{N_p-1} L(x_{t+j|t}, u_{t+j|t})$$

where $x_{t+j|t}$ denotes j step ahead state prediction using the model (1), given the input sequence $u_{t|t}, \dots, u_{t+j-1|t}$ and the ‘‘initial’’ state $x_{t|t} = x_t$. $U = [u_{t|t}^T, \dots, u_{t+N_c-1|t}^T]^T$ is the vector of the control moves to be optimized. The remaining predicted control moves $[u_{t+N_c|t}, \dots, u_{t+N_p-1|t}]$ can be computed with different strategies, e.g. by setting $u_{t+j|t} = u_{N_c-1|t}$ or $u_{t+j|t} = K x_{t+j|t}$, $\forall j \in [N_c, N_p - 1]$, where K is a suitable matrix.

The NMPC control law is then obtained applying the following RH strategy:

- 1) At time instant t , get x_t .
- 2) Solve the optimization problem:

$$\min_U J(U, x_{t|t}, N_p) \quad (2a)$$

s. t.

$$x_{t+j|t} \in \mathbb{X}, j = 1, \dots, N_p \quad (2b)$$

$$u_{t+j|t} \in \mathbb{U}, j = 0, \dots, N_p \quad (2c)$$

- 3) Apply the first element of the solution sequence U to the optimization problem as the actual control action $u_t = u_{t|t}$.
- 4) Repeat from step 1) at time $t + 1$.

It is assumed that the optimization problem (2) is feasible over a set $\mathcal{F} \subseteq \mathbb{R}^n$ which will be referred to as the ‘‘feasibility set’’. The application of such RH procedure implicitly defines the predictive controller as a nonlinear static function κ^0 of the state, i.e.:

$$u_t = \kappa^0(x_t)$$

It is supposed that the nominal control law κ^0 is continuous over the feasibility set \mathcal{F} . Such property has been investigated e.g. in [12], where the continuity of the control law is studied in the context of general finite-horizon nonlinear optimal control. Other results can be found in [13]. Note that stronger regularity assumptions (e.g. differentiability) cannot be made, since even in the simple case of linear dynamics, linear constraints and quadratic objective function, κ^0 is a piece-wise linear continuous

function (see e.g. [5] and [6]).

III. APPROXIMATED NMPC LAWS: ACCURACY RESULTS

In the standard NMPC formulation, the nominal control law κ^0 is evaluated by solving the optimization problem (2) on-line. However, a limitation in the practical use of NMPC is the presence of fast plant dynamics, for which the required sampling time may be too low for real-time optimization. A viable solution to this problem is the use of an approximated control function $\hat{\kappa} \approx \kappa^0$, derived off-line, whose on-line evaluation time is smaller.

It is considered that $\hat{\kappa}$ is defined over a compact set \mathcal{X} , containing the origin in its interior, such that:

$$\hat{\kappa} : \mathcal{X} \rightarrow \mathbb{R}, \mathcal{X} \subseteq \mathcal{F}$$

Moreover, $\hat{\kappa}$ is computed on the basis of the knowledge of a finite number ν of exact control moves, i.e.:

$$\tilde{u}^k = \kappa^0(\tilde{x}^k), k = 1, \dots, \nu \quad (3)$$

where the state values \tilde{x}^k are suitably chosen and define the set:

$$\mathcal{X}_\nu = \{\tilde{x}^k, k = 1, \dots, \nu\} \subseteq \mathcal{X}$$

In [8], it is shown that if $\hat{\kappa}$ has the following *key properties*:

- i) Input constraint satisfaction:

$$\hat{\kappa}(x) \in \mathbb{U} \quad \forall x \in \mathcal{X} \quad (4)$$

- ii) The pointwise approximation error $\Delta_{\hat{\kappa}}(x) = \kappa^0(x) - \hat{\kappa}(x)$ is bounded:

$$\|\Delta_{\hat{\kappa}}(x)\| \leq \zeta, \quad \forall x \in \mathcal{X} \quad (5)$$

where $\|\cdot\|$ is a suitable norm, e.g. Euclidean.

- iii) The bound $\zeta(\nu)$ converges to zero:

$$\lim_{\nu \rightarrow \infty} \zeta(\nu) = 0 \quad (6)$$

then guaranteed closed loop stability can be achieved, as well as arbitrary small performance degradation, in terms of Euclidean distance between the state trajectories obtained with the nominal and the approximated control laws.

As regards the asymptotic behaviour of ζ as $\nu \rightarrow \infty$, it is assumed that \mathcal{X}_ν is chosen such that the following property holds:

$$\lim_{\nu \rightarrow \infty} d_H(\mathcal{X}, \mathcal{X}_\nu) = 0 \quad (7)$$

where $d_H(X, X_\nu)$ is the Hausdorff distance between \mathcal{X} and \mathcal{X}_ν (see e.g. [14]):

$$d_H(\mathcal{X}, \mathcal{X}_\nu) = \max \left(\sup_{x \in \mathcal{X}} \inf_{\tilde{x} \in \mathcal{X}_\nu} (\|x - \tilde{x}\|_2), \sup_{\tilde{x} \in \mathcal{X}_\nu} \inf_{x \in \mathcal{X}} (\|\tilde{x} - x\|_2) \right)$$

Note that uniform gridding over \mathcal{X} satisfies condition (7).

Thus, in this paper sufficient conditions are given for a generic approximating function $\hat{\kappa}$, derived with any approximation method (e.g. interpolation, neural networks etc.), to have properties (4)-(6). For simplicity of notation, in the remaining of the paper it will be assumed that $\kappa^0 : \mathbb{R}^n \rightarrow \mathbb{R}$ and that $\mathbb{U} = \{u \in \mathbb{R} : \underline{u} \leq u \leq \bar{u}\}$, where $\underline{u}, \bar{u} \in \mathbb{R}$.

Since the nominal control law κ^0 is assumed to be continuous over \mathcal{F} , and both \mathcal{X} and \mathbb{U} are compact, then κ^0 is Lipschitz continuous over \mathcal{X} . Thus, κ^0 belongs to the following function set:

$$\kappa^0 \in \mathcal{A}_{L_{\kappa^0}} = \{\kappa : \mathcal{X} \rightarrow \mathbb{U}, |\kappa(x^1) - \kappa(x^2)| \leq L_{\kappa^0} \|x^1 - x^2\|_2, \forall x^1, x^2 \in \mathcal{X}\} \quad (8)$$

where L_{κ^0} is the Lipschitz constant of κ^0 over \mathcal{X} . The first contribution introduced in this paper is to derive sufficient conditions for any given approximating function $\hat{\kappa}$ to obtain a computable bound on the norm $|\Delta_{\hat{\kappa}}(x)|$ of the pointwise approximation error $\Delta_{\hat{\kappa}}(x)$:

$$\Delta_{\hat{\kappa}}(x) \doteq \kappa^0(x) - \hat{\kappa}(x)$$

From the knowledge of the ν exact control moves computed off-line (3), the exact values of $\Delta_{\hat{\kappa}}(\tilde{x})$ are known:

$$\Delta_{\hat{\kappa}}(\tilde{x}) = \tilde{u} - \hat{\kappa}(\tilde{x}), \forall \tilde{x} \in \mathcal{X}_\nu$$

The following Theorem shows how to compute a bound on $|\Delta_{\hat{\kappa}}(x)|$ on the basis of the knowledge of $\Delta_{\hat{\kappa}}(\tilde{x})$.

Theorem 1: Let κ^0 have property (8). If $\hat{\kappa}$ is Lipschitz continuous with Lipschitz constant $L_{\hat{\kappa}}$ and has property (4), then:

- i) the approximation error $\Delta_{\hat{\kappa}}$ is a Lipschitz continuous function over \mathcal{X} , with Lipschitz constant $L_{\Delta_{\hat{\kappa}}}$ bounded as:

$$L_{\Delta_{\hat{\kappa}}} \leq L_{\hat{\kappa}} + L_{\kappa^0} \quad (9)$$

- ii) $|\Delta_{\hat{\kappa}}(x)|$ is bounded:

$$|\Delta_{\hat{\kappa}}(x)| \leq \zeta, \forall x \in \mathcal{X}$$

- iii) A bound ζ can be computed as:

$$\zeta = \sup_{x \in \mathcal{X}} \max(\overline{\Delta_{\hat{\kappa}}}(x), -\underline{\Delta_{\hat{\kappa}}}(x)) \quad (10)$$

where

$$\begin{aligned} \overline{\Delta_{\hat{\kappa}}}(x) &\doteq \min[\overline{u} - \hat{\kappa}(x), \min_{\tilde{x} \in \mathcal{X}_\nu} (\Delta_{\hat{\kappa}}(\tilde{x}) + L_{\Delta_{\hat{\kappa}}} \|x - \tilde{x}\|_2)] \\ \underline{\Delta_{\hat{\kappa}}}(x) &\doteq \max[\underline{u} - \hat{\kappa}(x), \max_{\tilde{x} \in \mathcal{X}_\nu} (\Delta_{\hat{\kappa}}(\tilde{x}) - L_{\Delta_{\hat{\kappa}}} \|x - \tilde{x}\|_2)] \end{aligned} \quad (11)$$

- iv) if $L_{\hat{\kappa}} \leq L_{\Delta_{\hat{\kappa}}}$, the bound ζ (10) is the tightest on the basis of the available information on κ^0

Proof: See [15] ■

Remark 1: Note that if the approximation method employed to derive $\hat{\kappa}$ does not guarantee input constraint satisfaction, condition (4) can be imposed by modifying $\hat{\kappa}$ as follows:

$$\hat{\kappa}_S(x) = \begin{cases} \hat{\kappa}(x) & \text{if } \underline{u} \leq \hat{\kappa}(x) \leq \overline{u} \\ \underline{u} & \text{if } \hat{\kappa}(x) < \underline{u} \\ \overline{u} & \text{if } \hat{\kappa}(x) > \overline{u} \end{cases}$$

Remark 2: As regards the computation of the Lipschitz constant L_{κ^0} , the following estimate can be employed:

$$\hat{L}_{\kappa^0} = \inf (L : \tilde{u}^h + L \|\tilde{x}^h - \tilde{x}^k\|_2 \geq \tilde{u}^k, \forall k, h = 1, \dots, \nu) \quad (12)$$

As shown in [8], this estimate is such that:

$$\lim_{\nu \rightarrow \infty} \hat{L}_{\kappa^0} = L_{\kappa^0}$$

Remark 3: Depending on the properties of $\hat{\kappa}$, the Lipschitz constant $L_{\hat{\kappa}}$ can be computed analytically or numerically or using a procedure similar to (12).

Remark 4: Note that the bound (9) on the Lipschitz constant of the approximation error $\Delta_{\hat{\kappa}}(x)$ may be too conservative. Alternatively, an estimate $\hat{L}_{\Delta_{\hat{\kappa}}}$ of $L_{\Delta_{\hat{\kappa}}}$ can be computed using a procedure similar to (12).

According to Theorem 1, a bound $\zeta(\nu)$ on the approximation error can be computed for any continuous approximated control law $\hat{\kappa}$ and any value of ν , thus satisfying property (5). The next Theorem gives the additional condition needed to satisfy also property (6), i.e. the capability of guaranteeing an arbitrary small approximation error.

Theorem 2: Let \mathcal{X}_ν be chosen such that (7) holds. Let κ^0 satisfy property (8). If $\hat{\kappa}$ is Lipschitz continuous with Lipschitz constant $L_{\hat{\kappa}}$, such that (4) holds, and satisfies the following property (data interpolation):

$$\hat{\kappa}(\tilde{x}) = \tilde{u}, \forall \tilde{x} \in \mathcal{X}_\nu \quad (13)$$

then, in addition to the results i)–ii) of Theorem 1, the following hold:

- i) the bound ζ on the approximation error norm can be computed as:

$$\zeta = \sup_{x \in \mathcal{X}} \min[\max(\overline{u} - \hat{\kappa}(x), -\underline{u} + \hat{\kappa}(x)), \eta(x)] \quad (14)$$

where

$$\eta(x) = \min_{\tilde{x} \in \mathcal{X}_\nu} (L_{\Delta_{\hat{\kappa}}} \|x - \tilde{x}\|_2)$$

- ii) $\zeta(\nu)$ converges to zero:

$$\lim_{\nu \rightarrow \infty} \zeta(\nu) = 0$$

Proof: See [15] ■

Theorem 2 can be used to compute an upper bound on the error obtained using any approximated control law $\hat{\kappa}$ which interpolates the off-line computed data and to “tune” ν to guarantee a given desired accuracy. This is sufficient to guarantee that the closed-loop stability and performance properties obtained with control law $\hat{\kappa}$ are arbitrarily close to those of κ^0 , in terms of Euclidean distance between their respective closed-loop trajectories (see [8] and [9] for further details).

IV. OPTIMAL APPROXIMATIONS OF NMPC LAWS

In this Section, the problem of deriving approximating functions $\hat{\kappa}$ fulfilling the hypotheses of Theorem 2 is studied. Indeed, standard methods, e.g. based on expansions in term of suitable basis functions (polynomials, sigmoids, wavelets, etc.) could be used. However, it is well known that in general, as the number of basis functions needs to be increased in

order to achieve the interpolation condition (13), the approximation error $e(\kappa^0, \hat{\kappa}) = \|\kappa^0 - \hat{\kappa}\|_\infty = \sup_{x \in \mathcal{X}} |\kappa^0(x) - \hat{\kappa}(x)|$ may become very large (curse of dimensionality). Thus, the problem is to find, among all functions $\hat{\kappa}$ fulfilling the conditions of Theorem 2, an approximation $\hat{\kappa}$ of κ^0 which gives low (possibly minimal) approximation error. Let us define more precisely the optimization problem we investigate. The function κ^0 to be approximated is assumed to belong to the Feasible Function Set defined as:

$$FFS = \{\kappa : \mathcal{X} \rightarrow \mathbb{U} : \kappa \in \mathcal{A}; \kappa(\tilde{x}) = \tilde{u}, \forall \tilde{x} \in \mathcal{X}_\nu\} \quad (15)$$

where \mathcal{A} is some given subset of continuous functions. The aim is to find κ^{OPT} as the solution of the optimization problem:

$$\kappa^{\text{OPT}} = \arg \min_{\kappa \in FFS} \sup_{\kappa^0 \in FFS} e(\kappa^0, \hat{\kappa}) = \arg \min_{\kappa \in FFS} E(\kappa^0, \kappa)$$

where $E(\kappa^0, \kappa) = \sup_{\kappa^0 \in FFS} e(\kappa^0, \kappa)$ is the guaranteed (i.e. worst-case) approximation error. Note that such a κ^{OPT} , if found, satisfies the conditions required by Theorem 2 and has the minimal guaranteed approximation error $E(\kappa^0, \kappa^{\text{OPT}})$ achievable from the considered information on κ^0 , summarized in the FFS , which in turn depends on the known values (3) and on other (possibly qualitative) information described by \mathcal{A} .

A solution to this problem has been given in [8], in the case that, using the knowledge that κ^0 is Lipschitz continuous over \mathcal{X} , the set $\mathcal{A} = \mathcal{A}_{L_{\kappa^0}}$ defined by (8) is considered.

It is clear that the more detailed information on κ^0 is used, the lower is the guaranteed approximation error $E(\kappa^0, \kappa^{\text{OPT}})$. For example, the set \mathcal{X} can be subdivided in subsets \mathcal{X}^i over which κ^0 has Lipschitz constants $L_{\kappa^0}^i \leq L_{\kappa^0}$. Using the corresponding κ_i^{OPT} derived in [8] as approximating function of κ^0 on each subset \mathcal{X}^i could lead to significant reductions of the guaranteed approximation error, especially in the subregions where $L_{\kappa^0}^i \ll L_{\kappa^0}$. As the number of subdivisions grows, this approach allows to use information on the “local” Lipschitz constants of κ^0 .

A simpler approach is now presented, allowing to use such “local” information to derive an optimal approximation satisfying the conditions of Theorem 2, starting from a preliminary approximating function $\hat{\kappa}$ which satisfies conditions for Theorem 1 only.

Consider the residue function $\Delta_{\hat{\kappa}} = \kappa^0 - \hat{\kappa}$ which, on the basis of Theorem 1, is Lipschitz continuous over \mathcal{X} with Lipschitz constant $L_{\Delta_{\hat{\kappa}}}$. Then, the information available on κ^0 is summarized by the following FFS_Δ :

$$FFS_\Delta = \{\kappa : \mathcal{X} \rightarrow \mathbb{U}, \kappa - \hat{\kappa} \in \mathcal{A}_{L_{\Delta_{\hat{\kappa}}}}, \kappa(\tilde{x}) = \tilde{u}, \forall \tilde{x} \in \mathcal{X}_\nu\} \quad (16)$$

where

$$\mathcal{A}_{L_{\Delta_{\hat{\kappa}}}} = \{\Delta : \mathcal{X} \rightarrow \mathbb{R}, |\Delta(x^1) - \Delta(x^2)| \leq L_{\Delta_{\hat{\kappa}}} \|x^1 - x^2\|_2, \forall x^1, x^2 \in \mathcal{X}\} \quad (17)$$

Consider the following function:

$$\Delta_{\hat{\kappa}}^{\text{OPT}}(x) = \frac{1}{2} [\overline{\Delta}_{\hat{\kappa}}(x) + \underline{\Delta}_{\hat{\kappa}}(x)] \quad (18)$$

Theorem 3: For any given function $\hat{\kappa}$ satisfying the conditions of Theorem 1, if $L_{\hat{\kappa}} \leq L_{\Delta_{\hat{\kappa}}}$ the function $\tilde{\kappa}^{\text{OPT}} = \hat{\kappa} + \Delta_{\hat{\kappa}}^{\text{OPT}}$ has the following properties:

- i) $\tilde{\kappa}^{\text{OPT}} \in FFS_\Delta$
- ii) $\tilde{\kappa}^{\text{OPT}}$ is an optimal approximation of κ^0 with respect to the information $\kappa^0 \in FFS_\Delta$:

$$\begin{aligned} & \sup_{\kappa^0 \in FFS_\Delta} e(\kappa^0, \tilde{\kappa}^{\text{OPT}}) = \\ & = \inf_{\tilde{\kappa} \in FFS_\Delta} \sup_{\kappa^0 \in FFS_\Delta} e(\kappa^0, \tilde{\kappa}) = r_{\Delta, \infty} \end{aligned}$$

Proof: See [15] ■

According to Theorem 3, SM theory can be employed to improve the performance of a given approximating function $\hat{\kappa}$. The resulting approximated NMPC law $\tilde{\kappa}^{\text{OPT}}$ satisfies conditions for Theorem 2 to hold, and for a fixed value of ν it gives the minimum guaranteed approximation error. Note that the approach presented in [8], which employs an optimal SM approximation κ^{OPT} of a MPC law for linear systems, is a particular case of the results presented in this paper, i.e. using $\hat{\kappa} = 0$. In this case, the guaranteed error bound is given by the L_∞ -norm radius of information r_∞ of the set FFS (15) with $\mathcal{A} = \mathcal{A}_{L_{\kappa^0}}$ (8). It is worth investigating when the use of $\hat{\kappa} \neq 0$ improves the worst-case accuracy, giving lower guaranteed approximation errors. Indeed, since $|\kappa^0(x) - \kappa^{\text{OPT}}(x)| \leq r_\infty$ and $|\kappa^0(x) - \tilde{\kappa}^{\text{OPT}}(x)| \leq r_{\Delta, \infty}$, it can be noted that if:

$$r_{\Delta, \infty} < r_\infty \quad (19)$$

then the guaranteed accuracy obtained with $\tilde{\kappa}^{\text{OPT}}$ is higher than the one given by κ^{OPT} . As a consequence, a lower number ν of off-line computed values are sufficient for $\tilde{\kappa}^{\text{OPT}}$ to achieve given guaranteed stability and performance properties according to [8]. Lower ν numbers may lead to lower function evaluation times, depending on the computational burden of $\hat{\kappa}$. Condition (19) can be evaluated numerically, e.g. using the results of [16].

As a final remark, note that condition $L_{\hat{\kappa}} \leq L_{\Delta_{\hat{\kappa}}}$ can be checked by computing or estimating (e.g. using (12)) the Lipschitz constants $L_{\hat{\kappa}}$ and $L_{\Delta_{\hat{\kappa}}}$. Moreover, such assumption can be always satisfied using a preliminary approximating function $\hat{\kappa}$ whose complexity is not too high with respect to κ^0 , with the extreme case of $\hat{\kappa} = 0$, i.e. $L_{\hat{\kappa}} = 0$. For example, if $\hat{\kappa}$ is computed as the expansion of basis functions, it is possible to improve the obtained accuracy by gradually increasing the number of basis functions: in this case the value of $L_{\hat{\kappa}}$ may grow and condition $L_{\hat{\kappa}} \leq L_{\Delta_{\hat{\kappa}}}$ can be used as a stopping criterium, avoiding also data over-fitting. Then, the optimal SM approximation $\Delta_{\hat{\kappa}}^{\text{OPT}} \approx \Delta_{\hat{\kappa}}$ can be designed to further improve the performance of $\hat{\kappa}$.

V. NUMERICAL EXAMPLE

Consider the two-dimensional nonlinear oscillator obtained from the Duffing equation (see e.g. [17]):

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = u(t) - 0.6x_2(t) - x_1(t)^3 - x_1(t) \end{cases}$$

where the input constraint set \mathbb{U} is:

$$\mathbb{U} = \{u \in \mathbb{R} : |u| \leq 5\}$$

The following discrete time model to be used in the nominal MPC design has been obtained by forward difference approximation:

$$x_{t+1} = \begin{bmatrix} 1 & T_s \\ -T_s & (1 - 0.6T_s) \end{bmatrix} x_t + \begin{bmatrix} 0 \\ T_s \end{bmatrix} u_t + \begin{bmatrix} 0 & 0 \\ -T_s & 0 \end{bmatrix} x_t^3$$

with sampling time $T_s = 0.05$ s. The nominal MPC controller κ^0 is designed according to (2) with horizons $N_p = 100$, $N_c = 5$ and the following functions L and Φ :

$$L(x, u) = x^T Q x + u^T R u, \quad \Phi = 0$$

where

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 0.5$$

The following linear state inequality constraints define the considered set \mathbb{X} :

$$\mathbb{X} = \{x \in \mathbb{R}^2 : \|x\|_\infty \leq 3\}$$

The state prediction has been performed setting $u_{t+j|t} = u_{t+N_c-1|t}$, $j = N_c, \dots, N_p - 1$. The optimization problem (2) employed to compute $\kappa^0(x)$ has been solved using a sequential constrained Gauss-Newton quadratic programming algorithm (see e.g. [18]), where the underlying quadratic programs have been solved using the MatLab[®] function `quadprog`. The mean computational time of the on-line optimization was $4.3 \cdot 10^{-2}$ s, using MATLAB[®] 7 with an AMD Athlon(tm) 64 3200+ with 1 GB RAM.

Fig. 1 shows the obtained feasibility set \mathcal{F} and the set \mathcal{X} considered for the approximation, together with the constraint set \mathbb{X} . The level curves of the optimal cost function

$$J^*(x) = \min_U J(U, x)$$

are reported too: it can be noted that $J^*(x)$ is not convex, thus the technique proposed in [3] cannot be applied without ad-hoc modifications to guarantee closed loop stability and constraint satisfaction properties. On the other hand, the set membership technique proposed in this paper can be systematically employed since κ^0 results to be continuous. A set \mathcal{X}_ν of $\nu = 1 \cdot 10^4$ off-line computed exact control moves has been considered to derive the approximating functions. The values of $\tilde{x} \in \mathcal{X}_\nu$ have been chosen with uniform gridding over \mathcal{X} . The following approximating functions have been considered:

- i) Neural network approximation, obtained considering the set \mathcal{X}_ν in the design phase:

$$\hat{\kappa}_{\text{NS}}^{\text{NN}} = \sum_{i=1}^7 \alpha_i \tanh(\beta_i^1 x_1 + \beta_i^2 x_2 + \gamma_i) + \alpha_0$$

where $\alpha \in \mathbb{R}^8$, $\beta^1 \in \mathbb{R}^7$, $\beta^2 \in \mathbb{R}^7$ and $\gamma \in \mathbb{R}^7$ are suitable weights. To satisfy condition (4), function $\hat{\kappa}_{\text{NS}}^{\text{NN}}$ has been then modified as:

$$\hat{\kappa}^{\text{NN}}(x) = \begin{cases} \hat{\kappa}_{\text{NS}}^{\text{NN}}(x) & \text{if } -5 \leq \hat{\kappa}_{\text{NS}}^{\text{NN}}(x) \leq 5 \\ -5 & \text{if } \hat{\kappa}_{\text{NS}}^{\text{NN}}(x) < -5 \\ 5 & \text{if } \hat{\kappa}_{\text{NS}}^{\text{NN}}(x) > 5 \end{cases}$$

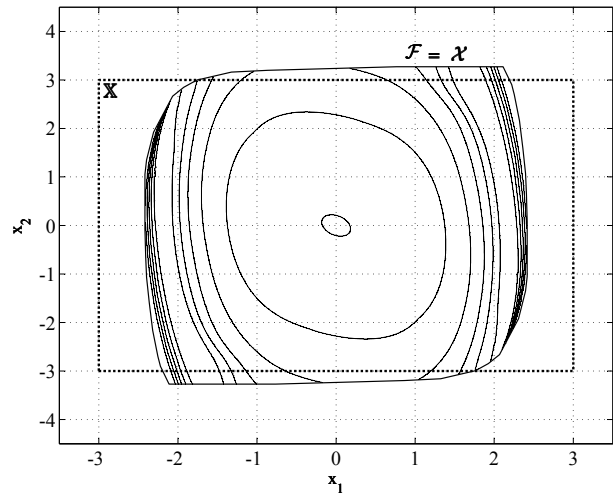


Fig. 1. Duffing oscillator example: sets \mathcal{F} and \mathcal{X} (thick solid line), constraint set \mathbb{X} (thick dotted line) and level curves of the optimal cost function $J^*(x)$.

- ii) Function $\tilde{\kappa}^{\text{OPT,NN}}$ obtained by adding to $\hat{\kappa}^{\text{NN}}$ the optimal SM approximation of the residue function $\kappa^0 - \hat{\kappa}^{\text{NN}}$, evaluated off-line at the points $\tilde{x} \in \mathcal{X}_\nu$
- iii) Function κ^{OPT} obtained directly by performing the optimal SM approximation of κ^0 , evaluated off-line at the points $\tilde{x} \in \mathcal{X}_\nu$

Fig. 2 shows the state trajectories obtained considering the initial condition $x_0 = [1, -3.1]^T$, outside the state constraints. It can be noted that all the approximated controllers are able to regulate the state to the origin and the related trajectories are practically superimposed. Moreover all the approximated controllers satisfy the state constraints. The

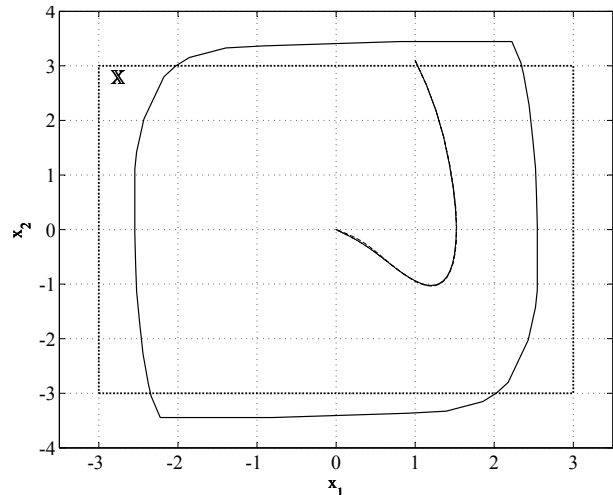


Fig. 2. Duffing oscillator example: state trajectories obtained with the nominal NMPC controller (solid), $\hat{\kappa}^{\text{NN}}$ (dashed), $\tilde{\kappa}^{\text{OPT,NN}}$ (dash-dotted) and κ^{OPT} (dotted). Initial condition: $x_0 = [1, -3.1]^T$

courses of the input variable u (Fig. 3) show that input constraints are always satisfied. To evaluate the performance and computational times of the considered control laws, 300 simulations have been performed starting from different

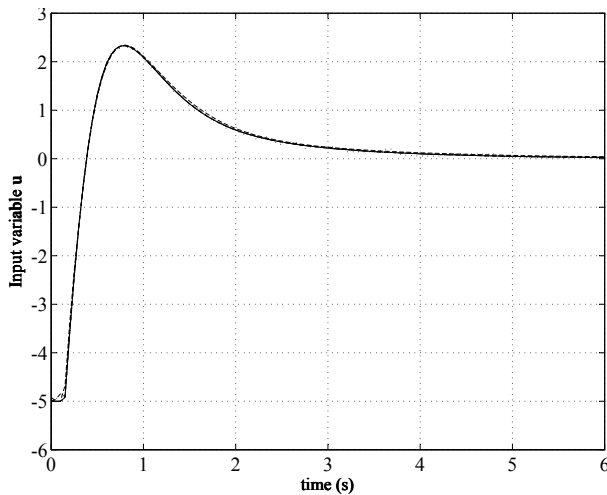


Fig. 3. Duffing oscillator example: courses of input variable u obtained with the nominal NMPC controller (solid), $\hat{\kappa}^{\text{NN}}$ (dashed), $\hat{\kappa}^{\text{OPT,NN}}$ (dash-dotted) and κ^{OPT} (dotted). Initial condition: $x_0 = [1, -3.1]^T$

initial conditions chosen with uniform gridding over \mathcal{X} . Each simulation lasted 600 time steps. The mean computational time \bar{t} , over all time steps of all simulations, obtained with each controller is reported in Table I. As a measure of control system performance, the Euclidean distance d between the closed loop state trajectories obtained with the nominal controller and any of the approximated ones has been considered at each time step. Then, the mean distance \bar{d} over all time steps of all simulations has been computed. The values of \bar{d} obtained with each approximated controller are reported in Table I too. Note that the neural network approximation $\hat{\kappa}^{\text{NN}}$ achieves the lowest value of \bar{t} , however its performance is the worst ($\bar{d} = 110^{-2}$). Function κ^{OPT} has better precision, but also higher computational times. Note that $\hat{\kappa}^{\text{OPT,NN}}$ is able to greatly improve the precision with respect to κ^{OPT} , with the same mean computational time. Indeed, using a lower value of ν in the computation of $\hat{\kappa}^{\text{OPT,NN}}$ a precision similar to that of κ^{OPT} can be obtained, but with faster computational times.

Thus, this example shows how the presented “local” SM approach is able to improve the performance of a given preliminary approximating function, achieving either the same precision of the optimal approach of [8], but with faster computation, or better precision with the same computational times.

TABLE I

DUFFING OSCILLATOR EXAMPLE: MEAN COMPUTATIONAL TIMES AND TRAJECTORY DISTANCES.

	$\hat{\kappa}^{\text{NN}}$	$\hat{\kappa}^{\text{OPT,NN}}$	κ^{OPT}
\bar{d}	110^{-2}	610^{-4}	410^{-3}
\bar{t}	210^{-5} s	110^{-3} s	110^{-3} s

VI. CONCLUSIONS

The use of approximated NMPC laws has been investigated, in order to reduce the computational times. A generic ap-

proximating function $\hat{\kappa}$, derived with any method, has been considered. At first, sufficient conditions have been given for $\hat{\kappa}$ to guarantee a finite computable bound on the approximation error (i.e. the difference with the nominal control law κ^0). Then, additional conditions have been obtained to make such a bound arbitrary small. This result makes it possible to derive guaranteed closed loop stability properties. Finally, it has been shown that set membership (SM) nonlinear function approximation theory can be employed to improve the performance of $\hat{\kappa}$. The resulting “fast” model predictive control law is given by the sum of $\hat{\kappa}$ with a SM approximated function and satisfies the above-mentioned conditions even if they are not met by $\hat{\kappa}$ alone. The only assumptions needed for the presented results to hold are the continuity of κ^0 and $\hat{\kappa}$. A nonlinear oscillator example has been used to show the effectiveness of the proposed methodology in terms of accuracy improvement.

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