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Power Spectral Geodesics and Tracking

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Abstract— **It is often reasonable to assume that a power spectral density can be used to reflect, locally in time, properties of a non-stationary time series; for instance, the short-time Fourier transform is based on this hypothesis and provides a succession of estimated power spectra over successive time intervals. In this paper we consider such families of power spectral densities, indexed by time, to represent changes in the spectral content of non-stationary processes. We propose as a model for the drift in spectral power over time, the model of mass transport; that is, at least locally, spectral power shifts along geodesics of a suitable mass-transport metric. We show that fitting spectral geodesics in the Wasserstein metric to data is a convex quadratic program. Finally, we highlight the effectiveness of this proposition in tracking features (e.g., spectral lines, peaks) of non-stationary processes.**

I. INTRODUCTION

Changes in the frequency content of a signal, whether this is speech, a bird song, the echo of a radar, or the whistle of an incoming train, often carry important and useful information. The desire to resolve and track "frequency components," over time, has led to a vast subject in signal processing known as *time-frequency analysis* [5], [6]. In this, one of the most basic questions is how to deal with time-variability and randomness at the same time. Ensemble averages are not available and time averages smooth out time-varying properties. A compromise is to postulate that the time-variability is relatively tame and that local averages can give accurate information on the frequency content. Thus, one is naturally led to a concept of "time-varying spectra." Traditionally, the most common tool for estimating time-varying spectra is the short-time Fourier transform (STFT). In the present paper, we will not be concerned with the exact correspondence between random time-signals and "time-varying spectra." Instead, we consider that a collection of local-in-time power spectra is available and we propose a non-parametric viewpoint for modeling time-changes. This viewpoint relies on a suitable metric and the corresponding geometry on the space (cone) of power spectral densities.

Driven by applications in speech, imaging, prediction of time-series, etc., a variety of measures have been devised/adapted to quantify distance between spectral density functions (see e.g., [10], [16], [11], [18]) often disregarding the fact that these failed to satisfy the triangular inequality (e.g., Itakura-Saito, Kullback-Leibler, etc.). In recent years there has been a renewed interest in developing natural spectral metrics and to study of the associated geometries [7], [8], [13], [14], [19], [20]. In the present work we consider one such particular metric, the Wasserstein metric. This and other transportation metrics quantify the mismatch between distributions as the optimal cost when transferring the corresponding mass of one to the other; the Wasserstein metric averages the square of the distance that mass needs to be transported. Transportation metrics have been adapted to quantify distance between distributions of unequal mass motivated by a variety of applications (e.g., see [2], [4], [9]).

Because power spectral densities of discrete-time random processes have support on an interval (e.g., $[-\pi,\pi]$), the geometry of mass transport (in one dimension) allows for a rather concrete description of geodesics (see [1], [21]). Our proposition is to view geodesics as the "simplest curves" that may link particular density functions, and thereby, as the simplest models for the evolution of "time-varying spectra". They can be thought of as the analog of straight lines in Euclidean geometry where points would take the place of density functions. Following such a reasoning, a "least-squares" fit of a geodesic to a collection of power spectra is analogous to Gauss' paradigm of fitting a celestial geodesic to observations of planetary motions. Carrying out this program for modeling time-varying spectra is the subject of the present paper.

More specifically, in Section II we discuss the optimal transportation problem. We define the Wasserstein metric, and present certain key facts about the corresponding geodesics. In Section III we formulate the problem of fitting geodesics to a collection of spectral density functions and explain how this can be expressed as a convex quadratic optimization program. Then, in Section IV we discuss representative examples (numerical experiments). In these, we first obtain STFT-estimates of time-varying spectra for simulated time-series. We fit geodesics to interpolate these estimated power spectra, and explain the relevance of such geodesics in capturing the underlying time-varying properties of the data.

II. THE WASSERSTEIN METRIC

Throughout we consider power spectral densities $f(\theta)$, $g(\theta)$, with $\theta \in [-\pi,\pi]$ and normalized to have integral equal to one. The case of non-normalized densities has been considered in [2], [9], [12].

The optimal transportation problem between f and g asks for the minimal cost of transferring mass $f(\theta)d\theta$ to a new location $\hat{\theta} = s(\theta) \in [-\pi,\pi]$ so as to equalize the two

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distributions, i.e., so that for any measurable set $S \subseteq [-\pi, \pi]$

$$\int_S f(\theta)d\theta = \int_{s(S)} g(\hat{\theta})d\hat{\theta}. \tag{1}$$

The transportation cost is given by the integral

$$\int_{-\pi}^{\pi} c(\theta, s(\theta))f(\theta)d\theta$$

for a choice of a measurable and non-negative cost function $c(\theta, \hat{\theta})$. For the special case where

$$c(\theta, \hat{\theta}) = |\theta - \hat{\theta}|^2,$$

the square root of the optimal cost is the Wasserstein 2-distance

$$W_2(f,g) := \inf\{\sqrt{\int_{-\pi}^{\pi} |\theta - s(\theta)|^2 f(\theta)d\theta} \ : \ \text{subject to (1)}\}$$

between f and g. It turns out that this is a metric [1], [21] and, moreover, there is an optimal map

$$s \, : \, \theta \mapsto \hat{\theta} \tag{2}$$

for which the infimum is achieved. In fact, this optimal map is non-decreasing and specified (uniquely) by

$$\int_{-\pi}^{\hat{\theta}} g(\sigma)d\sigma = \int_{-\pi}^{\theta} f(\sigma)d\sigma. \tag{3}$$

In any metric space, geodesics correspond (locally) to the shortest paths between points. Thus, in our case, a geodesic between two density functions f_0 and f_1 is an indexed family f_τ with $\tau \in [0,1]$, of minimal length

$$\int_0^1 W_2(f_\tau, f_{\tau+d\tau}).$$

A lot is known about W_2-geodesics; even in higher dimensions where they follow a gradient flow [21, page 252]. For the case at hand, where f_0, f_1 are one-dimensional densities, the W_2-geodesic connecting the two is unique and specified by

$$\int_{-\pi}^{(1-\tau)\theta + \tau s(\theta)} f_\tau(\sigma)d\sigma = \int_{-\pi}^{\theta} f_0(\sigma)d\sigma \tag{4}$$

where $s(\theta)$ is the optimizer in $W_2(f_0, f_1)$.

III. Geodesic "least-squares" fit

We begin with a sequence of power spectral densities

$$\mathscr{G} := \{g_{\tau_i}(\theta) \, : \, \theta \in [-\pi, \pi] \text{ for } i = 1, \ldots, n\},$$

where τ_i is an increasing sequence of time-indices, normalized so that $\tau_1 = 0$ and $\tau_n = 1$. These power spectra may typically be obtained from time-series data using STFT, and τ_i $(i = 1, \ldots, n)$ may represent the mid-points of the corresponding time-windows. We seek to interpolate this "points" by fitting a W_2-geodesic that deviates least from the estimated power spectra. Thus, our problem can be stated as follows:

Problem 1: Determine a W_2-geodesic f_τ, $\tau \in [0,1]$, which minimizes

$$J_{\mathscr{G}}(f_\tau) := \sum_{i=1}^{n} \left(W_2(f_{\tau_i}, g_{\tau_i})\right)^2.$$

A geodesic f_τ is completely specified by any two of f_0, f_1, or s for which

$$\int_{-\pi}^{s(\theta)} f_1(\sigma)d\sigma = \int_{-\pi}^{\theta} f_0(\sigma)d\sigma, \tag{5}$$

and can be determined using (4). Thus, the optimal choice of f_0, f_1, s needs to be determined from the data, i.e., the spectra $\mathscr{G}$ and the times τ_i $(i = 1, \ldots, n)$.

Computation of $W_2(f_{\tau_i}, g_{\tau_i})$ requires only the correspondence $\tilde{\theta} \mapsto \hat{\theta}$ for which

$$\int_{-\pi}^{\tilde{\theta}} g_{\tau_i}(\sigma)d\sigma = \int_{-\pi}^{\hat{\theta}} f_{\tau_i}(\sigma)d\sigma, \tag{6}$$

for $\tilde{\theta}, \hat{\theta} \in [-\pi, \pi]$. Combining (4) and (6), we have

$$\int_{-\pi}^{\tilde{\theta}} g_{\tau_i}(\sigma)d\sigma = \int_{-\pi}^{\hat{\theta} = (1-\tau_i)\theta + \tau_i s(\theta)} f_{\tau_i}(\sigma)d\sigma = \int_{-\pi}^{\theta} f_0(\sigma)d\sigma. \tag{7}$$

Thus,

$$J_{\mathscr{G}}(f_\tau) = \sum_{i=1}^{n} \int_{-\pi}^{\pi} \left(\tilde{\theta} - ((1-\tau_i)\theta + \tau_i s(\theta))\right)^2 g_{\tau_i}(\tilde{\theta})d\tilde{\theta}$$

where the correspondence

$$\tilde{\theta} \ \mapsto \ \theta \ \mapsto \ \hat{\theta} = ((1-\tau_i)\theta + \tau_i s(\theta))$$

can be unravelled from (7).

To simplify the above expression for $J_{\mathscr{G}}(f_\tau)$ we bring in mass distribution functions

$$F(\theta) = \int_{-\pi}^{\theta} f(\sigma)d\sigma,$$

denoted by capital letters; that is, F_{τ_i} is the integral of f_{τ_i}, and similarly for g_{τ_i}. Then (7) reduces to

$$G_{\tau_i}(\tilde{\theta}) = F_{\tau_i}((1-\tau_i)\theta + \tau_i s(\theta)) = F_0(\theta), \tag{8}$$

whereas the objective function can be re-written as

$$\begin{aligned} J_{\mathscr{G}}(f_\tau) &= \sum_{i=1}^{n} \int_{-\pi}^{\pi} \left(\tilde{\theta} - ((1-\tau_i)\theta + \tau_i s(\theta))\right)^2 dG_{\tau_i}(\tilde{\theta}) \\ &= \sum_{i=1}^{n} \int_{-\pi}^{\pi} \left(\tilde{\theta} - ((1-\tau_i)\theta + \tau_i s(\theta))\right)^2 dF_0(\theta), \end{aligned} \tag{9}$$

where the second equality is obtained directly by using (8). The function F_0 is non-decreasing and takes values in $[0,1]$. Thus, $J_{\mathscr{G}}(f_\tau)$ is expressed as a function of F_0 and s (which together determine f_τ). Despite that apparent complexity in how F_0 and s enter, the latter expression is amenable to a numerically attractive re-formulation as follows.

Numerical integration in (9) can be carried out along the axis where F_0 takes values. To do this, divide the F_0-range of values $[0,1]$ into N subintervals of equal length $1/N$ and denote by θ_k, $k = 0, 1, \ldots, N$ the values of θ for which

$$F_0(\theta_k) = \frac{k}{N}.$$

Similarly, denote by $\tilde{\theta}_{i,k}$ $(i = 1, \ldots, n, \ k = 0, 1, \ldots, N)$ the values for which

$$G_{\tau_i}(\tilde{\theta}_{i,k}) = \frac{k}{N},$$

and by $\hat{\theta}_k$ the values for which

$$F_1(\hat{\theta}_k) = \frac{k}{N}.$$

Thereby, $J_{\mathscr{G}}(f_\tau)$ is approximated by the following finite sum

$$\mathbb{J} = \frac{1}{N}\sum_{i=1}^{n}\sum_{k=1}^{N}\left(\tilde{\theta}_{i,k} - \left((1-\tau_i)\theta_k + \tau_i\hat{\theta}_k\right)\right)^2. \qquad (10)$$

The values of $\tilde{\theta}_{i,k}$ ($i = 1,\ldots,n$, $k = 0,1,\ldots,N$) can be readily computed from the problem data $\mathscr{G}$, and the only unknowns in this "discretization" of $J_{\mathscr{G}}(f_\tau)$ are the vector of θ's, namely θ_k, $k = 0,1,\ldots,N$ (which help determine F_0) and the vector of corresponding $\hat{\theta}$'s (which help determine F_1, and then s). Therefore, Problem 1 can be solved numerically via the following convex quadratic program with linear constraints:

$$\begin{aligned}\min\{\mathbb{J}:\ \text{subject to } & -\pi \le \theta_k \le \theta_{k+1} \le \pi \\ \text{and } & -\pi \le \hat{\theta}_k \le \hat{\theta}_{k+1} \le \pi \\ & \text{for } 0 \le k \le N-1\}. \qquad (11)\end{aligned}$$

The objective function is convex, so the optimal solution can be found efficiently [3]. For all numerical examples that follow, we used the software package SeDuMi, available through [15], to solve (11).

IV. Numerical examples

We present two examples. The first example demonstrates smoothing properties of geodesic fit, as expected, whereas the second highlights how such a technique behaves when tracking chirp signals in noise.

A. Example 1

In this first example we generate time-series data by driving a time-varying system with unit-variance white noise and then superimposing white measurement noise with variance equal to 2. The time-varying system consists of a succession of (15^{th}-order) auto-regressive filters chosen to match the spectral character of a W_2-geodesic between an ideal power spectum

$$f_{0,\text{ideal}}(\theta) = \left|\frac{1-0.5z^{-1}+0.6z^{-2}}{1+0.8z^{-1}+0.9z^{-2}}\right|^2_{z=e^{j\theta}}$$

and a final

$$f_{1,\text{ideal}}(\theta) = \left|\frac{1+0.5z^{-1}+0.6z^{-2}}{1-0.8z^{-1}+0.9z^{-2}}\right|^2_{z=e^{j\theta}}.$$

These are shown in Figure 1. Then, we use STFT with a window of 128 points and an overlapping between successive windows by 64 points to obtain a collection of power spectral $\mathscr{G}$ as before.

The spectrogram obtained by STFT is shown in Figure 2. Figure 3 shows the time-series data (in the first row) and then, below, it compares STFT power spectra ($g_{\tau_i}(\theta)$) in the second row with corresponding spectra obtained via a geodesic fit ($f_{\tau_i}(\theta)$). It is clear that the geodesic path captures quite accurately the drift of power in the spectrum over time. Furthermore, the corresponding "frozen time" spectra $f_{\tau_i}(\theta)$ for $i = 1,\ldots,n$ appear to reproduce quite accurately the expected power distribution at the particular points in time. On the other hand, due to the small signal to noise ratio (SNR) the STFT seem quite unreliable.

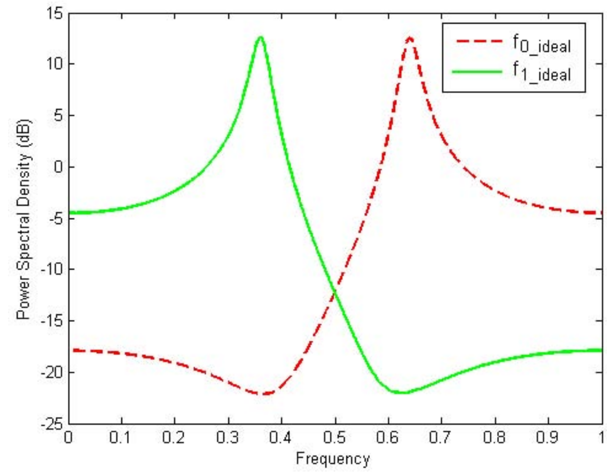

Fig. 1. Power spectra $f_{0,\text{ideal}}$ and $f_{1,\text{ideal}}$ as functions of θ/π.

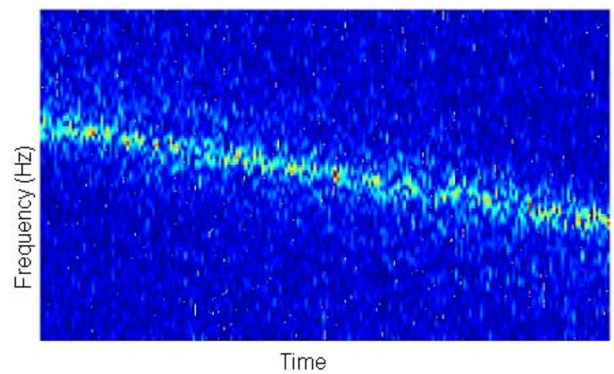

Fig. 2. STFT spectrogram.

B. Example 2

In this example, we generate two chirp signals with additive noise. The spectrogram using STFT is shown in Figure 4. Because of the apparent discontinuity in the path of spectral lines, as we allow the chirp to exceed the Nyquist frequency, we determine a piecewise geodesic approximation (every 20 psd's of the STFT). Figure 5, as before, shows the time-series and right below, compares the spectra obtained by the STFT with "frozen time" samples of the estimated geodesic path. The apparent improvement in SNR between the STFT spectra g_{τ_i}'s and the geodesic samples f_{τ_i}'s is rather evident. A final example along a similar vain is shown in Figure 6 and 7. Here, the frequency of the chirp follows a more complicated trajectory. Tracking requires shorter piecewise segments for constructing geodesic approximations. The result is shown in Figure 6, and similarly, it compares favorably to the STFT spectra with regard to SNR.

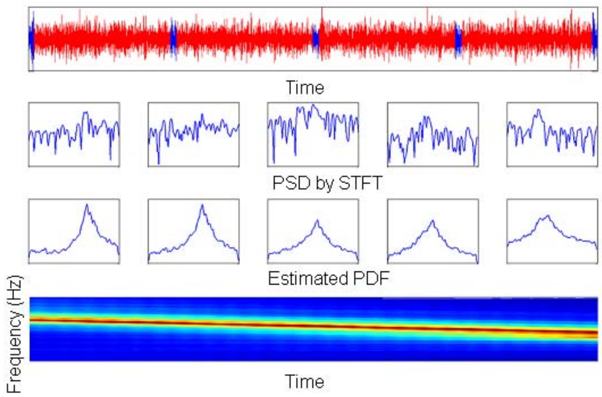

Fig. 3. First row: time-series data; second row: STFT spectra based on the highlighted parts of the time-series; third row: samples of geodesic fit to STFT spectra; fourth row: estimated geodesic path.

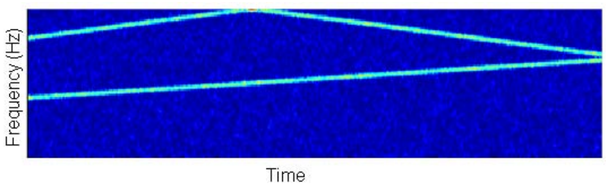

Fig. 4. STFT spectrogram

V. Concluding Remarks

In recent years there has been an interest to endow the space of power spectral densities with a natural metric (see [7], [14], [13], [9]). This has been motivated by a desire to develop tools for quantitative spectral analysis and modeling. Besides the relevance of metrics in quantifying modeling uncertainty, comparing spectra, etc., a metric topology brings up the concept of geodesics. These are the analogs of the straight lines of Euclidean geometry and represent the simplest models of paths across the space of density functions. In the present paper we sought to explore the concept of spectral geodesics for tracking features of a "time-varying" power spectrum. Such "time-varying spectra" are typically associated with non-stationary time-series [5], [6] modeling of which has always been somewhat of a conundrum in signal analysis. In this work, we first formulate the problem of approximating spectra with geodesics utilizing the Wasserstein W_2-metric. We then show that this "geodesic fit" problem is amenable to standard numerical tools of convex optimization. We use numerical examples to highlight the potential of the concept of a geodesic between spectra as a model for time-variability. The data for our formulations can be provided by standard spectral analysis techniques and, in particular, by the short-time Fourier transform. It is perhaps a distant goal, but always worth pondering, how to circumvent that intermediate step and compute geodesics directly based on time-domain data.

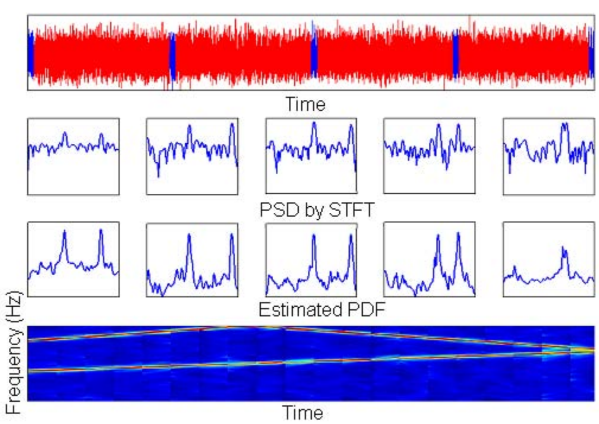

Fig. 5. Top row: time-series with two chirp signals and additive noise; second row: STFT spectra corresponding to windows marked with blue; third row: estimated geodesic-fit samples; last row: the estimate geodesic path.

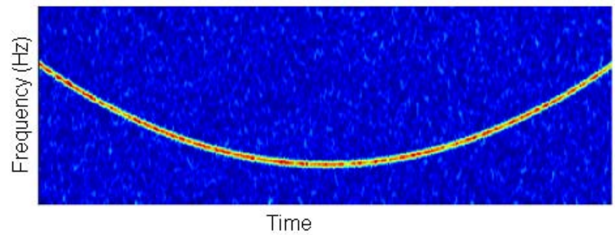

Fig. 6. STFT spectrogram.

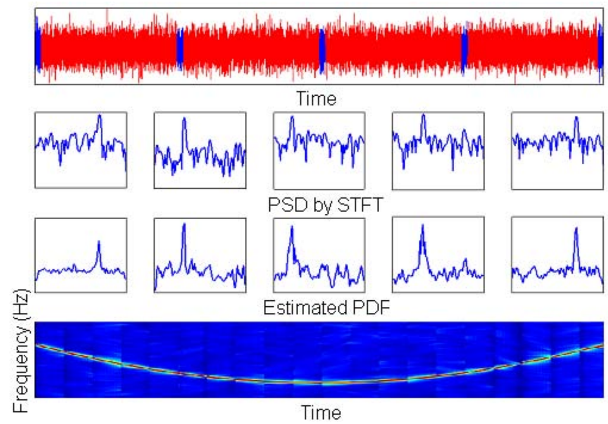

Fig. 7. Top row: time-series with a "quadratic chirp signal" and additive noise; second row: STFT spectra corresponding to windows marked with blue; third row: estimated geodesic-fit samples; last row: the estimate geodesic path.

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