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An Impossibility Theorem on Feedback Based on Stochastic Embedding

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Abstract— **Feedback is a key concept in control systems. A fundamental theoretical problem has been to understand the maximum capability of feedback in controlling uncertain dynamical systems. This needs not only to answer what the feedback can do, but also to answer, the more difficult and basic question, what feedback cannot do. An impossibility theorem on feedback will describe the fundamental limitations of the feedback principle in dealing with uncertainties by all possible feedbacks laws. In this paper, we will use a stochastic embedding approach to address this problem. This approach is based on extensions of the Cramer-Rao inequality to uncertain nonlinearly parameterized dynamical systems, which will then lead to a new impossibility theorem on the feedback capability for the control of a basic class of discrete-time nonlinearly parameterized uncertain dynamical systems.**

I. INTRODUCTION

It is well known that feedback is a key concept in control systems, which is mainly used to deal with uncertainties in the dynamical systems to be controlled. Robust control and adaptive control are two typical techniques for feedback design in the presence of structural uncertainties. It is conceivable that adaptive control has the ability to deal with larger class of uncertainties since an on-line estimation loop is usually included in the feedback control design.

There has been much progress in adaptive control of linear systems (cf. e.g., [1],[2],[3],[4]), or nonlinear systems with nonlinearity having linear growth rate (cf. e.g.[11], [20]). Furthermore, it is also possible to design globally stablizing adaptive controls for a wide class of nonlinear continuous-time systems[5], but fundamental difficulties arise for adaptive control of nonlinear discrete-time systems, partly because the high gain or nonlinear damping methods that are crucial in the continuous-time case are no longer effective in the discrete-time case. Similarly, for sampled-data control of nonlinear uncertain systems, the design of stabilizing sampled-data feedback is possible if the sampling rate is high enough (cf.e.g., [17] and [18]), but difficulties will again emerge if the sampling rate is a prescribed value (may not be small enough), even for nonlinear systems with nonlinearity having a linear growth rate (cf.[19]) .

Given the above difficulties that we encountered in the adaptive control of discrete-time (or sampled-data) nonlinear systems, one may curious to know whether or not such difficulties are caused by the inherent limitations of the feedback principle. To investigate this fundamental problem, we have to place ourselves in a framework that is somewhat beyond those of the traditional robust control and adaptive control. This is because we need to answer not only what feedback can do, but also the more difficult question what feedback cannot do. This means that we need to study the fundamental limitations of the full feedback mechanism which includes all (nonlinear and time-varying) feedback laws.

An initial step in this direction was made in [6], where the following basic model is considered :

$$y_{t+1} = \theta f(y_t) + u_t + w_{t+1}, \tag{1}$$

where θ is an unknown scalar parameter, $\{w_t\}$ is a Gaussian white noise sequence, and where $f(\cdot)$ is a known nonlinear function having a nonlinear growth rate characterized by [1]

$$|f(x)| = \Theta(|x|^b) \quad \text{with} \quad b \geq 1. \tag{2}$$

It was found that the system is globally stabilizable by feedback if and only if $b < 4$ (see, [6], [13]). Obviously, this critical case on the feedback capability gives a direct"impossibility theorem" on the maximum capability of feedback for the case where $b \geq 4$. It is worth pointing out that such "impossibility theorem" obviously holds also for any (more general) class of uncertain systems that includes the basic model class as a subclass. In subsequent works, this "impossibility theorem" was extended to systems with multiple unknown parameters by introducing a polynomial criterion(see, [10],[11]). More recently, [15] proved that the polynomial rule of [10] does indeed provide a necessary and sufficient condition for global feedback stabilization of a wide class of nonlinear systems with bounded multiple unknown parameters and with bounded noises, by using a purely deterministic method, which is usually quite complicated in the study of the impossibility theorems.

However, all the above mentioned impossibility results on feedback consider linearly parameterized models, i.e., the unknown parameter enters into the systems in a linear way. To investigate the more general nonlinearly parameterized systems, we will use a stochastic embedding approach in this paper. It will lead to a new impossibility theorem on the feedback capability for the control of a basic classes of discrete-time uncertain nonlinearly parameterized dynamical systems. This theorem shows that the key factor in determining the capability and limitations of the feedback mechanism is the growth rate of the sensitivity function (with respect to the uncertain parameter).

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[1] $|f(x)| = \Theta(|x|^b)$ means $0 < \liminf_{x\to\infty} \frac{|f(x)|}{|x|^b} \leq \limsup_{x\to\infty} \frac{|f(x)|}{|x|^b} < \infty$.

This paper is organized as follows. In the next section, we will present the main results of the paper, followed by a description of the stochastic embedding approach in Section III. The proof of the main results will be given in Section IV, and some concluding remarks will be given in Section V.

II. MAIN RESULT

To explore the fundamental limits of the feedback capability, we consider the following nonlinearly parameterized control system :

$$y_{t+1} = f(\theta, y_t) + u_t + w_{t+1}, \tag{3}$$

where $\theta \in \mathbb{R}^p, p \geq 1$, is an unknown parameter vector in the p-dimensional Euclidean space, y_t, u_t and w_t are the system output, input and noise signals, respectively, $f(\cdot,\cdot) : \mathbb{R}^{p+1} \to \mathbb{R}$ is a known nonlinear function. We first assume that the noise sequence satisfies the following condition:

A1) The noise sequence is an arbitrarily bounded sequence with an upper bound $w > 0$, i.e.,

$$\sup_{t\geq 1} |w_t| \leq w. \tag{4}$$

To establish our main result of the paper, we need a structural condition on the nonparametric function $f(\cdot,\cdot)$ as follows.

A2) The sensitivity function of θ defined by $f'(\theta, x) \triangleq \dfrac{\partial f(\theta, x)}{\partial \theta} \triangleq (f_1'(\theta, x), \cdots, f_p'(\theta, x))^\tau$ exists, which has the following growth rates as $x \to \infty$,

$$|f_i'(\theta, x)| = \Theta(|x|^{b_i}), \quad 1 \leq i \leq p \tag{5}$$

uniformly in θ, where the exponents b_i are arranged in a decreasing order $b_1 > b_2 > \cdots > b_p > 0$, with $b_1 > 1$.

We would like to know if the above uncertain systems can be stabilized by feedback in the same averaging sense. First, we state two standard definitions(cf.e.g., [12]).

Definition 2.1: A sequence $\{u_t\}$ is called a feedback control law if at any time $t \geq 0$, u_t is a (causal) function of all the observations up to the time t: $\{y_i, i \leq t\}$, i.e.,

$$u_t = h_t(y_0, \cdots, y_t) \tag{6}$$

where $h_t(\cdot) : I\!R^{t+1} \to I\!R^1$ can be any Lebesgue measurable (nonlinear) mapping.

Definition 2.2: The system (3) is said to be **not** globally stabilizable by feedback, if for every feedback control law $\{h_t(\cdot)\}$, there always exist some $y_0 \in \mathbb{R}$, $\theta \in \mathbb{R}^p$, and some noise sequence $\{w_t\}$ satisfying A1), such that the corresponding outputs of the closed-loop system are not bounded, i.e.,

$$\sup_{t\geq 0} |y_t| = \infty. \tag{7}$$

The main result of this paper is the following impossibility theorem on feedback.

Theorem 2.1: The uncertain system (3) under the Assumptions A1)-A2) is not globally stabilizable by feedback if there exists some $x \in (1, b_1)$ such that

$$P(x) \leq 0, \tag{8}$$

where $P(x)$ is a polynomial defined by

$$\begin{aligned} P(x) \;&=\; x^{p+1} - b_1 x^p + (b_1 + b_3 + \cdots + b_p)x^{p-1} \\ &\quad - b_3 x^{p-3} - \cdots - b_p x. \end{aligned}$$

The proof of this theorem is given in Section IV.

Remark 2.1: Note that if the parameter θ were known, then the System (3) could be stabilized by the simple feedback law $u_t = -f(\theta, y_t)$. Hence, the above limit on the capability of feedback is mainly caused by the uncertainty of the parameter θ, and by the limitations of identification to be given by Proposition 3.1 in the next section. Furthermore, Theorem 2.1 shows that such a limit of feedback is determined by the growth rate of the sensitivity function $\partial f(\theta, \cdot)/\partial\theta$, since the criterion polynomial $P(x)$ is determined by its growth exponents b_i. For example, in the scalar parameter case where $p = 1$, the above polynomial reduces to

$$P(x) = x^2 - b_1 x + b_1$$

and it is easy to see that there is $x \in (1, b_1)$ such that $P(x) \leq 0$ if and only if $b_1 \geq 4$, which means that in the case where $b_1 \geq 4$, no globally stabilizing feedback exists. Theorem 2.1 extends the existing impossibility results in [6][13] from linear parametric case to the present nonlinear parametric case.

Remark 2.2: If there exists some $\theta^* \in \mathbb{R}^p$ and some $t^* \geq 1$ such that for any $t \geq t^*$, we have $P_t'(\theta) \leq P_t'(\theta^*)$, where

$$P_t'(\theta) \triangleq I + M_1 t^4 \sum_{i=0}^{t-1} f'(\theta, \phi_i) f^{'\tau}(\theta, \phi_i),$$

then it can be shown in a similar way that Theorem 2.1 holds for the polynomial

$$\begin{aligned} P(x) \;&=\; x^{p+1} - b_1 x^p + (b_1 - b_2)x^{p-1} + \cdots \\ &\quad + (b_{p-1} - b_p)x + b_p. \end{aligned} \tag{9}$$

Furthermore, in the case where the function $f(\theta, x)$ is reduced to a linearly parameterized function $\theta^\tau f(x)$ with $\theta \in R^p$, the assumption $P_t'(\theta) \leq P_t'(\theta^*)$ is automatically satisfied since the matrix $P_t'(\theta)$ is free of θ. In this case, it has been shown in [15] that, the condition $P(x) > 0, x \in (1, b_1)$, is also sufficient for global feedback stabilization of this class of uncertain systems with bounded disturbances, where $P(x)$ is defined by (9).

Remark 2.3: If, instead of the A2), the sensitivity function has a linear growth rate: $\|f'(\theta, x)\| = O(\|x\|)$, as $x \to \infty$, then it can be shown [16] that the system (3) is globally stabilizable by feedback.

Remark 2.4: If the unknown parameter θ belongs to a compact set, a similar proof shows that Theorem 2.1 is also true.

III. STOCHASTIC EMBEDDING

Theorem 2.1 is a deterministic result in nature, but it will be established using a stochastic embedding approach in the paper. It is interesting to note that the stochastic embedding approach used in most of the existing literature (see, e.g. [9]) normally arrives at an assertion that is hold outside an exceptional set of probability zero. Here, we will arrive at a pure deterministic assertion for our impossibility theorem.

We start with the following more general nonlinearly parameterized model:

$$y_{t+1} = f(\theta, \phi_t) + w_{t+1}, \tag{10}$$

where $\theta \in \mathbb{R}^p, p \geq 1$ is an unknown parameter vector, $\phi_t = (y_t, u_t; \cdots ; y_0, u_0)$, u_t and w_t are the system regression vector, feedback law and noise signal respectively, and where $f(\cdot, \cdot) \in C^1 : \mathbb{R}^{2(t+1)+p} \to \mathbb{R}$ is a differentiable nonlinear function.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\theta \in \mathbb{R}^p$ be a random vector and $\{w_t\}_{t=1}^{\infty}$ be a stochastic process on this probability space. The stochastic embedding idea is to construct a special class of θ and $\{w_t\}_{t=1}^{\infty}$ on this probability space, such that their sample pathes are consistent with those of our assumptions and they are easily applicable to some Cramer-Rao-like inequalities for dynamical systems. This can be done by choosing a suitable class of probability density functions(p.d.f.) as follows.

Let us take $\theta \in \mathbb{R}^p$ to have the following $p-$ dimensional Gaussian p.d.f. $p(\theta)$ defined by $N(0, I)$:

$$p(\theta) = \frac{1}{(2\pi)^{p/2}} \exp\{-\frac{1}{2}\|\theta\|^2\}. \tag{11}$$

Also, let us take $\{w_t\}$ to be an independent sequence which is independent of θ with w_t having a Gaussian p.d.f. $q_t(z)$ defined by $N\left(0, \frac{1}{t^2}\right)$:

$$q_t(z) = \frac{t}{\sqrt{2\pi}} \exp\left(-\frac{z^2 t^2}{2}\right), \tag{12}$$

Obviously, $\{w_t\}$ satisfies A1) almost surely for large enough t, since

$$\lim_{t\to\infty} w_t = 0, \quad a.s.$$

We will show that in the above stochastic framework, for every feedback control $u_t \in \mathcal{F}_t^y \triangleq \sigma\{y_i, 0 \leq i \leq t\}$, there always exists an initial condition y_0 and a set $D \subset \Omega$ with positive probability such that for any $\omega \in D$, $\theta(\omega) \in \mathbb{R}^p$ and $\{w_t(\omega)\}$ is bounded, and that the corresponding output signal y_t of the closed-loop control system has a dynamical lower bound. This will naturally gives the corresponding results in the deterministic framework of this paper.

To get the above result, we need a key lemma on a conditional Cramer-Rao inequality for dynamical systems (see, eg., [7], [11]). To this end, we first define some notations which will be used throughout the sequel.

$f_t \triangleq f(\theta, \phi_t)$, where ϕ_t is defined in (10).

$E_x y \triangleq E\{y|x\}$, where x, y are random vectors.

$\hat{f}(\theta, \phi_t) \triangleq E[f(\theta, \phi_t)|\mathcal{F}_t^y]$;

$P_t(\theta) \triangleq I + M_1 t^4 \sum_{i=0}^{t-1} E[f'(\theta, \phi_i) f'^{\tau}(\theta, \phi_i)|\mathcal{F}_t^y]$;

$Q_{t+1}(\theta) \triangleq P_t(\theta) + E[f'(\theta, \phi_t)|\mathcal{F}_t^y] E^{\tau}[f'(\theta, \phi_t)|\mathcal{F}_t^y]$,

where $\mathcal{F}_t^y \triangleq \sigma\{y_1, \cdots, y_t\}$ and $M_1 > 0$ is some random variable.

We first give a Cramer-Rao inequality like result for the best prediction of the uncertain function $f(\theta, \phi_t)$ given $\{y_1, \cdots, y_t\}$.

Proposition 3.1: Let θ be a parameter vector with p.d.f. $p(\theta)$ defined in (11), and be independent of $\{w_k\}$ which is an i.i.d. random sequence with p.d.f. $q(z)$ defined in (12). Then for the dynamical equation (10) with arbitrarily deterministic initial value y_0, we have

$$\begin{aligned} & E_x[f(\theta, \phi_t) - \hat{f}(\theta, \phi_t)]^2 \\ \geq \ & \frac{1}{2} E_x^{\tau} f'(\theta, \phi_t) P_t^{-1}(\theta) E_x f'(\theta, \phi_t). \end{aligned} \tag{13}$$

where $x \triangleq \{y_1, \cdots, y_t\}$.

Based on this Proposition, we now try to get a dynamical lower bound to the output process. For this, we need the following additional assumption.

A3) For any $\varepsilon > 0$, there exists some function $h(\varepsilon)$ such that for any $\phi \in R^{2t+2}$ and $\|\phi\| \geq h(\varepsilon)$, the set Δ_ε defined by

$$\Delta_\varepsilon \triangleq \{\theta \in \mathbb{R}^p : |f(\theta, \phi)| < \varepsilon \max_{\theta} |f'(\theta, \phi)|\}, \tag{14}$$

satisfies $L(\Delta_\varepsilon) \leq K\varepsilon$ for all large ϕ, where $L(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}^p$ and $K > 0$ is some constant.

Proposition 3.2: Under the conditions of Proposition 3.1 and A3), if the regression $\|\phi_t\| \geq h\left(\frac{\delta}{(t+1)^2}\right)$ for all t, where δ is some constant we defined latter, then there exists some set $D \subset \Omega$ with $P(D) > 0$, such that on this set,

$$\begin{aligned} y_{t+1}^2 \quad \geq \quad & \frac{1}{K_1(t+1)^4 + 4} \left[\frac{1}{2} \frac{\det Q_{t+1}(\theta)}{\det P_t(\theta)} \right. \\ & \left. -(K_1(t+1)^4 + 4)K_2 - \frac{3}{2}\right], \end{aligned} \tag{15}$$

holds for all $t \geq 1$, where $K_1, K_2 > 0$ are some constants.

The proofs of Propositions 3.1 and 3.2 are given in the next section. Proposition 3.2 is a key result to be used in establishing the impossibility theorem stated in the last section. It can be further simplified under additional conditions on the nonlinear structure $f(\cdot, \cdot)$.

IV. PROOF OF THE MAIN RESULTS

To prove Theorem2.1, we need to prove Propositions 3.1 and 3.2. The proof of Propositions 3.1 is composed of several lemmas to be given below. The first lemma below is a standard conditional Cramer-Rao inequality (see, e.g.[7], [11]).

Lemma 4.1: Let x be random vector, and let θ be a parameter vector with p.d.f. $p(\theta)$ defined in (11). Then for any measurable vector function $g(x,\theta)$ having partial derivatives of first order w.r.t. θ, and let $E_x g(x,\theta)$ and $E_x \frac{\partial g(x,\theta)}{\partial\theta}$ exist. Then we have

$$\begin{aligned} & E_x[g(x,\theta) - E_x g(x,\theta)]^2 \\ \geq\ & E_x \frac{\partial g(x,\theta)}{\partial\theta}\left\{E_x\left[\frac{\partial \log p(x,\theta)}{\partial\theta}\cdot\frac{\partial^\tau \log p(x,\theta)}{\partial\theta}\right]\right\}^{-1} \\ & \times E_x^\tau \frac{\partial g(x,\theta)}{\partial\theta}. \end{aligned}$$

Applying this lemma to the dynamical system defined by (10), we can further get the following result.

Lemma 4.2: Under the conditions of Proposition 3.1, we have for $t \geq 1$

$$\begin{aligned} & E_x[f(\theta,\phi_t) - E_x f(\theta,\phi_t)]^2 \\ \geq\ & \frac{1}{2}E_x^\tau f'(\theta,\phi_t)[E_x F_t(\theta)]^{-1}E_x f'(\theta,\phi_t), \end{aligned} \quad (16)$$

where $x \triangleq \{y_1,\cdots,y_t\}$ and

$$F_t(\theta) \triangleq \sum_{k=1}^{t}\frac{\partial \log q(y_k - f_{k-1})}{\partial\theta}\cdot\sum_{k=1}^{t}\frac{\partial^\tau \log q(y_k - f_{k-1})}{\partial\theta} + I,$$

with $p(\cdot)$ and $q(\cdot)$ being the p.d.f. of the parameter θ and noise w_t respectively.

Proof. Directly applying Lemma 4.1, we have

$$\begin{aligned} & E_x[f(\theta,\phi_t) - E_x f(\theta,\phi_t)]^2 \\ \geq\ & E_x^\tau \frac{\partial f(\theta,\phi_t)}{\partial\theta}\left\{E_x\left[\frac{\partial \log p(x,\theta)}{\partial\theta}\cdot\frac{\partial^\tau \log p(x,\theta)}{\partial\theta}\right]\right\}^{-1} \\ & \times E_x \frac{\partial f(\theta,\phi_t)}{\partial\theta} \\ =\ & E_x^\tau \frac{\partial f(\theta,\phi_t)}{\partial\theta}\times \\ & \left\{E_x\left[\frac{\partial[\log p(x|\theta) + \log p(\theta)]}{\partial\theta}\right.\right. \\ & \left.\left.\cdot\frac{\partial^\tau[\log p(x|\theta) + \log p(\theta)]}{\partial\theta}\right]\right\}^{-1} E_x\frac{\partial f(\theta,\phi_t)}{\partial\theta}. \end{aligned}$$

Note that by the Bayes rule and the dynamical equation (10),

$$\begin{aligned} p(x|\theta) &= p(y_1, y_2.\cdots, y_t|\theta) \\ &= p(y_1|\theta, y_0)p(y_2|\theta, y_0, y_1) \\ &\quad \cdots p(y_t|\theta, y_0,\cdots, y_{t-1}) \\ &= q(y_1 - f_0)\cdot q(y_2 - f_1)\cdots q(y_t - f_{t-1}). \end{aligned}$$

Then, by the matrix Schwarz inequality

$$\begin{aligned} & \left(\sum_{k=1}^{t}\frac{\partial \log q(y_k - f_{k-1})}{\partial\theta} + \frac{\partial \log p(\theta)}{\partial\theta}\right) \\ & \cdot\left(\sum_{k=1}^{t}\frac{\partial \log q(y_k - f_{k-1})}{\partial\theta} + \frac{\partial \log p(\theta)}{\partial\theta}\right)^\tau \\ & \leq 2\left(\sum_{k=1}^{t}\frac{\partial \log q(y_k - f_{k-1})}{\partial\theta}\cdot\right. \\ & \left.\sum_{k=1}^{t}\frac{\partial^\tau \log q(y_k - f_{k-1})}{\partial\theta} + \frac{\partial \log p(\theta)}{\partial\theta}\cdot\frac{\partial^\tau \log p(\theta)}{\partial\theta}\right), \end{aligned}$$

and the fact that [11]

$$E_x\frac{\partial \log p(\theta)}{\partial\theta}\cdot\frac{\partial^\tau \log p(\theta)}{\partial\theta} = -E_x\frac{\partial^2 \log p(\theta)}{\partial\theta^2}$$

we can arrive at (16) directly by noting that

$$-\frac{\partial^2 \log p(\theta)}{\partial\theta^2} \leq I, \quad a.s.$$

which can be easily verified by the definition of $p(\theta)$. ∎

Lemma 4.3: Under the conditions of Proposition 3.1, we have

$$F_t(\theta) \leq M_1\sum_{k=0}^{t-1} f'(\theta,\phi_k)f'^\tau(\theta,\phi_k) + I,$$

where $F_t(\theta)$ is defined in Lemma 4.2 and $M_1 > 1$ is some random variable.

Proof. Since $q_k(y_k - f_{k-1}) = \frac{k}{\sqrt{2\pi}}\exp\{-\frac{k^2}{2}(y_k - f_{k-1})^2\}$, $k = 1, 2,\cdots, t$, we have

$$\begin{aligned} \frac{\partial \log q_k(y_k - f_{k-1})}{\partial\theta} &= \frac{\partial}{\partial\theta}\{-\frac{k^2}{2}(y_k - f_{k-1})^2\} \\ &= k^2 w_k f'(\theta,\phi_{k-1}). \end{aligned}$$

Note that by the definition of $q_t(z)$ in (12), for some random variable $M_1 > 1$,

$$\sum_{k=1}^{t} w_k^2 < M_1.$$

Hence, by the matrix Schwarz inequality, we have

$$\begin{aligned} & \left(\sum_{k=1}^{t} f'(\theta,\phi_{k-1})k^2 w_k\right)\left(\sum_{k=1}^{t} f'(\theta,\phi_{k-1})k^2 w_k\right)^\tau \\ \leq\ & t^4\left(\sum_{k=1}^{t} f'(\theta,\phi_{k-1})f'^\tau(\theta,\phi_{k-1}^\tau)\right)\left(\sum_{k=1}^{t} w_k^2\right) \\ \leq\ & M_1 t^4\sum_{k=1}^{t} f'(\theta,\phi_{k-1})f'^\tau(\theta,\phi_{k-1}^\tau) \qquad \text{on } D_0, \end{aligned}$$

which gives the lemma by the definition of $F_t(\theta)$. ∎

To prove Proposition 3.2, we first prove the following lemma.

Lemma 4.4: There exists a set $D \subset \Omega$ with positive porobability such that on D

$$E[y_{t+1}^2|\mathcal{F}_t^y] \le \left(4 + K_1(t+1)^4\right)(y_{t+1}^2 + K_2) + 1$$

for any $t \ge 0$, where $K_1, K_2 > 0$ are some constants.

Proof. To prove this lemma, we define

$$\Delta_t \triangleq \left\{\theta \in \mathbb{R}^p : |f(\theta,\phi_t)| < \frac{\delta}{(t+1)^2} f^*(\phi_t)\right\},$$
$$0 < \delta < \frac{1}{2KP}, \quad t \ge 0,$$

where, $f^*(\phi_t) \triangleq \max_\theta \|f'(\theta,\phi_t)\|$, $P \triangleq \sup_{\theta\in\mathbb{R}^p} p(\theta) = \frac{1}{(2\pi)^{p/2}}$, and K is defined in A3).

Recursively define $\Theta_{t+1} \triangleq \Theta_t - \Delta_t, t = 0, 1, \cdots$, where $\Theta_0 = \mathbb{R}^p$. Let $\Theta_\infty \triangleq \lim_{t\to\infty} \Theta_t, D \triangleq \{\omega : \theta \in \Theta_\infty\}$.

So, by Assumption A3) and the condition of this lemma, we have $L(\Delta_t) \le \frac{K\delta}{(t+1)^2}$, and

$$\begin{aligned} P(\{\omega : \theta \in \bigcup_{t=0}^{\infty} \Delta_t\}) &\le \sum_{t=0}^{\infty} P(\{\omega : \theta \in \Delta_t\}) \\ &\le PK \sum_{t=0}^{\infty} \frac{\delta}{(t+1)^2} < 1, \end{aligned}$$

which implies

$$P(D) \ge 1 - P(\{\omega : \theta \in \bigcup_{t=0}^{\infty} \Delta_t\}) > 0.$$

Next, note that $E[X|\mathcal{F}_t^y]$ is a.s. bounded for any integrable random variable X by [21, p.145]. Now, let $\omega^* \in D$ be any fixed point. Then by the definitions of D and Δ_t, we have

$$\begin{aligned} & E_x[f(\theta,y_t) - f(\theta(\omega^*),\phi_t)]^2(\omega^*) \\ = \;& E_x[(\theta - \theta(\omega^*))^\tau f'(\xi,\phi_t)]^2(\omega^*) \\ \le \;& E_x\|\theta - \theta(\omega^*)\|^2 \cdot \max_\xi \|f'(\xi,\phi_t)\|^2(\omega^*) \\ \le \;& \frac{(t+1)^4}{\delta^2} f^2(\theta,\phi_t) E_x\|\theta - \theta(\omega^*)\|^2(\omega^*) \\ \le \;& \frac{(t+1)^4 M_2(\omega^*)}{\delta^2} f^2(\theta,\phi_t)(\omega^*), \end{aligned}$$

where $M_2(\omega^*)$ is a constant. Consequently, by noting that $|w_t|^2 \le K_2$, $a.s.$ for some random constant $K_2 > 0$, we have for any $\omega^* \in D$,

$$\begin{aligned} & [E_x y_{t+1}^2](\omega^*) \\ = \;& E_x f^2(\theta,\phi_t)(\omega^*) + 1 \\ \le \;& 2f^2(\theta(\omega^*),\phi_t)(\omega^*) \\ & +2E_x[f(\theta_t,\phi_t) - f(\theta(\omega^*),\phi_t)]^2(\omega^*) + 1 \\ \le \;& \left(2 + \frac{2M_2(t+1)^4}{\delta^2}\right) f^2(\theta(\omega^*),\phi_t)(\omega^*) + 1 \\ = \;& \left(2 + \frac{2M_2(t+1)^4}{\delta^2}\right) [y_{t+1}(\omega^*) - w_{t+1}(\omega^*)]^2 + 1 \\ = \;& \left(4 + \frac{4M_2(t+1)^4}{\delta^2}\right) (y_{t+1}^2(\omega^*) + K_2) + 1 \\ \le \;& \left(4 + K_1(t+1)^4\right)(y_{t+1}^2(\omega^*) + K_2) + 1, \end{aligned}$$

where $K_1 = \frac{4M_2}{\delta^2}$ is a constant. Hence the proof is completed. ■

Proof of Proposition 3.2: First of all, it is easy to see that $E[w_{t+1}|\mathcal{F}_t^y] = Ew_{t+1} = 0$ by (12). By (10) we know that

$$y_{t+1} = [f(\theta,\phi_k) - \hat{f}(\theta,\phi_k)] + \hat{f}(\theta,\phi_k) + w_{t+1}, \quad (17)$$

where $\hat{f}(\theta,\phi_k)$ are defined as in Proposition 3.1. Consequently, by the fact $E[f(\theta,\phi_k) - \hat{f}(\theta,\phi_k)|\mathcal{F}_t^y] = 0$ and $E[w_{t+1}|\mathcal{F}_t^y] = 0$ it follows that for any $u_t \in \mathcal{F}_t^y$,

$$\begin{aligned} & E[y_{t+1}^2|\mathcal{F}_t^y] \\ = \;& E\{[f(\theta,\phi_k) - \hat{f}(\theta,\phi_k)]^2|\mathcal{F}_t^y\} + \hat{f}^2(\theta,\phi_k) \\ & +E[w_{t+1}^2|\mathcal{F}_t^y] \\ \ge \;& E\{[f(\theta,\phi_k) - \hat{f}(\theta,\phi_k)]^2|\mathcal{F}_t^y\}. \end{aligned} \quad (18)$$

Then by Proposition 3.1, we have on D,

$$\begin{aligned} & E_x y_{t+1}^2 \\ \ge \;& \frac{1}{2}(E_x^\tau f'(\theta,\phi_t) P_t^{-1}(\theta) E_x f'(\theta,\phi_t) + 1) - \frac{1}{2} \\ = \;& \frac{1}{2}\frac{\det\{P_t(\theta) + E_x f'(\theta,\phi_t) E_x^\tau f'(\theta,\phi_t)\}}{\det P_t(\theta)} - \frac{1}{2} \\ = \;& \frac{1}{2}\frac{\det Q_{t+1}(\theta)}{\det P_t(\theta)} - \frac{1}{2}. \end{aligned}$$

This together with Lemma 4.4 shows that (15) is true. ■

Proof of Theorem 2.1 In the stochastic framework, note that every feedback law $u_t = h_t(y_0, \cdots, y_t)$ in the form of (6) in Definition 2.1 is measurable $\mathcal{F}_t^y$. By Proposition 3.2, for every feedback law u_t, there at least exists some sample point $\omega^* \in D \subset \Omega$ with $\theta(\omega^*)$ and $w_t(\omega^*)$ for all $t \ge 1$ such that for arbitrary y_0, the inequality (15) holds for $y_{t+1}^2(\omega^*)$.

For simplicity, we continue to use θ and $\{w_t\}$ to denote the parameter $\theta(\omega^*)$ and noise sequence $w_t(\omega^*)$ as above. In fact we can prove A2) implies A3), and hence the by Proposition 3.2, the corresponding output sequence satisfies on D

$$\begin{aligned} y_{t+1}^2 \;\ge\; & \frac{1}{K_1(t+1)^4 + 4}\left[\frac{1}{2}\frac{\det Q_{t+1}(\theta)}{\det P_t(\theta)}\right. \\ & \left. -(K_1(t+1)^4 + 4)K_2 - \frac{3}{2}\right], \end{aligned}$$

where

$$P_t(\theta) \triangleq I + M_1 t^4 \sum_{i=0}^{t-1} E[f'(\theta,y_i) f'^\tau(\theta,y_i)|\mathcal{F}_t^y].$$

and

$$Q_{t+1}(\theta) \triangleq P_t(\theta) + E[f'(\theta,y_t)|\mathcal{F}_t^y] E^\tau[f'(\theta,y_t)|\mathcal{F}_t^y].$$

Since $f_i'(\theta,y_t) = \Theta(y_t^{b_i})$ uniformly in θ, we have

$$E[f_i'(\theta,y_t)|\mathcal{F}_t^y] = \Theta(y_t^{b_i}),$$

then by a similar proof as for Lemma 2.4.1 in [11] and the p.d.f of w_t, we know that under our assumption $\{|y_t|\}$ bounded, for some $\delta' > 0$ and $t \ge 1$, if $|y_i| \ge |y_{i-1}|^{1+\delta'}, i =$

$1, 2, \cdots, t$, then the matrix determinants will satisfy for large $|y_0| \geq 1$,

$$\det(P_t) = O(t^{4p}|y_{t-1}|^{2(b_1+\cdots+b_p)}),$$
$$|y_t|^{2b_1} \cdots |y_{t-p+1}|^{2b_p} = O(\det(Q_{t+1})).$$

Furthermore, let $\lambda \in (1, b_1)$ be a solution of inequality (9). By almost the same proof as that for (81)-(85) in [11], we can prove by an induction that for some random variable $\mu > 0$,

$$|y_{t+1}| \geq \mu |y_t|^\lambda,$$

whenever $|y_0| \geq 1$ is large enough. Hence $\{y_t\}$ diverges to infinity exponentially fast. The proof is completed. ■

V. CONCLUDING REMARKS

The primary motivation of this paper is to explore the fundamental limitations of the feedback principle in stabilizing discrete-time nonlinear dynamical systems with parameterized uncertainties. By using a stochastic embedding approach, we have in this paper established an impossibility theorem on the feedback in controlling a basic class of uncertain dynamical systems. A remarkable observation is that the key factor that determines the fundamental limitations of the feedback principle is the growth rate of the sensitivity function of the uncertain parameter. Evidently, such an impossibility theorem naturally gives an limitation on adaptive control of discrete-time nonlinear systems. For further investigation, it would be interesting to investigate more general nonlinearly parameterized dynamical systems.

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