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Proportional-Integral Observer Design for Nonlinear Systems

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***Abstract*— A new dissipative method to design observers for a large class of nonlinear systems has been introduced recently by the author. It generalizes and includes several well-known observer design methods in the literature. In this paper a procedure to design Proportional-Integral Observers based on the dissipativity theory is presented. Properties of these observers are discussed.**

I. INTRODUCTION

Recently [1], [2] the author has proposed a Dissipative Design of Observers for nonlinear systems that can be transformed into the form

$$\Sigma : \begin{cases} \dot{x} = Ax + G\psi(\sigma) + \varphi(t, y, u) + Dw \ , \\ y = Cx \ , \ \sigma = Hx, \quad x(0) = x_0 \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is a known input, $w \in \mathbb{R}^q$ is an unknown perturbation, $y \in \mathbb{R}^p$ is the measured output, and $\sigma \in \mathbb{R}^r$ is a (not necessarily measured) linear function of the state. $\varphi(t, y, u)$ is an arbitrary nonlinear function of the time, the input and the output. $\psi(\sigma)$ is a q-dimensional vector that depends on the variable σ. ψ and φ are assumed to be locally Lipschitz in σ or y, continuous in u, and piecewise continuous in t, so that existence and uniqueness of solutions is guaranteed. Although not strictly necessary, it will be assumed for simplicity that the trajectories of interest of Σ are defined for all positive times.

For the most important cases the design can be reduced to Linear Matrix Inequalities (LMI), that are numerically very well behaved, and have became standard in the field. In the perturbation free case, i.e. $w = 0$, this method generalizes and encompasses several other design methods: the High-Gain methodology [3], [4], the Thau observers [5], and the observers for Lipschitz nonlinear systems [6], well-known in the literature.

Proportional-Integral (PI) Observers are very well known in the literature, in particular for linear systems, because of their robustness properties against constant perturbations [7], [8], [9], [10], [11], [12], [13], [14], [15], [16]. This is basically a consequence of the internal model principle, since the integral action represents an internal model of unknown constant perturbations, a fact that is widely used in robust regulation. For LTI systems [7], [15], [16] PI-Observers are used for robust control under parameter perturbations, whereas LTR recovery is the objective in [8]. The possibility of robustly estimation of both states and perturbation with PI-Observers is emphasized in [10]. An adaptive PI-Observer is introduced in [13]. In [9] it is shown that a PIO can be used to attenuate the sensor noise. Simultaneous disturbance attenuation and fault detection is the main objective in [14], where fading observers are used instead of PIOs. PI-Observers for linear descriptor systems [11], [12] or for nonlinear systems [9], [17], [18], [19], [20], with Lipschitz nonlinearities and linear observer gains, have been recently proposed.

The objective of this paper is to show how to design a PI-Observer when a Proportional (P) Observer is known, using dissipative properties. It is also shown that the PI-Observer can estimate perfectly the state and the unknown constant perturbation, when this satisfies a matching condition. In contrast to most results [9], [17], [18], [19], [20] our design does not require Lipschitzness of the nonlinearities of the plant, it allows to use nonlinear proportional and integral observer gains, and these can be selected in a simple and generic manner, assuring the convergence of the PI-Observer, once the P-observer has been designed. Our design uses truly integral action to reject constant perturbations instead of the fading observer proposed in [19] to attenuate disturbances. Moreover, the dissipativity theory is used in a different manner than in [19], [20].

II. PRELIMINARIES

A. Dissipative systems

From the general dissipativity theory [21], [22], [23], [24] (see also [25]) the following results are of relevance here. Consider the LTI continuous time system

$$\Sigma_L : \begin{cases} \dot{x} = Ax + Bu \ , \quad x(0) = x_0 \\ y = Cx \ , \end{cases} \tag{2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^q$, and $y \in \mathbb{R}^m$ are the state, the input and the output vectors, respectively. Let us consider quadratic *supply rates*

$$\omega(y, u) = y^T Q y + 2 y^T S u + u^T R u \ , \tag{3}$$

with Q, R symmetric.

Definition 1: System Σ_L is said to be *state strictly dissipative (SSD)* with respect to the supply rate $\omega(y, u)$, or for short (Q, S, R)-SSD, if there exist a matrix $P = P^T > 0$, and $\epsilon > 0$ such that

$$\begin{bmatrix} PA + A^T P + \epsilon P \ , & PB \\ B^T P & 0 \end{bmatrix} - \begin{bmatrix} C^T Q C & C^T S \\ S^T C & R \end{bmatrix} \leq 0 \ . \tag{4}$$

For quadratic systems, i.e. $m = q$, passivity corresponds to the supply rate $\omega(y, u) = y^T u$. This definition assures the existence of a quadratic positive definite *storage function*

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$V(x) = x^T P x$ such that along any trajectory of the system $\dot{V}(x(t), u(t)) \leq -\epsilon V(x(t)) + \omega(y(t), u(t))$.

A time-varying memoryless nonlinearity $\psi : [0, \infty) \times \mathbb{R}^q \to \mathbb{R}^m$, $y = \psi(t, u)$, piecewise continuous in t and locally Lipschitz in u, such that $\psi(t, 0) = 0$, is said to be dissipative with respect to the supply rate $\omega(y, u)$ (3), or for short (Q, S, R)-D, if it satisfies $\omega(\psi(t, u), u) \geq 0$ for every $t \geq 0$, and $u \in \mathbb{R}^q$.

Remark 2: Note that the classical sector conditions [25] for square nonlinearities, i.e. $m = q$, can be represented in this form. If ψ is in the sector $[K_1, K_2]$, i.e. $(y - K_1 u)^T (K_2 u - y) \geq 0$, then it is (Q, S, R)-D, with $(Q, S, R) = \left(-I, \frac{1}{2}(K_1 + K_2), -\frac{1}{2}\left(K_1^T K_2 + K_2^T K_1\right)\right)$. If ψ is in the sector $[K_1, \infty]$, i.e. $(y - K_1 u)^T u \geq 0$, then it is $\left(0, \frac{1}{2} I, -\frac{1}{2}\left(K_1 + K_1^T\right)\right)$-D.

A generalization of the circle criterion of absolute stability for non square systems can be easily obtained, and it will be used in the sequel.

Lemma 3: Consider the feedback interconnection

$$\begin{array}{l} \dot{x} = Ax + Bu \ , \quad x(0) = x_0 \\ y = Cx \ , \ u = -\psi(t, y) \ . \end{array} \tag{5}$$

If the linear system (C, A, B) is $\left(-R_N, S_N^T, -Q_N\right)$-SSD, then the equilibrium point $x = 0$ of (5) is globally exponentially stable for every (Q_N, S_N, R_N)-D nonlinearity.

B. A strong Lyapunov function

To analyze the convergence properties of PI-Observers it will be required to study conditions for the asymptotic stability of the interconnection of a nonlinear globally asymptotically stable system in the forward loop with an integrator in the feedback. This general class of systems is very important in adaptive control and identification [26], [27] and it is usually studied, from a passive perspective, as the negative feedback interconnection of two passive subsystems. In this case the sum of the storage functions constitutes a weak Lyapunov function, that is, one whose time derivative is only negative semidefinite, even in the cases when asymptotic stability can be assured. We will be interested here in a special class with a time-invariant interconnection. The novelty of our result here is that we will provide explicit conditions for the global exponential stability of the whole system and we will give a *strong* Lyapunov function that ensures this.

Consider the following system

$$\Xi : \left\{ \begin{array}{ll} \dot{x} = f(x, t) + Bk(z) \ , & x \in \mathbb{R}^n \\ \dot{z} = Cx \ , & z \in \mathbb{R}^p \end{array} \right. \tag{6}$$

where $f(x, t)$ is locally Lipschitz in x and measurable in t, $k : \mathbb{R}^p \to \mathbb{R}^p$ is locally Lipschitz continuous and it is the gradient of a scalar, positive definite, decrescent, radially unbounded, continuously differentiable function $W(z)$, i.e. $k^T(z) = \frac{\partial W(z)}{\partial z}$, $k(0) = 0$, and B, C are constant matrices of appropriate dimensions. Assume that $f(0, t) = 0$, and that the system $\dot{x} = f(x, t)$ has zero as a globally uniformly asymptotically stable equilibrium point, and that there is a quadratic Lyapunov function $V(x) = x^T P x$, with P symmetric and positive definite, such that

$$\dot{V}(x) = \frac{\partial V(x)}{\partial x} f(x, t) \leq -\epsilon V(x) \quad ,$$

with $\epsilon > 0$. From a passivity approach it follows that, if $B = -P^{-1} C^T$ then the function $V^*(x, z) = V(x) + W(z)$, that is the sum of the individual storage functions, satisfies

$$\dot{V}^*(x, z) \leq -\epsilon V(x) - x^T C^T k(z) + \frac{\partial W(z)}{\partial z} Cx = -\epsilon V(x) \ .$$

From this property it follows the uniform stability of the equilibrium point, the boundedness of the trajectories and the asymptotic convergence to zero of x. To assure the uniform asymptotic stability of the origin further conditions, as those in the Theorem of Matrosov [27] have to be used. V^* is therefore a *weak* Lyapunov function for the system. Our aim is to propose a *strong* Lyapunov function, that assures the GUAS of the origin, under some additional assumptions.

Proposition 4: Consider the system (6) satisfying the given conditions. Suppose further that: (i) $f(x, t)$ is globally bounded by $\|f(x, t)\| \leq (\lambda + l\mu(\|x\|)) \|x\|$, where $\lambda > 0$, $l \geq 0$ and $\mu(\cdot)$ a monoton increasing function, (ii) that the Jacobian matrix of $k(z)$ is continuous and uniformly upper and lower bounded, and (iii) that C has full row rank. Under these conditions

$$U(x, z) = \delta(V^*(x, z)) + x^T C^T k(z) \ ,$$

with $\delta(\cdot)$ some suitable K_∞ function, is a strong Lyapunov function for the system. Moreover, under these assumptions the equilibrium point is globally uniformly asymptotically and locally exponentially stable.

Proof: Note that from the mean value theorem (see [28]) it follows that $2W(z) = z^T H(tz) z$, $k(z) = H(\tau z) z$, for some $t, \tau \in (0, 1)$, where $H(z) = \frac{\partial^2 W(z)}{\partial z^2}$ is the Hessian matrix of $W(z)$. By assumption, $c_1 \mathbb{I} \leq H(z) \leq c_2 \mathbb{I}$ for all $z \in \mathbb{R}^p$ and some positive constants c_1, c_2. We show first that for a suitable δ function U is positive definite and decrescent, i.e. there exist K_∞ functions α_1, α_2 such that $\alpha_1(\|(x, z)\|) \leq U(x, z) \leq \alpha_2(\|(x, z)\|)$. For this consider a continuously differentiable K_∞ function $\delta(r) = \bar{\delta} \int_0^r (\lambda + l\mu(\beta\sigma))^2 d\sigma$, with $\bar{\delta}, \beta > 0$. Note that

$$\begin{aligned} U(x, z) = \delta\left(x^T P x + W(z)\right) + x^T C^T k(z) \geq \\ \geq \bar{\delta} x^T P x + x^T C^T k(z) + \bar{\delta} W(z) \geq \\ \geq \bar{\delta} \lambda_{\min}(P) \|x\|^2 + c_1 \|C\| \|x\| \|z\| + \bar{\delta} c_1 \|z\|^2 \end{aligned}$$

it follows easily that U is positive definite for some value of $\bar{\delta}$ sufficiently large, with $\alpha_1(r)$ a quadratic function. In a similar manner it can be shown that U is decrescent, but in general $\alpha_2(r)$ is not a quadratic function. Next we show that the derivative of U is negative definite. Recall that by hypothesis $\|f(x, t)\| \leq (\lambda + l\mu(\|x\|)) \|x\|$, for all $t \geq 0$

and $x \in \mathbb{R}^n$.

$$\begin{aligned}\dot{U}(x,z) &\le -\delta'(V^*(x,z))\,\epsilon V(x) + k^T(z)\,Cf(x,t) + \\ &- k^T(z)\,CP^{-1}C^T k(z) + x^T C^T \frac{\partial k(z)}{\partial z} Cx \\ &\le -\bar{\delta}\epsilon\lambda_{\min}(P)\,(\lambda + l\mu(\beta V^*(x,z)))^2 \|x\|^2 \\ &+ c_2(\lambda + l\mu(\|x\|))\,\|C\|\,\|x\|\,\|z\| + \\ &- c_1^2 \left\|CP^{-1}C^T\right\| \|z\|^2 + c_2\|C\|^2\|x\|^2 \\ &\le -\alpha_3(\|(x,z)\|)\ .\end{aligned}$$

It follows easily that $\dot{U}$ is negative definite for some values of β and $\bar{\delta}$ sufficiently large. Moreover, since (locally) α_1 and α_3 are quadratic functions, global uniform asymptotic and local exponential stability follows [29]. ∎

Remark 5: The condition on matrix C to be of full row rank implies the requirement that $p \le n$, i.e. the number of integrators is at most equal to the state dimension of the nonlinear subsystem. Although, this is not necessary for the stability of the interconnection, it is in fact necessary to have asymptotic stability. This can be easily seen in the linear time invariant case, when $f(x,t) = Ax$ and $k(z) = Kz$. If $p > n$ the system matrix of system (6) will be singular, so that it cannot be Hurwitz matrix, whatever the values of A, B, C or K are. In the nonlinear time-invariant case, i.e. $f(x,t) = f(x)$, the necessity of $p \le n$ for exponential stability follows simply from linearization.

Remark 6: A twice continuously differentiable function $W(z)$ is *uniformly convex* in $\mathbb{R}^p$ [28], that is, there is a constant $c > 0$ such that, for all $z, y \in \mathbb{R}^p$ and $0 < \alpha < 1$ it is satisfied that $\alpha W(z) + (1-\alpha)W(y) - W(\alpha z + (1-\alpha)y) \ge c\alpha(1-\alpha)\|z-y\|^2$ if and only if its second order derivative (the Hessian matrix) is *uniformly positive definite* , i.e. $0 < c_1\mathbb{I} \le H(z)$ for all $z \in \mathbb{R}^p$.

III. Dissipative Observer Design

The Dissipative Design of Observers is a method recently proposed in [1], [2]. Its basic idea is to decompose the observer error dynamics into dissipative subsystems, and, using the dissipative theory, design the output injection in such a way, that the error dynamics converges. For the particular class of systems described by the form (1) the results are particularly suitable for calculations, since in this case the system is naturally decomposed as a LTI system with a memoryless nonlinearity in the feedback loop, for which checking dissipativity is particularly simple. For the nominal system Σ, i.e. with $w = 0$, a full order observer of the form

$$\Omega : \begin{cases} \dot{\hat{x}} = A\hat{x} + L(\hat{y} - y) + G\psi(\hat{\sigma} + N(\hat{y} - y)) + \\ \qquad +\varphi(t,y,u)\ , \quad \hat{x}(0) = \hat{x}_0 \\ \hat{y} = C\hat{x}\ ,\ \hat{\sigma} = H\hat{x} \end{cases} \tag{7}$$

is proposed, where matrices $L \in \mathbb{R}^{n\times p}$, and $N \in \mathbb{R}^{r\times p}$ have to be designed. Defining the state estimation error by $e \triangleq \hat{x} - x$, the output estimation error by $\tilde{y} \triangleq \hat{y} - y$, and the functional estimation error by $\tilde{\sigma} \triangleq \hat{\sigma} - \sigma$, the dynamics of e can be written as

$$\Xi : \begin{cases} \dot{e} = A_L e + G\nu\ , \quad e(0) = e_0 \\ z = H_N e\ ,\ \nu = -\phi(z,\sigma)\ , \end{cases} \tag{8}$$

where $A_L \triangleq A + LC$, $H_N \triangleq H + NC$, $z \triangleq H_N e$, and a new nonlinearity $\phi(z,\sigma) \triangleq \psi(\sigma) - \psi(\sigma + z)$. Note that $\phi(0,\sigma) = 0$ for all σ. In general, the error dynamics (8) is not autonomous, since it is driven by the system (1) through the linear function of the state $\sigma = Hx$. ϕ is therefore a time varying nonlinearity, whose time variation depends on the state trajectory of the plant.

The observer design consists in finding matrices L and N, if they exist, so that Ξ satisfies the conditions of Lemma 3. For this it is necessary to assume that the nonlinearity ϕ satisfy one or several supply rates ω:

Assumption 7: ϕ is (Q_i, S_i, R_i)-D for some finite set of non positive semidefinite quadratic forms $\omega_i(\phi, z) = \phi^T Q_i \phi + 2\phi^T S_i z + z^T R_i z \ge 0$, $\forall \sigma, i = 1, \cdots, M$.

If ϕ satisfies Assumption 7, then it is $\sum_{i=1}^{M} \theta_i (Q_i, S_i, R_i)$-D for every $\theta_i \ge 0$, i.e. ϕ is dissipative with respect to the supply rate $\omega(\phi, z) = \sum_{i=1}^{M} \theta_i \omega_i(\phi, z)$. In this case the design is as follows:

Theorem 8: [1], [2], [30] Suppose that assumption 7 is satisfied. If there are matrices L and N, and a vector $\theta = (\theta_1, \cdots, \theta_M)$, $\theta_i \ge 0$, such that the linear subsystem of Ξ is $\left(-R_\theta, S_\theta^T, -Q_\theta\right)$-SSD, with $(Q_\theta, S_\theta, R_\theta) = \sum_{i=1}^{M} \theta_i (Q_i, S_i, R_i)$, that is if there exist a matrix $P = P^T > 0$, matrices K, W, and $\epsilon > 0$ such that

$$\begin{bmatrix} PA_L + A_L^T P + \epsilon P + H_N^T R_\theta H_N\ , & PG - H_N^T S_\theta^T \\ G^T P - S_\theta H_N & Q_\theta \end{bmatrix} \le 0 \tag{9}$$

then Ω is a global exponential observer for Σ, $V(e) = e^T P e$ is a Lyapunov function for Ξ and $\dot{V} \le -\epsilon V(e)$.

Remark 9: All the results are valid if ψ depends on the time, and/or a measurable signal υ, as for example the input or the output of the plant. This is also true if system matrices are time-varying (or parameter-varying), but P and ϵ are constant.

Remark 10: The observer design relies on finding (if they exist) matrices L and N, a vector $\theta = (\theta_1, \cdots, \theta_M)$, $\theta_i \ge 0$, a matrix $P = P^T > 0$, and $\epsilon > 0$ such that the inequality (9) is satisfied. In general this is a nonlinear matrix inequality feasibility problem. However, when N is fixed it becomes Linear Matrix Inequality (LMI) feasibility problem, for which a solution can be effectively found by several algorithms in the literature [31], [32].

Remark 11: The proposed method generalizes, unifies and improves several methods previously proposed in the literature. Some of them are [1]: (i) The Circle criterion design [33], [34]. (ii) Lipschitz observer design [5], [6]. (iii) High-Gain observer design [3], [4].

IV. A Dissipative Proportional-Integral Observer Design

The observer designed in Theorem 8 is *proportional*, since only a static nonlinear function of the estimation error is injected. It is well known that the injection of an integral term of the estimation error greatly improves the robustness properties of the observer. In what follows, a (nonlinear) proportional-integral term will be included in the dissipative observer, and the properties of such observer will be studied using the dissipativity theory.

A *Proportional Integral Observer (PIO)* for system (1) is a dynamical system Ω_{PI} that has as inputs the input u and the output y of Σ, and its output $\hat{x}$ is an estimation of the state x of Σ. A full order PI observer for Σ of the form

$$\Omega_{PI}: \begin{cases} \dot{\hat{x}} = A\hat{x} + L\tilde{y} + G\psi(\hat{\sigma} + N\tilde{y}) + \varphi(t, y, u) + \\ \quad +E[\varkappa_I(\xi) + \varkappa_P(K\tilde{y})] \ , \quad \hat{x}(0) = \hat{x}_0 \\ \dot{\xi} = K\tilde{y} \ , \qquad \xi(0) = \xi_0 \\ \hat{y} = C\hat{x} \ , \ \hat{\sigma} = H\hat{x} \ , \ \tilde{y} = \hat{y} - y \ , \end{cases} \tag{10}$$

is proposed, where matrices $L \in \mathbb{R}^{n\times p}$, $N \in \mathbb{R}^{r\times p}$, $K \in \mathbb{R}^{q\times p}$, and $E \in \mathbb{R}^{n\times q}$, and the functions $\varkappa_I : \mathbb{R}^q \to \mathbb{R}^q$, and $\varkappa_P : \mathbb{R}^q \to \mathbb{R}^q$ have to be designed.

The dynamics of the error system Ξ_{PI} can be written as the feedback interconnection of two systems:

$$\Xi_{PI}: \begin{cases} \Xi_1: \begin{cases} \dot{e} = A_L e - G\phi(H_N e, \sigma) + Ev - Dw \ , \\ \tilde{\gamma} = KCe \ , e(0) = e_0 \end{cases} \\ \Xi_2: \begin{cases} \dot{\xi} = \tilde{\gamma} \ , \quad \xi(0) = \xi_0 \\ v = \varkappa_I(\xi) + \varkappa_P(\tilde{\gamma}) \ , \end{cases} \end{cases} \tag{11}$$

The objective of the design is to render the (closed) set $\{(e,\xi) \in \mathbb{R}^{n\times q} \mid e = 0\}$ asymptotically stable for system Ξ_{PI} in the nominal case, i.e. $w = 0$. Since $V_1(e) = e^T Pe$ is a Lyapunov function for the system we have for Ξ_1 that $\dot{V}_1 \le -\epsilon V_1(e) - e^T PE(-v)$. If K and E are selected such that $e^T PE = -\tilde{\gamma}^T = -e^T C^T K^T$, then it follows that Ξ_1 is strictly state passive from $(-v) \to \tilde{\gamma}$.

It will be shown in the sequel that if $\varkappa_I(\cdot)$ and $\varkappa_P(\cdot)$ are selected appropriately, then the subsystem Ξ_2 is also passive from $\tilde{\gamma} \to v$. Consider a C^1 function $V_2(\xi) \ge 0$ for all $\xi \in \mathbb{R}^q$, and $V_2(0) = 0$. Select

$$\varkappa_I(\xi) = \left(\frac{\partial V_2(\xi)}{\partial \xi}\right)^T .$$

Then it is clear that $V_2(\xi) = \int_0^\xi \varkappa_I(z) \cdot dz$. Moreover, if $\varkappa_P(\cdot)$ is such that $\varkappa_P^T(\tilde{\gamma})\tilde{\gamma} \ge 0$ for all $\tilde{\gamma}$, then along the trajectories of Ξ_2 it is satisfied

$$\dot{V}_2(\xi) = \varkappa_I^T(\xi)\dot{\xi} = \varkappa_I^T(\xi)\tilde{\gamma} = -\varkappa_P^T(\tilde{\gamma})\tilde{\gamma} + v^T\tilde{\gamma} \le v^T\tilde{\gamma} \ .$$

It follows then that the time derivative of the storage function $V(e,\xi) = V_1(e) + V_2(\xi)$ along the solutions of Ξ_{PI} is

$$\begin{aligned} \dot{V}(e,\xi) &\le -\epsilon V_1(e) - \tilde{\gamma}^T v - \varkappa_P^T(\tilde{\gamma})\tilde{\gamma} + v^T\tilde{\gamma} \\ &\le -\epsilon V_1(e) - \varkappa_P^T(\tilde{\gamma})\tilde{\gamma} \ . \end{aligned}$$

This ensures that $e(t) \to 0$ as $t \to \infty$. Moreover, if $V_2(\xi)$ is radially unbounded, then the state (e,ξ) will be bounded.

If convergence of the equilibrium point $(e,\xi) = 0$ is desired, further conditions are required. A set of such conditions are given in the next theorem, together with a strong Lyapunov function to ensure this.

Theorem 12: Suppose that the conditions of Theorem 8 are satisfied, and that $\psi(\sigma)$ and $\varkappa_P(\tilde{\gamma})$ are globally bounded by some K_∞ functions. If given any $K \in \mathbb{R}^{q\times p}$ such that KC has full row rank, and a C^2 uniformly convex in $\mathbb{R}^q$, positive definite, decrescent function $V_2(\xi)$, with $V_2(0) = 0$, one selects $E = -P^{-1}C^T K^T$, $\varkappa_I(\xi) = \left(\frac{\partial V_2(\xi)}{\partial \xi}\right)^T$ and $\varkappa_P^T(\tilde{\gamma})\tilde{\gamma} \ge 0$, with $\varkappa_I(\xi)$ globally Lipschitz, then Ω_{PI} (10) is a globally asymptotic and locally exponential stable PI-observer for Σ. Moreover, the function $U(e,\xi) = \delta\left(e^T Pe + V_2(\xi)\right) + e^T C^T K^T \varkappa_I(\xi)$ is a strong Lyapunov function for $\delta(\cdot)$ some suitable K_∞ function.

Proof: Let us rewrite the system as

$$\Xi_{PI}: \begin{cases} \dot{e} = f(e,\sigma) - P^{-1}C^T K^T \varkappa_I(\xi) \ , \quad e(0) = e_0 \\ \dot{\xi} = KCe \ , \quad \xi(0) = \xi_0 \end{cases}$$

where

$$f(e,\sigma) \triangleq A_L e - G\phi(H_N e, \sigma) - P^{-1}C^T K^T \varkappa_P(KCe) \ . \tag{12}$$

From the Hypothesis it follows that $f(e,\sigma)$ is globally bounded by some K_∞ function uniformly in σ. Consider $V(e) = e^T Pe$. Its time derivative along the solutions of system $\dot{e} = f(e,\sigma)$ is

$$\begin{aligned} \dot{V}(e) &= e^T Pf(e,\sigma) = e^T P(A_L e - G\phi(H_N e,\sigma)) + \\ &\quad -e^T C^T K^T \varkappa_P(KCe) \le -\epsilon V(e) \ . \end{aligned}$$

The result then follows directly from Proposition 4. ■

Remark 13: A particular, but important, case is the one when $\varkappa_I(\xi) = K_I\xi$, $K_I = K_I^T > 0$ and $\varkappa_P(\tilde{\gamma}) = K_P\tilde{\gamma}$, $K_P > 0$, are linear functions. In this case $V_2(\xi) = \frac{1}{2}\xi^T K_I \xi$. However, it is also possible to use nonlinear and discontinuous functions $\varkappa_P(\tilde{\gamma})$.

Remark 14: Note the enormous flexibility in selecting the PI-gains allowed by the Theorem: if $\varkappa_I(\xi)$ and $\varkappa_P(\tilde{\gamma})$ are appropriate gains, then so do $\gamma_1\varkappa_I(\xi)$ and $\gamma_2\varkappa_P(\tilde{\gamma})$ for arbitrary positive gains γ_1 and γ_2.

V. Robustness properties of the PI-Observer

Now consider the perturbed case, when in (1) $w \ne 0$. Suppose that the matching condition $D = -P^{-1}C^T K^T = E$ is satisfied, and define $\bar{\xi} = \varkappa_I^{-1}(w)$, that exists for every w, since $\varkappa_I$ is globally invertible, and $e_\xi = \xi - \bar{\xi}$. The error dynamics (11) can be written as

$$\Xi_{PI}: \begin{cases} \dot{e} = f(e,\sigma) + D\left(\varkappa_I(\bar{\xi} + e_\xi) - \varkappa_I(\bar{\xi})\right) \ , \\ \dot{e}_\xi = KCe + \zeta \ , \ e(0) = e_0, e_\xi(0) = e_{\xi 0} \end{cases}$$

where $f(e,\sigma)$ is given by (12), and $\zeta = d\bar{\xi}/dt = \left(\frac{\partial \varkappa_I(\bar{\xi})}{\partial \xi}\right)^{-1} \dot{w}$ is related to the time derivative of w. The robust convergence of the PI-Observer is assured by the next Theorem.

Theorem 15: Suppose that the conditions of Theorem 12 are satisfied, that $\psi(\sigma)$ and $\varkappa_P(\tilde{\gamma})$ are globally Lipschitz, and that the matching condition $D = -P^{-1}C^T K^T$ is satisfied. Under these conditions, if one selects $E = D$, the PI-Observer Ω_{PI} (10) is a globally exponentially stable PI-observer for Σ, so that $\hat{x} \to x$ and $\xi \to \bar{\xi}$, when the perturbation w is constant (or converges to a constant value). When w is a time-varying signal, the observation error (e, e_ξ) converges to a neighborhood of the origin, with a radius depending on the time derivative of w, that is, the observer is practically stable.

Proof: From the globally Lipschitzness of $f(e, \sigma)$ follows the global exponential stability of the PI-Observer error, from which, by standard arguments [25], it follows its ISS with respect to the perturbation ζ, from which the results of the Theorem are obtained. ■

Remark 16: When the (unknown) perturbation is constant, its value can be determined from the state of the integral term, since $w = \varkappa_I(\bar{\xi})$. If w changes slowly, this estimation is correct up to a small error. When the matching condition $D = -P^{-1}C^T K^T$ is satisfied, the effect of the perturbation can be compensated exactly by the integral term of the PI-observer.

Remark 17: An alternative way of designing a PI-Observer by the dissipative method consists in two steps. First, the plant model will be extended by a model of the perturbation. Since in this case the perturbation is constant a model is given by an integrator. In the second step a dissipative observer can be designed for the extended plant model. This alternative method constitutes an extension of most known results [9], [17], [18], [19], [20]. It allows also to alleviate the main drawback of the proposed method in this paper: the satisfaction of the matching condition for eliminating the perturbation.

Using the convergence and robustness properties of the PI-Observer a separation result in the spirit of our previous result [30] can be proved for a closed loop system with constant perturbations.

VI. Example

To illustrate the design procedure and the performance of the proposed method, consider a system composed of two carts, where x_1 (x_3) and x_2 (x_4) are the position and velocity of Cart 1 (Cart 2), respectively. Carts 1 and 2 are connected by a linear spring with constant k_1 and a linear damper with damping coefficient d_1. Cart 1 is fixed to the wall through a nonlinear spring, with spring function $\delta(x_1) = k_2 \tanh(x_1) + k_3 x_1$, with k_2, k_3 positive constants, and a nonlinear damper, with friction represented by $\xi(x_2) = F_m x_2^2 \operatorname{sign}(x_2) + \mu_v x_2$, where $F_m \geq 0$ and $\mu_v \geq 0$ are viscous friction constants. On Cart 2 acts an external force u. A state space representation of the system is given by (1), where $\psi(\sigma) = \delta(\sigma_1) + \xi(\sigma_2)$ and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_1 & -d_1 & k_1 & d_1 \\ 0 & 0 & 0 & 1 \\ k_1 & d_1 & -k_1 & -d_1 \end{bmatrix} B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} G = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} .$$

The nonlinearity $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$, $\psi(\sigma) = \delta(\sigma_1) + \xi(\sigma_2)$ represents the total force acting on Cart 1, consisting of the sum of the forces due to the nonlinear spring, depending on the elongation σ_1, and the nonlinear friction, that is a function of the velocity σ_2. Note that none of the methods in Remark 11 can be used to design a P-Observer for this system, since the nonlinearity has two inputs and one output and is not globally Lipschitz. Therefore none of the known methods to design PI-Observers can be used in this case.

$\phi(z, \sigma) = \phi_1(z_1, \sigma_1) + \phi_2(z_2, \sigma_2)$ is written as the sum of the individual incremental functions $\phi_1(z_1, \sigma_1) = \delta(\sigma_1) - \delta(\sigma_1 + z_1)$ and $\phi_2(z_2, \sigma_2) = \xi(\sigma_2) - \xi(\sigma_2 + z_2)$. It is possible to show that ϕ is (Q, S, R) −D with

$$Q = 0 \ , \ S = \begin{bmatrix} 0 & -\frac{1}{2} \end{bmatrix} \ , R = \begin{bmatrix} (k_2 + k_3)^2 & 0 \\ 0 & \left(\frac{1}{4} - \mu_v\right) \end{bmatrix} .$$

For numerical calculations and simulations $k_1 = 1$, $d_1 = 1$, $k_2 = 5$, $k_3 = 1$, $\mu_v = 1$, $F_m = 2$, have been taken. The following values of P, ϵ, N and L satisfy the MI (9) and are used to design a P-Observer:

$$P = \begin{bmatrix} 19.375 & 0 & -44.75 & -18.875 \\ 0 & 0.5 & -7.5 & 0 \\ -44.750 & -7.5 & 300 & 25 \\ -18.875 & 0 & 25 & 25.875 \end{bmatrix} \epsilon = 1.75$$

$$N = \begin{bmatrix} 5 \\ -15 \end{bmatrix} L^T = [-324.5, -797.5, -53.2, -194.6] \ .$$

A nonlinear PI-Observer is designed proposing $\varkappa_I(\xi) = 3000\left(\xi + \xi^3 \operatorname{sat}(0.001\xi)\right)$ and $\varkappa_P(\tilde{\gamma}) = 50\left(10\tilde{\gamma} + \tilde{\gamma}^3\right)$, that satisfy the conditions of Theorem 12. Simulation results are presented in Figures 1 and 2. The input function $u(t)$ is selected as a step function that changes from -1.9 (Nt) to 8.1 (Nt) at $t = 5.8$ (s). The initial conditions are $x_0 = 0$ for the plant, $x_{P0} = \begin{bmatrix} 2 & 4 & 1 & -5 \end{bmatrix}$ for the P-Observer and $x_{PI0} = -x_{P0}$ for the PI-Observer. They have been selected different to improve the visibility of the graphics. In Figure 1 the observation errors for both observers are presented for the nominal system without perturbation, whereas in Figure 2 the observation errors are seen when a (matched) constant perturbation is added to the plant. It is clear that the response of both observers for the nominal case are very similar, so that no loss of convergence velocity is appreciated by the introduction of the Integral term in the observer. When the constant perturbation of magnitude 500 is added, the asymptotic insensibility of the PI-Observer is clearly appreciated in Figure 2, that shows a large steady state deviation of the P-Observer.

VII. Conclusions

Given a P-Observer, designed by an arbitrary method, in this paper a procedure is proposed to design a PI-Observer, based on dissipativity properties. In this form it is easy to design nonlinear proportional and integral gains ensuring the stability of the PI-Observer. Its main drawback is the requirement of a matching condition for the perturbation. These PI-Observers have the usual properties of systems with integral terms, that are robust against constant perturbations, and they can be used, in principle, for robust regulation purposes. For this kind of observers the separation property derived in [30] for the basic dissipative observers can be

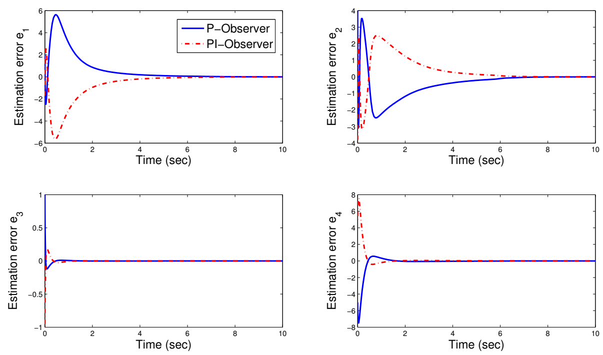

Fig. 1. Estimation error of the P- and PI-Observers without perturbation.

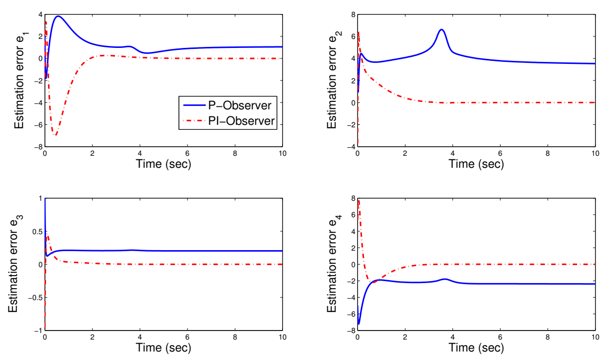

Fig. 2. Estimation error of the P- and PI-Observers with perturbation.

extended to systems with constant perturbations. A further extension would consist in selecting the observer gains to attenuate the effect of the perturbation on the estimation error.

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