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Perturbation theory of boundary value problems and approximate controllability of perturbed boundary control problems

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***Abstract*— A semigroup approach for the well-posedness of perturbed nonhomogeneous abstract boundary value problems is developed in this paper. This allows us to introduce a useful variation of constant formula for the solutions. Drawing from this formula, necessary and sufficient conditions for the approximate controllability of such systems are obtained, using the feedback theory of well-posed and regular linear systems developed by Salamon, Staffans and Weiss.**

***Index Terms*— Boundary value problem, perturbation theory, approximate controllability, regular systems.**

I. Introduction

Perturbation theory has been proved very useful to solve many Cauchy problems that can be regarded as perturbations of well-established problems [1, Chap.III]. The well-known classes of perturbations for strongly continuous semigroups are the Miyadera–Voigt and the Desch–Schappacher perturbations [1]. In the last few years, a general perturbation theory for infinite dimensional linear systems had been introduced, in Hilbert spaces [2] and in Banach spaces [3, Chap.7]. All these perturbations are distributional. In many cases, however, it happens that the perturbation acts at the boundary of systems, hence the concept of perturbed boundary-value problems, [4], [5]. These problems result from the feedback theory of boundary control problems [3], [5] , where the control is 0.

The following nonhomogeneous perturbed boundary value problem is considered in this paper:

$$\begin{aligned} \dot{w}(t) &= \mathcal{A}_m w(t) + f(t), \quad t \geq 0, \quad w(0) = \varpi \in \mathcal{X}, \\ \mathcal{N} w(t) &= \mathcal{M} w(t) + g(t) \end{aligned} \tag{1}$$

where $\mathcal{A}_m : \mathcal{X}_m \to \mathcal{X}$ and $\mathcal{N}, \mathcal{M} : \mathcal{X}_m \to \mathcal{U}$ are linear operators with Banach spaces $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{X}_m$ being a dense domain of $\mathcal{X}$ endowed with a norm $|\cdot|$, which is finer than the norm $\|\cdot\|$ of $\mathcal{X}$ such that $(\mathcal{X}_m, |\cdot|)$ is complete, and $f : [0,\infty) \to \mathcal{X}$, $g : [0,\infty) \to \mathcal{U}$ are locally p-integrable functions. Many partial differential equations can be reformulated as (1). These include examples of boundary and point controls, neutral systems, which are motivated by important applications in aeroelastic systems, thermoelastic plates, etc.

According to [4], [5], assume

(**H1**) $\mathcal{A} = \mathcal{A}_m$ with domain $\mathcal{D}(\mathcal{A}) := \mathrm{Ker}\mathcal{N}$ generates a C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ on $\mathcal{X}$,
(**H2**) $\mathrm{Im}\mathcal{N} = \mathcal{U}$.

These assumptions show that the boundary value problem (1) with $\mathcal{M} = 0$ and $f = 0$ is well-posed in the sense that it can be reformulated as a well-posed open loop system on the state space $\mathcal{X}$ and control space $\mathcal{U}$; see e.g. [6], [7], [8], [9], [10], [11] and the references therein. One can think of $\mathcal{M}$ as a perturbation of the boundary operator $\mathcal{N}$. The well-posedness of (1) with $f = 0$ and $g = 0$ has been studied in [5] using boundary control systems theory and in [4] for the case of bounded perturbations $\mathcal{M}$ using Hille–Yosida theorem [1]. In this paper, weaker assumptions on $\mathcal{M}$ will be introduced so that the system (1) is equivalent to a well-posed open loop system and a variation of constants formula for the solution of system (1) will be developed, using the feedback theory of regular linear systems [2]. This formula will then be used to develop conditions on approximate controllability of systems that can be reformulated as system (1), e.g. population dynamic systems and neutral differential equations in Banach spaces.

Controllability, which reflects the reachability of a point in the state space, is an important concept in systems theory. This has been a very active area for many years; see e.g. [12], [13], [14], [15], [16], [17], [18], [19] for this property about time-delay systems, which belong to a special class of infinite-dimensional systems and widely exist in engineering [20]. In the classical finite-dimensional case, it is possible to define controllability in different ways but they are all equivalent and lead to the same concept. In the infinite-dimensional case, this is no longer true. In this paper, the concept of approximate controllability is studied, in the sense

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that, for a given arbitrary $\varepsilon > 0$, it is possible to steer the state from the origin to the neighborhood, with a radius ε, of all points in the state space. This is weaker than the concept of exact controllability, where $\varepsilon = 0$. These concepts was first studied by Fattorini [8] for distributed-parameter systems; see [7] for details and some applications to systems with state delays. The contribution of this paper is to develop the approximate controllability of perturbed boundary control problems, based on the perturbation theory of boundary value problems.

The organization of the paper is as follows. The notion of regular systems and their closed-loop systems is recalled in Section II. The well-posedness of the homogeneous boundary value problem associated with system (1) is developed in Section III; a variation of constants formula for the solution of system (1) is established in Section IV and then the approximate controllability of perturbed boundary control problems is investigated in Section V. Conclusions are made in Section VI.

II. Preliminaries: An overview of Salamon–Weiss systems

In this section, the framework of infinite dimensional well-posed and regular linear systems in the Salamon–Weiss sense is recalled. See [5], [3], [21], [22], [2], [23] for more details.

Throughout this section X, U, Y are Banach spaces, $p > 1$ is a real number and $(A, \mathcal{D}(A))$ is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X. The type of $T(t)$ is defined as $\omega_0(A) := \inf\{t^{-1}\log\|T(t)\| : t > 0\}$. The domain $\mathcal{D}(A)$, which is a Banach space, is endowed with the graph norm $\|x\|_A := \|x\| + \|Ax\|$, $x \in \mathcal{D}(A)$. Denote the resolvent set of A by $\rho(A)$ and the resolvent operator of A by $R(\mu, A) := (\mu - A)^{-1}$ for $\mu \in \rho(A)$. The completion of X with respect to the norm $\|x\|_{-1} = \|R(\mu, A)x\|$ for $x \in X$ and some $\mu \in \rho(A)$ is a Banach space denoted by X_{-1}, which is called the *extrapolation space* of A. For any Banach spaces E and F, denote the Banach space of all linear bounded operators from E to F by $\mathcal{L}(E, F)$ with $\mathcal{L}(E) := \mathcal{L}(E, E)$. Let $f : \mathbb{R} \to X$ be a function and let I be an interval in $\mathbb{R}$. Then the restriction $f|_I$ of f to the interval I is defined as $f|_I = \begin{cases} f & \text{on the interval } I, \\ 0 & \text{otherwise.} \end{cases}$

Definition 1: $B \in \mathcal{L}(U, X_{-1})$ is called an *admissible control operator* for A if for all $t > 0$ and $u \in L^p([-r, 0], U)$ the *control map*

$$\Phi(t)u := \int_0^t T(t-s)Bu(s)\,ds \tag{2}$$

takes values in X.

If B is an admissible control operator, then the closed graph theorem shows that $\Phi(t) \in \mathcal{L}(L^p([0, t], U), X)$. Moreover,

$$\Phi(t+s)u = \Phi(t)(u(\cdot + s)|_{[0,t]}) + T(t)\Phi(s)(u|_{[0,s]}) \tag{3}$$

for $t, s \geq 0$ and $u \in L^p([0, s+t], U)$. The pair (T, Φ) is called a *control system* represented by the operator B; see [22] for more details.

Definition 2: Let (T, Φ) be the control system represented by the operator B. Define the *reachability space*

$$\mathcal{R} := \cup_{t\geq 0}\mathrm{Ran}\Phi(t).$$

Then, $(A,\, B)$ is said to be *approximately controllable* if $\mathcal{R}$ is dense in X. Here, Ran is the range of an operator.

This means that, for a given arbitrary $\varepsilon > 0$, it is possible to steer the state from the origin to the neighborhood, with a radius ε, of all points in the state space.

Definition 3: $C \in \mathcal{L}(\mathcal{D}(A), Y)$ is called an *admissible observation operator* for A if

$$\int_0^\tau \|CT(t)x\|^p\,dt \leq \gamma^p\|x\|^p \tag{4}$$

for all $x \in \mathcal{D}(A)$ and constants $\tau \geq 0$ and $\gamma := \gamma(\tau) > 0$.

Let C be an admissible observation operator for A. According to (4), the map $\Psi_\infty x := CT(\cdot)x$, defined on $\mathcal{D}(A)$, extends to a bounded operator $\Psi_\infty : X \to L^p_{loc}(\mathbb{R}_+, Y)$. For any $x \in X$ we set $\Psi(t)x := \Psi_\infty x$ on $[0, t]$. This shows that $\{\Psi(t) : t \geq 0\} \subset \mathcal{L}(X, L^p_{loc}(\mathbb{R}_+, X))$ and

$$\begin{aligned} &\Psi(t+s)x = \Psi(s)x \quad \text{on } [0, s], \\ &\Psi(t+s)x = [\Psi(t)T(s)x](\cdot - s) \quad \text{on } [s, s+t] \end{aligned} \tag{5}$$

for $t, s \geq 0$ and $x \in X$. We call $(T(t), \Psi(t))_{t\geq 0}$ an *observation system* [21].

Definition 4: The *Yosida extension* of an operator $C \in \mathcal{L}(\mathcal{D}(A), Y)$ with respect to A is the operator defined by

$$\begin{aligned} \tilde{C}x &:= \lim_{\sigma\to+\infty} C\sigma R(\sigma, A)x, \\ \mathcal{D}(\tilde{C}) &:= \Big\{x \in X : \lim_{\sigma\to+\infty} C\sigma R(\sigma, A)x \text{ exists in } Y\Big\}. \end{aligned} \tag{6}$$

If C is an admissible observation operator for A then

$$T(t)x \in \mathcal{D}(\tilde{C}) \quad \text{and} \quad (\Psi_\infty x)(t) = \tilde{C}T(t)x \tag{7}$$

for all $x \in X$ and almost every (a.e.) $t \geq 0$ [21, Thm. 4.5].

Definition 5: Let B and C be admissible control and observation operators for A, respectively, and $(T(t), \Phi(t))_{t\geq 0}$ and $(T(t), \Psi(t))_{t\geq 0}$ be the corresponding control and observation systems. Then (A, B, C) is said to generate a *well-posed system* on X, U, Y if there exist operators $\mathbb{F}(t) \in$

$\mathcal{L}(L^p([0,t],U), L^p([0,t],Y))$ such that

$$[\mathbb{F}(t+s)u](\tau) = [\mathbb{F}(t)(u(\cdot+s)|_{[0,t]})](\tau-s) + [\Psi(t)\Phi(s)(u|_{[0,s]})](\tau-s) \tag{8}$$

for $\tau \in [s, s+t]$, $t, s \geq 0$, and $u \in L^p([0,s+t],U)$. This well-posed system is denoted as $\Sigma := (T(t), \Phi(t), \Psi(t), \mathbb{F}(t))$ and the operators $\mathbb{F}(t)$ are called *input–output operators*.

An important subclass of the well-posed systems, called regular systems, is defined as follows [2].

Definition 6: Assume that (A,B,C) is the generator of a well-posed system Σ on X,U,Y with input–output operators $(\mathbb{F}(t))_{t\geq 0}$. Define

$$(\mathbb{F}_\infty u)(t) := (\mathbb{F}(\tau)u)(t), \quad t \in [0,\tau].$$

The system Σ is called *regular* (with feedthrough $D=0$) if

$$\lim_{t\to 0^+} \frac{1}{t}\int_0^t (\mathbb{F}_\infty u_0)(\sigma)\,d\sigma = 0$$

exists in Y for the constant function $u_0(t) = u, u \in U, t \geq 0$. Alternatively, (A,B,C) is called a regular triple.

Assume that

$$\begin{cases} \dot{w}(t) = Aw(t) + Bu(t), & t \geq 0, \\ y(t) = Cw(t), & t \geq 0, \end{cases} \tag{9}$$

with $B \in \mathcal{L}(U, X_{-1})$ and $C \in \mathcal{L}(\mathcal{D}(A), Y)$ is a regular system. Let $\tilde{C}$ be the Yosida extension of C with respect to A and $\Phi(t)$ be the control map as in (2). Then the state trajectory $w(\cdot)$ and output function $y(\cdot)$ of the system (9) satisfy [2], [23]:

(1) For any $t \geq 0$ and $u \in L^p([0,t],U)$,

$$w(t) = T(t)w(0) + \Phi(t)u.$$

(2) For almost every $t \geq 0$,

$$w(t) \in \mathcal{D}(\tilde{C}) \quad \text{and} \quad y(t) = \tilde{C}w(t).$$

(3) For $\mu \in \rho(A)$,

$$\operatorname{Im}\big(R(\mu, A_{-1})B\big) \subset \mathcal{D}(\tilde{C}) \tag{10}$$

and the transfer function of the system (9) is given by

$$G(\mu) = \tilde{C}R(\mu, A_{-1})B. \tag{11}$$

According to [2], there exists $\gamma > 0$ such that

$$\sup_{\mathrm{Re}\mu > \gamma} \|G(\mu)\| < \infty. \tag{12}$$

Definition 7: Let Σ be a well-posed linear system on X,U,Y with input–output operators $\mathbb{F}(t)$, $t \geq 0$. An operator $\Gamma \in \mathcal{L}(Y,U)$ is called an *admissible feedback* for Σ if the operator $I - \mathbb{F}(t)\Gamma$ has uniformly bounded inverse.

Theorem 8: [23] Let (A,B,C) generates a regular linear system Σ with $\Gamma \in \mathcal{L}(Y,U)$ as an admissible feedback, and $\tilde{C}$ be the Yosida extension of C with respect to A. Then $(A_\Gamma, B, \tilde{C})$ generates a regular linear system Σ^Γ, where

$$A_\Gamma := A_{-1} + B\Gamma\tilde{C},$$
$$\mathcal{D}(A_\Gamma) = \big\{x \in D(\tilde{C}) : (A_{-1} + B\Gamma\tilde{C})x \in X\big\}.$$

The C_0-semigroup $T_\Gamma := (T_\Gamma(t))_{t\geq 0}$ on X generated by A_Γ satisfies $\operatorname{Im}(T_\Gamma(t)) \subset \mathcal{D}(\tilde{C})$ for a.e. $t \geq 0$, and

$$T_\Gamma(t)x = T(t)x + \int_0^t T_{-1}(t-\tau)B\Gamma\tilde{C}T_\Gamma(\tau)x\,d\tau$$

for any $t \geq 0$ and $x \in X$.

III. Homogeneous boundary value problems

Considering the system (1) with $f = g = 0$, that is,

$$\begin{aligned} \dot{w}(t) &= \mathcal{A}_m w(t), \quad t \geq 0, \quad w(0) = \varpi, \\ \mathcal{N}w(t) &= \mathcal{M}w(t), \qquad t \geq 0. \end{aligned} \tag{13}$$

Define the operator

$$\mathfrak{A}x = \mathcal{A}_m x, \quad \mathcal{D}(\mathfrak{A}) = \big\{x \in \mathcal{D}(\mathcal{A}_m) : \mathcal{N}x = \mathcal{M}x\big\}. \tag{14}$$

This operator is already used by Salamon [5] and Greiner [4] to prove the well-posedness of the initial value problem (13). In the present work, we introduce weaker conditions on $\mathcal{M}$ to prove that the operator $\mathfrak{A}$ is a generator in $\mathcal{X}$ and, moreover, characterize its spectrum using the feedback theory of regular linear systems. Before going into details, some known results concerning boundary control systems are recalled. Consider

$$\begin{aligned} \dot{w}(t) &= \mathcal{A}_m w(t), \quad t \geq 0, \quad w(0) = \varpi, \\ \mathcal{N}w(t) &= u(t), \qquad t \geq 0, \end{aligned} \tag{15}$$

where $u(t)$ is the boundary input, coupled with the observation equation

$$y(t) = \mathcal{M}w(t), \qquad t \geq 0. \tag{16}$$

The feedback law $y(t) = u(t)$ retrieves the abstract boundary value problem (13). In order to investigate the admissibility of this feedback law, the boundary input–output system (15)–(16) is transformed into a distributed-parameter linear system.

Assume that the conditions (H1) and (H2) are satisfied. Greiner [4, Lemma 1.2] and Salamon [5] showed that the following direct sum holds:

$$\mathcal{D}(\mathcal{A}_m) = \mathcal{D}(\mathcal{A}) \oplus \operatorname{Ker}(\mu - \mathcal{A}_m), \qquad \mu \in \rho(\mathcal{A}), \tag{17}$$

and the following inverse, called the *Dirichlet operator*,

$$\mathfrak{D}_\mu := \Big(\mathcal{N}_{|\operatorname{Ker}(\mu - \mathcal{A}_m)}\Big)^{-1} : \mathcal{U} \to \operatorname{Ker}(\mu - \mathcal{A}_m) \tag{18}$$

exists and is bounded. For $\mu \in \rho(\mathcal{A})$, there is

$$(\mathcal{A}_m - \mathcal{A}_{-1}) = (\mu - \mathcal{A}_{-1})\mathfrak{D}_\mu \mathcal{N} \quad \text{on} \quad \mathcal{D}(\mathcal{A}_m).$$

Define

$$\mathcal{B} := (\mu - \mathcal{A}_{-1})\mathfrak{D}_\mu, \qquad \mu \in \rho(\mathcal{A}). \tag{19}$$

Then $\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{X}_{-1})$, $\mathrm{Im}(\mathcal{B}) \cap \mathcal{X} = \{0\}$ and

$$(\mathcal{A}_m - \mathcal{A}_{-1})_{|\mathcal{D}(\mathcal{A}_m)} = \mathcal{B}\mathcal{N}. \tag{20}$$

According to (19) and the resolvent equation

$$R(\lambda, \mathcal{A}) - R(\mu, \mathcal{A}) = (\mu - \lambda)R(\lambda, \mathcal{A})R(\mu, \mathcal{A}),$$

there is

$$\mathfrak{D}_\lambda - \mathfrak{D}_\mu = (\mu - \lambda)R(\lambda, \mathcal{A})\mathfrak{D}_\mu \quad \text{for} \quad \lambda, \mu \in \rho(\mathcal{A}). \tag{21}$$

Using (20), one can see that the system (15)–(16) is equivalent to the following distributed-parameter system

$$\begin{aligned} \dot{w}(t) &= \mathcal{A}_{-1}w(t) + \mathcal{B}u(t), \quad t \geq 0, \quad w(0) = \varpi, \\ y(t) &= \mathcal{C}w(t), \qquad t \geq 0, \end{aligned} \tag{22}$$

where $\mathcal{C} := \mathcal{M}J$ with J the canonical injection from $\mathcal{D}(\mathcal{A})$ to $\mathcal{D}(\mathcal{A}_m)$. The boundary system (15)–(16) is called well-posed if and only if $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ generates a well-posed linear system on $\mathcal{X}, \mathcal{U}, \mathcal{U}$.

Lemma 1: Assume that (H1)–(H2) are satisfied and $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ generates a regular linear system with transfer function $G(\mu)$ and that

$$\lim_{\sigma \to +\infty} \mathcal{M}\mathfrak{D}_\sigma z = 0, \qquad z \in \mathcal{U}. \tag{23}$$

Then

$$\mathcal{D}(\mathcal{A}_m) \subset \mathcal{D}(\tilde{\mathcal{C}}), \quad \mathcal{M} = \tilde{\mathcal{C}} \quad \text{on} \quad \mathcal{D}(\mathcal{A}_m), \tag{24}$$

where $\widetilde{\mathcal{C}}$ is the Yosida extension of $\mathcal{C}$ with respect to $\mathcal{A}$. In this case,

$$G(\mu) = \mathcal{M}\mathfrak{D}_\mu, \qquad \mu \in \rho(\mathcal{A}). \tag{25}$$

Proof: By the above assumptions and by combining (19) with (10) we have $\mathrm{Im}(\mathfrak{D}_\mu) \subset \mathcal{D}(\tilde{\mathcal{C}})$ for some $\mu \in \rho(\mathcal{A})$. Since $\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\tilde{\mathcal{C}})$, by (17) we have $\mathcal{D}(\mathcal{A}_m) \subset \mathcal{D}(\tilde{\mathcal{C}})$. Take $z \in \mathcal{U}$ and a real number $\sigma > \omega_0(\mathcal{A})$ and $\mu \in \rho(\mathcal{A})$. The identity (21) shows that

$$\lim_{\sigma \to +\infty} \mathcal{M}\mathfrak{D}_\sigma z = \mathcal{M}\mathfrak{D}_\mu z - \lim_{\sigma \to +\infty} \mathcal{C}\sigma R(\sigma, \mathcal{A})\mathfrak{D}_\mu z.$$

According to the assumption (23), there is

$$\mathcal{M}\mathfrak{D}_\mu = \tilde{\mathcal{C}}\mathfrak{D}_\mu \quad \text{on} \quad \mathcal{U}. \tag{26}$$

Since $z - \mathfrak{D}_\mu \mathcal{N} z \in \mathcal{D}(\mathcal{A})$ for $z \in \mathcal{D}(\mathcal{A}_m)$,

$$\mathcal{M}(z - \mathfrak{D}_\mu \mathcal{N} z) = \mathcal{C}(z - \mathfrak{D}_\mu \mathcal{N} z) = \tilde{\mathcal{C}}(z - \mathfrak{D}_\mu \mathcal{N} z).$$

Now (24) follows by (26) and, finally, (25) follows by (11). ■

Theorem 9: Assume that (H1)–(H2) are satisfied and $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ generates a regular linear system with the identity operator $\Gamma = I : \mathcal{U} \to \mathcal{U}$ as an admissible feedback operator. Moreover, assume that (23) holds. Then the operator $\mathfrak{A}$ defined in (14) generates a strongly continuous semigroup $\mathcal{T}_I := (\mathcal{T}_I(t))_{t \geq 0}$ on $\mathcal{X}$, satisfying $\mathrm{Im}\big(\mathcal{T}_I(\tau)\big) \subset \mathcal{D}(\tilde{\mathcal{C}})$ for almost every $\tau \geq 0$. Moreover,

$$\mathcal{T}_I(t)x = \mathcal{T}(t)x + \int_0^t \mathcal{T}_{-1}(t - \tau)\mathcal{B}\tilde{\mathcal{C}}\mathcal{T}_I(\tau)x \, d\tau \tag{27}$$

for all $t \geq 0$ and $x \in \mathcal{X}$.

Proof: The above assumptions and Theorem 8 show that the operator

$$\begin{aligned} \mathcal{A}_I &:= \mathcal{A}_{-1} + \mathcal{B}\tilde{\mathcal{C}}, \\ \mathcal{D}(\mathcal{A}_I) &:= \big\{x \in \mathcal{D}(\tilde{\mathcal{C}}) : (\mathcal{A}_{-1} + \mathcal{B}\tilde{\mathcal{C}})x \in \mathcal{X}\big\} \end{aligned} \tag{28}$$

generates a strongly continuous semigroup $\mathcal{T}_I := (\mathcal{T}_I(t))_{t \geq 0}$ on $\mathcal{X}$ satisfying (27). Now we prove that the operator $\mathcal{A}_I$ coincides with $\mathfrak{A}$. For $\mu \in \rho(\mathcal{A})$ and $x \in \mathcal{D}(\mathcal{A}_I)$,

$$\mathcal{A}_I x = \mathcal{A}_{-1}(x - \mathfrak{D}_\mu \tilde{\mathcal{C}}x) + \mu \mathfrak{D}_\mu \tilde{\mathcal{C}}x \in \mathcal{X},$$

due to (19). Thus $x - \mathfrak{D}_\mu \tilde{\mathcal{C}}x \in \mathcal{D}(\mathcal{A})$. Moreover, $\mathcal{N}x = \mathcal{N}\mathfrak{D}_\mu \tilde{\mathcal{C}}x = \tilde{\mathcal{C}}x = \mathcal{M}x$, by (24). This shows that $\mathcal{D}(\mathcal{A}_I) \subset \mathcal{D}(\mathfrak{A})$. By combining (20) and (24), it follows that

$$\begin{aligned} \mathfrak{A}x &= \mathcal{A}_m x = \mathcal{A}_{-1}x + \mathcal{B}\mathcal{N}x = \mathcal{A}_{-1}x + \mathcal{B}\mathcal{M}x \\ &= \mathcal{A}_{-1}x + \mathcal{B}\tilde{\mathcal{C}}x = \mathcal{A}_I x. \end{aligned}$$

The converse follows as above and hence $\mathfrak{A} = \mathcal{A}_I$. ■

Proposition 1: Assume the assumptions in Theorem 9 be satisfied. For $\mu \in \rho(\mathcal{A})$, the following statements are equivalent:

(a). $\mu \in \rho(\mathfrak{A})$.
(b). $(I - \mathcal{M}\mathfrak{D}_\mu)$ is invertible.

Moreover, for $\mu \in \rho(\mathfrak{A}) \cap \rho(\mathcal{A})$,

$$R(\mu, \mathfrak{A}) = (I - \mathfrak{D}_\mu \mathcal{M})^{-1} R(\mu, \mathcal{A}). \tag{29}$$

Proof: Let Σ be the regular linear system generated by $(\mathcal{A}, \mathcal{B}, \mathcal{C})$, Σ^I be its closed-loop system under the admissible feedback operator I and $\mu \in \rho(\mathcal{A})$. According to the proof of Theorem 9, the operator $\mathfrak{A}$ coincides with the generator of Σ^I. Then, by [24, Theorem 1.2] $\mu \in \rho(\mathfrak{A})$ if and only if $I - G(\mu)$ is invertible, where $G(\mu)$ is the transfer function of Σ. Lemma 1 shows that $G(\mu) = \mathcal{M}\mathfrak{D}_\mu$. This proves the first part of the proposition.

Now let $\mu \in \rho(\mathfrak{A}) \cap \rho(\mathcal{A})$. By taking Laplace transform on both sides of (27), there is

$$\begin{aligned} R(\mu, \mathfrak{A}) &= R(\mu, \mathcal{A}) + R(\mu, \mathcal{A}_{-1})\mathcal{B}\tilde{\mathcal{C}}R(\mu, \mathfrak{A}) \\ &= R(\mu, \mathcal{A}) + \mathfrak{D}_\mu \mathcal{M} R(\mu, \mathfrak{A}), \end{aligned}$$

due to (19) and (24). Hence,

$$(I - \mathfrak{D}_\mu \mathcal{M})R(\mu, \mathfrak{A}) = R(\mu, \mathcal{A}). \tag{30}$$

According to [3, p.744], for $\mu \in \rho(\mathfrak{A}) \cap \rho(\mathcal{A})$, the inverse of $I - \mathfrak{D}_\mu \mathcal{M}$ exists and

$$(I - \mathfrak{D}_\mu \mathcal{M})^{-1} = I + \mathfrak{D}_\mu (I - \mathcal{M}\mathfrak{D}_\mu)^{-1}\mathcal{M}. \tag{31}$$

Hence, (29) holds from (30). This completes the proof. ■

IV. Nonhomogeneous boundary value problems

Following the previous section, this section is devoted to the well-posedness of the boundary value problem (1).

Lemma 2: Assume the assumptions of Theorem 9 be satisfied. Set $\mathfrak{B} := (\ \mathcal{B} \quad I\ )$ and $\mathfrak{C} := (\ \mathcal{C} \quad 0\ )^\top$. Then $(\mathcal{A}, \mathfrak{B}, \mathfrak{C})$ generates a regular linear system $\mathbf{\Sigma}$ on $\mathcal{X}, \mathbb{U}, \mathbb{U}$, with the identity operator on $\mathbb{U} := \mathcal{U} \times \mathcal{X}$ as an admissible feedback.

Proof: Let $\Sigma := (\mathcal{T}, \Phi, \Psi, \mathbb{F})$ be the system generated by the triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$. It is clear that the triple $(\mathcal{A}, I_\mathcal{X}, \mathcal{C})$ generates a regular linear system $\Sigma^0 = (\mathcal{T}, \Phi^0, \Psi, \mathbb{F}^0)$ with $\Phi^0(t)u = (\mathcal{T} * u)(t)$ and $\mathbb{F}^0(t) = \tilde{\mathcal{C}}\Phi^0(\cdot)$ on $[0, t]$ ($\mathbb{F}^0$ is well defined due to [25, Prop.3.3]). Define

$$\mathbf{T}(t) = T(t), \mathbf{\Phi}(t) = (\Phi(t)\ \Phi^0(t)), \mathbf{\Psi}(t) = \left(\begin{smallmatrix} \Psi(t) \\ 0 \end{smallmatrix} \right)$$

and

$$\mathbf{F}(t) = \begin{pmatrix} \mathbb{F}(t) & \mathbb{F}^0(t) \\ 0 & 0 \end{pmatrix}$$

for $t \geq 0$. Then the system $\mathbf{\Sigma} := (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F})$ is generated by the triple $(\mathcal{A}, \mathfrak{B}, \mathfrak{C})$ and has the identity operator on $\mathcal{U} \times \mathcal{X}$ as an admissible feedback. ■

Theorem 10: Assume the assumptions of Theorem 9 be satisfied. Then system (1) is equivalent to the distributed-parameter control system

$$\dot{w}(t) = \mathfrak{A}w(t) + \mathcal{B}g(t) + f(t), \quad t \geq 0, \quad w(0) = \varpi \in \mathcal{X},$$

and has a unique strong solution $w : \mathbb{R}_+ \to \mathcal{X}$ satisfying $w(t) \in \mathcal{D}(\tilde{\mathcal{C}})$ for a.e. $t \geq 0$ and

$$w(t) = \mathcal{T}_I(t)\varpi + \int_0^t (\mathcal{T}_I)_{-1}(t - \tau)[\mathcal{B}g(\tau) + f(\tau)]\, d\tau \tag{32}$$

for $t \geq 0$ and $\varpi \in \mathcal{X}$.

Proof: Let Σ and $\mathbf{\Sigma}$ be the regular linear systems introduced in the proof of Lemma 2. If $y(t)$ and $\mathbf{y}(t)$ denote the output functions of Σ and $\mathbf{\Sigma}$, respectively, then $\mathbf{y}(t) = (y(t), 0)^\top$ for a.e. $t \geq 0$. Let, by Lemma 2, $\mathbf{\Sigma}^I := (\mathbf{T}_I, \mathbf{\Phi}_I, \mathbf{\Psi}_I, \mathbf{F}_I)$ be the closed-loop system of $\mathbf{\Sigma}$. Note that $\mathcal{A}_{-1} + \mathfrak{B}\tilde{\mathfrak{C}} = \mathcal{A}_{-1} + \mathcal{B}\tilde{\mathcal{C}}$. Then by the proof of Theorem 9, the operator $\mathfrak{A}$ coincides with the generator of $\mathbf{\Sigma}^I$. Hence $\mathbf{T}_I(t) = \mathcal{T}_I(t)$, where $\mathcal{T}_I$ is the C_0-semigroup on $\mathcal{X}$ generated by $\mathfrak{A}$; see Theorem 9. Let $\mathbf{u}$ be the input of $\mathbf{\Sigma}$ and introduce the feedback law $\mathbf{u} = \mathbf{y} + \mathbf{u}_c$, where $\mathbf{u}_c$ is another suitable input. Then, the state trajectory of $\mathbf{\Sigma}^I$ is given by

$$w(t) = \mathcal{T}_I(t)\varpi + \mathbf{\Phi}_I(t)\mathbf{u}_c, \qquad t \geq 0. \tag{33}$$

Take $\mathbf{u}_c = (g, f)^\top$, then $\mathbf{u} = (u, f)^\top$ with $u = y + g$ as the input of Σ. This means that the closed-loop system $\mathbf{\Sigma}^I$ with input $\mathbf{u}_c = (g, f)^\top$ is equivalent to system (1). Substitute $\mathbf{u}_c = (g, f)^\top$ into (33), then the strong solution of system (1) can be obtained as

$$\begin{aligned} w(t) &= \mathcal{T}_I(t)\varpi + \int_0^t (\mathcal{T}_I)_{-1}(t - \tau)\mathfrak{B}\mathbf{u}_c(\tau)\, d\tau \\ &= \mathcal{T}_I(t)\varpi + \int_0^t (\mathcal{T}_I)_{-1}(t - \tau)[\mathcal{B}g(\tau) + f(\tau)]\, d\tau. \end{aligned}$$

This completes the proof. ■

Remark 11: In this paper, only the strong solutions of system (1) when $\varpi \in \mathcal{X}$ are discussed. For classical solutions, new extrapolation spaces associated with $\mathcal{X}$ and more conditions on the smoothness of f will be needed. This will be discussed in another paper.

V. Approximate controllability of perturbed boundary control problems

Let U be a Banach space, $B \in \mathcal{L}(U, \mathcal{X})$ and $K \in \mathcal{L}(U, \mathcal{U})$. Taking

$$f = Bu \quad \text{and} \quad g = Ku,$$

where $u \in L^p_{loc}(\mathbb{R}_+, U)$ is a control function, then system (1) becomes the following boundary control problem

$$\begin{aligned} \dot{w}(t) &= \mathcal{A}_m w(t) + Bu(t), \quad t \geq 0, \quad w(0) = \varpi, \\ \mathcal{N}w(t) &= \mathcal{M}w(t) + Ku(t). \end{aligned} \tag{34}$$

According to Theorem 10, the above is equivalent to the distributed-parameter control system

$$\dot{w}(t) = \mathfrak{A}w(t) + (B + \mathcal{B}K)u(t), \quad t \geq 0, \quad w(0) = \varpi, \tag{35}$$

with $\mathcal{B}$ defined in (19).

Definition 12: Let assumptions of Theorem 9 be satisfied. The boundary value problem (1) with $f = Bu$ and $g = Ku$ is said to be *approximately controllable* if the open-loop system $(\mathfrak{A}, B + \mathcal{B}K)$ is (in the sense of Definition 2).

Theorem 13: Let assumptions of Theorem 9 be satisfied. The boundary value problem (1) with $f = Bu$ and $g = Ku$ is approximately controllable if and only if, for $\mu \in \rho(\mathfrak{A}) \cap \rho(\mathcal{A})$ and $\varphi \in \mathcal{X}'$, the fact that

$$\langle (I - \mathfrak{D}_\mu \mathcal{M})^{-1}(\mathfrak{D}_\mu K + R(\mu, \mathcal{A})B)u, \varphi \rangle = 0, \text{ for } \forall u \in U, \tag{36}$$

implies that $\varphi = 0$.

Proof: According to Definition 2, the open-loop system $(\mathfrak{A}, B + \mathcal{B}K)$ is approximately controllable if and only if

$$Cl\left(\bigcup_{t \geq 0} \left\{ \int_0^t (\mathcal{T}_I)_{-1}(t-\tau)(B+\mathcal{B}K)u(\tau)\, d\tau : u \in L^p_{loc}(\mathbb{R}_+, U) \right\} \right) = \mathcal{X},$$

where $Cl(\cdot)$ means the closure of a set. The above is equivalent to the fact that, for $t \geq 0$ and any $u \in L^p_{loc}(\mathbb{R}_+, U)$,

$$\left\langle \int_0^t (\mathcal{T}_I)_{-1}(t-\tau)(B + \mathcal{B}K)u(\tau)\, d\tau, \varphi \right\rangle = 0 \tag{37}$$

implies $\varphi = 0$, for $\varphi \in \mathcal{X}'$. Now, for $\mu \in \rho(\mathfrak{A}) \cap \rho(\mathcal{A})$, after taking Laplace transform to (37), the fact that, for $\varphi \in \mathcal{X}'$,

$$\left\langle R(\mu, \mathfrak{A}_{-1})(B + \mathcal{B}K)\hat{u}(\mu), \varphi \right\rangle = 0 \tag{38}$$

implies that $\varphi = 0$. Here, $\hat{u}(\mu) \in U$ is the Laplace transform of $u \in L^p_{loc}(\mathbb{R}_+, U)$. Conversely, by the uniqueness of Laplace transform one can see that (38) implies (37). According to (29) and (19),

$$\begin{aligned} R(\mu, \mathfrak{A}_{-1})(B + \mathcal{B}K) &= (I - \mathfrak{D}_\mu \mathcal{M})^{-1} R(\mu, \mathcal{A}_{-1})(B + \mathcal{B}K) \\ &= (I - \mathfrak{D}_\mu \mathcal{M})^{-1}(\mathfrak{D}_\mu K + R(\mu, \mathcal{A})B). \end{aligned}$$

This completes the proof. ■

VI. Conclusion

This paper shows a useful variation of constant formula for nonhomogeneous perturbed boundary value problems, using the feedback theory of regular linear systems. It is then applied to investigate the approximate controllability of perturbed boundary control problems. The obtained result can be applied to investigate the approximate controllability of systems that can be reformulated as perturbed boundary control problems. As an example, the approximate controllability of general neutral systems with state and input delays in Banach spaces is investigated in [26].

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