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Path-by-Path Optimal Control of Switched and Markovian Jump Linear Systems

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Abstract— **A receding-horizon-type LQG control problem for switched linear systems is proposed and applied to the case of Markov switching. The objective is to determine a linear dynamic output feedback control law that minimizes a finite-horizon quadratic cost over all admissible future switching paths subject to almost sure uniform stability of the closed-loop system. A solution is determined by running a series of semidefinite programs offline until a saturation in achievable performance is reached.**

I. INTRODUCTION

Switched systems [1], [2] and Markovian jump systems [3], [4] model multi-modal systems under nondeterministic switching between different modes of operation. They appear in many contexts such as networked control systems subject to signal quantization or feedback delays [5], [6], macroeconomic models switching among different economic phases [7], and distributed networks of autonomous vehicles undergoing network topology changes [8]. On the other hand, receding-horizon control, where a finite-horizon optimization is performed at each time step over a moving horizon window, is useful in approximately solving hard infinite-horizon optimal control problems [9]–[11], for online optimization based model predictive control under hard state-control constraints [12], and when short-term optimization is emphasized over infinite-horizon planning [13]–[15].

Previous results on receding-horizon-type control of Markovian jump systems and switched systems appear in [7], [16]–[22]. With the exception of [22], however, these results require a sufficiently long control horizon to guarantee closed-loop stability. Moreover, they are restricted to controllers that depend only on the (estimated) current mode of operation even though controllers with memory of past modes are known to outperform these controllers [23], [24].

In this paper, we propose synthesis results for dynamic output feedback controllers that overcome these aforementioned limitations of existing work. Even though finite-horizon optimization for a given horizon length involves looking ahead future switching paths, the resulting controller coefficients depend only on the past switching path (of a finite length). This ensures not only the causality of the controller but also guarantees (almost sure) uniform exponential stability of the closed-loop system without introducing conservatism [25]–[27]. Moreover, stability and optimality are guaranteed for each horizon length. In particular, instead of optimizing the uniform output regulation performance as in [22], we seek to achieve Pareto-optimal path-by-path output regulation, which leads to a novel result in receding-horizon-type control of Markovian jump linear systems.

Notation. The Euclidean vector norm on $\mathbb{R}^n$ is denoted by $\|\cdot\|$, so that $\|\mathbf{x}\| = \sqrt{\mathbf{x}^{\mathrm{T}}\mathbf{x}}$ for $\mathbf{x} \in \mathbb{R}^n$. If $\mathbf{X}$, $\mathbf{Y} \in \mathbb{R}^{n\times n}$ are symmetric, then we write $\mathbf{X} < \mathbf{Y}$ (resp. $\mathbf{X} \le \mathbf{Y}$) to mean that $\mathbf{X} - \mathbf{Y}$ is negative definite (resp. negative semidefinite).

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II. DEFINITIONS AND PRELIMINARY RESULTS

A. Switched Linear Systems

Given a positive integer N, infinite sequences in $\{1,\dots,N\}$ shall be called *switching sequences*; denoted by $\{1,\dots,N\}^\infty$ is the set of all switching sequences. Similarly, any finite sequence in $\{1,\dots,N\}$ belonging to $\{1,\dots,N\}^{L+1}$ for some nonnegative integer L shall be called a *switching path* of length L. In particular, zero-length switching paths are called *modes*. With a nonempty subset Θ of $\{1,\dots,N\}^\infty$ understood, members of Θ shall be said to be *admissible*.

Let $\mathbf{A}_i \in \mathbb{R}^{n\times n}$, $\mathbf{B}_i \in \mathbb{R}^{n\times m}$, $\mathbf{C}_i \in \mathbb{R}^{l\times n}$, and $\mathbf{D}_i \in \mathbb{R}^{l\times m}$, $i = 1, \dots, N$, be given. If Θ is a nonempty subset of $\{1,\dots,N\}^\infty$, and if $\mathscr{G}$ is the indexed family

$$\mathscr{G} = \{(\mathbf{A}_i, \mathbf{B}_i, \mathbf{C}_i, \mathbf{D}_i) \colon i = 1,\dots,N\}, \tag{1}$$

then the pair $(\mathscr{G},\Theta)$ is called a discrete-time *switched linear system*, and defines the family of state-space representations

$$\begin{aligned}\mathbf{x}(t+1) &= \mathbf{A}_{\theta(t)}\mathbf{x}(t) + \mathbf{B}_{\theta(t)}\mathbf{w}(t),\\ \mathbf{z}(t) &= \mathbf{C}_{\theta(t)}\mathbf{x}(t) + \mathbf{D}_{\theta(t)}\mathbf{w}(t)\end{aligned} \tag{2}$$

over all $\theta = (\theta(0), \theta(1), \dots) \in \Theta$. Here, $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{w}(t) \in \mathbb{R}^m$, and $\mathbf{z}(t) \in \mathbb{R}^l$ are the state, disturbance input, and error output, respectively, of the switched system at time t. If Θ is a singleton, $(\mathscr{G},\Theta)$ is a linear time-varying system; if $N = 1$, then $(\mathscr{G},\Theta)$ is a linear time-invariant system. On the other hand, the pair $(\mathscr{G},\{1,\dots,N\}^\infty)$ is the *discrete linear inclusion*, where every switching sequence is admissible.

B. Markovian Jump Linear Systems

Let $\mathbf{P} = (p_{ij}) \in [0,1]^{N\times N}$ be a row-stochastic matrix such that $\sum_{j=1}^{N} p_{ij} = 1$ for all $i \in \{1,\dots,N\}$. Let $\mathbf{p} = (p_i) \in [0,1]^{1\times N}$ be a row-vector such that $\sum_{i=1}^{N} p_i = 1$. The pair $(\mathbf{P},\mathbf{p})$ defines the homogeneous *Markov chain* with transition probability matrix $\mathbf{P}$ and initial distribution $\mathbf{p}$. Realizations of $(\mathbf{P},\mathbf{p})$ are switching sequences. Let $\Theta(\mathbf{P},\mathbf{p})$ be the set of all realizations $\theta = (\theta(0), \theta(1), \dots)$ of $(\mathbf{P},\mathbf{p})$ such that

$p_{\theta(t)\theta(t+1)} > 0$ for all t and $p_{\theta(0)} > 0$. Switching sequences in $\Theta(\mathbf{P},\mathbf{p})$ shall be called *admissible* with respect to $(\mathbf{P},\mathbf{p})$.

If $(\mathbf{P},\mathbf{p})$ is a Markov chain and if $\mathscr{G}$ is as in (1), then the triple $(\mathscr{G},\mathbf{P},\mathbf{p})$ defines the discrete-time *Markovian jump linear system*, whose state-space representation is given by (2) for each realization θ of $(\mathbf{P},\mathbf{p})$. It suffices for our purpose to identify the Markovian jump linear system $(\mathscr{G},\mathbf{P},\mathbf{p})$ with the switched linear system $(\mathscr{G},\Theta(\mathbf{P},\mathbf{p}))$.

C. Uniform Exponential Stability

Our stability notion for switched linear systems requires uniformity in the exponential decay rate of the state over all admissible switching sequences. Let $\mathscr{G}$ be as in (1), and let Θ be a nonempty subset of $\{1,\dots,N\}^\infty$.

Definition 1: The switched linear system $(\mathscr{G},\Theta)$ is said to be *uniformly exponentially stable* if there exist $c > 1$ and $\lambda \in (0,1)$ such that, whenever $\mathbf{w}(t) = 0$ for all $t \in \{0,1,\dots\}$, the system of difference equations (2) satisfies

$$\|\mathbf{x}(t)\| \le c\lambda^{t-s}\|\mathbf{x}(s)\| \tag{3}$$

for all t, $s \in \{0,1,\dots\}$ with $s \le t$, for all $\mathbf{x}(s) \in \mathbb{R}^n$, and for all $\theta \in \Theta$.

It is well-known that the discrete linear inclusion $(\mathscr{G},\{1,\dots,N\}^\infty)$ is asymptotically stable under all switching sequences if and only if the joint spectral radius of $\{\mathbf{A}_1,\dots,\mathbf{A}_N\}$ is strictly less than one [28], and that asymptotically stable discrete linear inclusions are in fact uniformly exponentially stable—see, e.g., [25], [29]. In general, for nonempty $\Theta \subset \{1,\dots,N\}^\infty$, an exact characterization of uniform exponential stability is given by an increasing union of linear matrix inequalities [25], [26]. For $(\theta(0),\theta(1),\dots) \in \{1,\dots,N\}^\infty$, set $\theta(-1) = \theta(-2) = \cdots = 0$. Then, for nonnegative integers L and nonempty $\Theta \subset \{1,\dots,N\}^\infty$, define

$$\mathscr{L}_L(\Theta) = \{(\theta(t-L),\dots,\theta(t))\colon \theta \in \Theta, t \ge 0\}.$$

and

$$\begin{aligned}\mathscr{W}_L(\Theta) &= \mathscr{L}_L(\Theta) \cap \{1,\dots,N\}^{L+1}\\ &= \{(\theta(t),\dots,\theta(t+L))\colon \theta \in \Theta, t \ge 0\}.\end{aligned}$$

For notational convenience, set $\{1,\dots,N\}^0 = \{0\}$. The following lemma appears in [22] and its proof is immediate from the results in [25], [26].

Lemma 2: The switched linear system $(\mathscr{G},\Theta)$ is uniformly exponentially stable if and only if there exist a nonnegative integer M and matrices $\mathbf{Y}_{(j_1,\dots,j_M)} > \mathbf{0}$, $(j_1,\dots,j_M) \in \{1,\dots,N\}^M$, such that

$$\mathbf{A}_{i_M}^{\mathrm{T}}\mathbf{Y}_{(i_0,\dots,i_{M-1})}\mathbf{A}_{i_M} - \mathbf{Y}_{(i_1,\dots,i_M)} < \mathbf{0}$$

for all $(i_0,\dots,i_M) \in \mathscr{W}(\Theta)$.

On the other hand, as is done in the series of papers [25]–[27], our notion of stability for Markovian jump linear systems requires almost sure uniformity in the exponential decay rate of the state. Let $\mathscr{G}$ be as in (1) and let $(\mathbf{P},\mathbf{p})$ be a Markov chain.

Definition 3: The Markovian jump linear system $(\mathscr{G},\mathbf{P},\mathbf{p})$ is said to be *almost surely uniformly exponentially stable* if, whenever $\mathbf{w}(t) = 0$ for all $t \in \{0,1,\dots\}$, there exists a $c > 1$ and $\lambda \in (0,1)$ such that, with probability one, inequality (3) holds for all t, $s \in \{0,1,\dots\}$ with $s \le t$ and for all $\mathbf{x}(s) \in \mathbb{R}^n$.

As pointed out in [27], almost sure uniform exponential stability is a deterministic notion, and hence we are able to extend the classical LQG control result to Markovian jump linear systems without introducing conservatism in stability analysis. The following lemma is from [25]:

Lemma 4: The Markovian jump linear system $(\mathscr{G},\mathbf{P},\mathbf{p})$ is almost surely uniformly exponentially stable if and only if the switched linear system $(\mathscr{G},\Theta(\mathbf{P},\mathbf{p}))$ is uniformly exponentially stable.

D. Path-by-Path Performance

Let $\mathscr{G}$ be as in (1), and let $\Theta \subset \{1,\dots,N\}^\infty$ be nonempty. Assume initial state $\mathbf{x}(0) = 0$, and let $\mathbf{w} = (\mathbf{w}(0),\mathbf{w}(1),\dots)$ be the zeromean white noise sequence satisfying

$$\mathrm{E}[\mathbf{w}(t)] = 0 \quad \text{for all } t; \tag{4a}$$

$$\mathrm{E}[\mathbf{w}(t)\mathbf{w}(s)^{\mathrm{T}}] = \begin{cases}\mathbf{I}, & \text{if } t = s;\\ \mathbf{0}, & \text{otherwise.}\end{cases} \tag{4b}$$

Here, $\mathrm{E}[\cdot]$ denotes the expectation. If the error output sequence $\mathbf{z}$ is generated under a switching sequence $\theta \in \Theta$, then we write $\mathbf{z} = (\mathbf{z}_\theta(0),\mathbf{z}_\theta(1),\dots)$. We are concerned with a collection of finite-horizon quadratic performance measures.

Definition 5: Given a nonnegative integer T, the switched linear system $(\mathscr{G},\Theta)$ is said to satisfy *T-step path-by-path performance levels* $\gamma_{(i_0,\dots,i_T)} > 0$, $(i_0,\dots,i_T) \in \mathscr{W}_T(\Theta)$, if, under the condition that $\mathbf{x}(0) = 0$ and $\mathbf{w}$ is a random sequence satisfying (4), the system of difference equations (2) yields

$$\frac{1}{T+1}\sum_{t=t_0}^{t_0+T}\mathrm{E}\,\|\mathbf{z}_\theta(t)\|^2 < \gamma^2_{(\theta(t_0),\dots,\theta(t_0+T))}$$

for all $t_0 \in \{0,1,\dots\}$ and for all $\theta \in \Theta$.

This path-by-path performance specification generalizes the case of a single uniform performance level [22], where $\gamma_{(i_0,\dots,i_T)} = \gamma$ for some $\gamma > 0$ and for all $(i_0,\dots,i_T) \in \mathscr{W}_T(\Theta)$. As illustrated in the context of disturbance attenuation in [26], we will see shortly that a convex combination of path-by-path performance levels not only gives us a refinement of the uniform performance specification, but it also enables us to extend the switched systems results to Markovian jump systems.

Let $(\mathbf{P},\mathbf{p})$ be a Markov chain with $\mathbf{P} = (p_{ij})$ and $\mathbf{p} = (p_i)$. We say that $\mathbf{p}$ is $\mathbf{P}$-*invariant* if $\mathbf{pP} = \mathbf{p}$. There exists a unique $\mathbf{P}$-invariant distribution $\mathbf{p}$ when $\mathbf{P}$ is irreducible. If $\mathbf{p}$ is $\mathbf{P}$-invariant and if switching sequences θ are realizations of $(\mathbf{P},\mathbf{p})$, then, for each $(i_0,\dots,i_T) \in \mathscr{W}_T(\Theta(\mathbf{P},\mathbf{p}))$, the probability that $(\theta(t),\dots,\theta(t+T)) = (i_0,\dots,i_T)$ is given by

$$\pi_{(i_0,\dots,i_T)} = p_{i_0}p_{i_0 i_1}\cdots p_{i_{T-1}i_T} \tag{5}$$

for all t, $T \in \{0,1,\dots\}$. The sum of the probabilities $\pi_{(i_0,\dots,i_T)}$ over $(i_0,\dots,i_T) \in \mathbf{W}_T(\Theta(\mathbf{P},\mathbf{p}))$ is equal to one, and $\pi_{(i_0,\dots,i_T)}$ give us the *T-step probability distribution* of $(\mathbf{P},\mathbf{p})$.

Definition 6: Given a nonnegative integer T, the Markovian jump linear system $(\mathscr{G},\mathbf{P},\mathbf{p})$, where $\mathbf{p}$ is $\mathbf{P}$-invariant, is said to satisfy a *T-step average performance level* $\gamma > 0$ if, under the condition that $\mathbf{x}(0) = 0$ and $\mathbf{w}$ is a random sequence satisfying (4), the system of difference equations (2) yields

$$\frac{1}{T+1}\sum_{t=t_0}^{t_0+T} \mathrm{E}\,\|\mathbf{z}(t)\|^2 < \gamma^2$$

for all $t_0 \in \{0,1,\dots\}$, where $\mathrm{E}[\cdot]$ denotes the expectation with respect to θ and $\mathbf{w}$.

III. ANALYSIS RESULTS

A. Analysis of Switched Linear Systems

Let us first characterize uniformly exponentially stable switched linear systems satisfying given path-by-path performance levels. A straightforward generalization of the uniform performance result in [22] serves this purpose.

Let $\mathscr{G}$ be as in (1). If $\mathbf{Y}_{\theta,t}^{(t_0)} \geq \mathbf{0}$ satisfy

$$\mathbf{A}_{\theta(t)}\mathbf{Y}_{\theta,t}^{(t_0)}\mathbf{A}_{\theta(t)}^{\mathrm{T}} - \mathbf{Y}_{\theta,t+1}^{(t_0)} = -\mathbf{B}_{\theta(t)}\mathbf{B}_{\theta(t)}^{\mathrm{T}} \tag{6}$$

for all $\theta \in \Theta$ and for all t_0, $t \in \{0,1,\dots\}$ with $t_0 \leq t$, subject to the initial condition $\mathbf{Y}_{t_0}^{(t_0)} = \mathbf{0}$, then we have

$$\mathbf{Y}_{\theta,t+1}^{(t_0+1)} \leq \mathbf{Y}_{\theta,t+1}^{(t_0)} \tag{7}$$

for all $\theta \in \Theta$ and t_0, $t \in \{0,1,\dots\}$ with $t_0 \leq t$, and

$$\mathrm{E}\,\|\mathbf{z}_\theta(t)\|^2 = \mathrm{tr}\left(\mathbf{C}_{\theta(t)}\mathbf{Y}_{\theta,t}^{(0)}\mathbf{C}_{\theta(t)}^{\mathrm{T}} + \mathbf{D}_{\theta(t)}\mathbf{D}_{\theta(t)}^{\mathrm{T}}\right) \tag{8}$$

for all $\theta \in \Theta$ and for all $t \in \{0,1,\dots\}$.

Lemma 7: Let $\mathbf{Y}_{\theta,t+1}^{(t_0)}$ be as in (6). Then

$$\mathbf{Y}_{\theta,t+1} \geq \mathbf{Y}_{\theta,t+1}^{(t_0)}$$

for $t \in \{t_0, t_0+1, \dots\}$, whenever $\mathbf{Y}_{\theta,t_0} \geq \mathbf{0}$ and

$$\mathbf{A}_{\theta(t)}\mathbf{Y}_{\theta,t}\mathbf{A}_{\theta(t)}^{\mathrm{T}} - \mathbf{Y}_{\theta,t+1} \leq -\mathbf{B}_{\theta(t)}\mathbf{B}_{\theta(t)}^{\mathrm{T}}$$

for $t \in \{t_0, t_0+1, \dots\}$.

Proof: The result is immediate from definitions. ■

Theorem 8: Let $\mathscr{G}$ be as in (1), and let $\Theta \subset \{1,\dots,N\}^\infty$ be nonempty. Given a nonnegative integer T, the switched linear system $(\mathscr{G},\Theta)$ is uniformly exponentially stable and satisfies T-step path-by-path performance levels $\gamma_{(i_0,\dots,i_T)} > 0$, $(i_0,\dots,i_T) \in \mathscr{W}_T(\Theta)$, if and only if there exist a nonnegative integer M and matrices $\mathbf{Y}_{(j_1,\dots,j_M)} > \mathbf{0}$ such that

$$\mathbf{A}_{i_M}\mathbf{Y}_{(i_0,\dots,i_{M-1})}\mathbf{A}_{i_M}^{\mathrm{T}} - \mathbf{Y}_{(i_1,\dots,i_M)} < -\mathbf{B}_{i_M}\mathbf{B}_{i_M}^{\mathrm{T}} \tag{9a}$$

for all $(i_0,\dots,i_M) \in \mathscr{W}_M(\Theta)$, and such that

$$\frac{1}{T+1}\sum_{t=M}^{M+T} \mathrm{tr}\left(\mathbf{C}_{i_t}\mathbf{Y}_{(i_{t-M},\dots,i_{t-1})}\mathbf{C}_{i_t}^{\mathrm{T}} + \mathbf{D}_{i_t}\mathbf{D}_{i_t}^{\mathrm{T}}\right) < \gamma_{(i_M,\dots,i_{M+T})}^2 \tag{9b}$$

for all $(i_0,\dots,i_{M+T}) \in \mathscr{L}_{M+T}(\Theta)$ with $(i_M,\dots,i_{M+T}) \in \mathscr{W}_T(\Theta)$.

Proof: The proof is based on Lemma 2 and Lemma 7, and it is identical to that of [22, Theorem 1] except that we now have multiple performance levels $\gamma_{(i_M,\dots,i_{M+T})}$ instead of a single γ. ■

B. Analysis of Markovian Jump Linear Systems

The following result identifies the T-step average performance level of a Markovian jump linear system $(\mathscr{G},\mathbf{P},\mathbf{p})$ as a convex combination of the T-step path-by-path performance levels of the corresponding switched linear system $(\mathscr{G},\Theta(\mathbf{P},\mathbf{p}))$, where the coefficients for the convex combination are the T-step probabilities of the Markov chain $(\mathbf{P},\mathbf{p})$.

Theorem 9: Let $(\mathbf{P},\mathbf{p})$ be a Markov chain, where $\mathbf{p}$ is $\mathbf{P}$-invariant. Given a nonnegative integer T, the Markovian jump linear system $(\mathscr{G},\mathbf{P},\mathbf{p})$ is almost surely uniformly exponentially stable and satisfies T-step average performance level $\gamma > 0$ if and only if the switched linear system $(\mathscr{G},\Theta(\mathbf{P},\mathbf{p}))$ is uniformly exponentially stable and satisfies T-step path-by-path performance levels $\gamma_{(i_0,\dots,i_T)} > 0$, $(i_0,\dots,i_T) \in \mathscr{W}_T(\Theta)$, such that

$$\sum_{(i_0,\dots,i_T)\in\mathscr{W}_T(\Theta(\mathbf{P},\mathbf{p}))} \pi_{(i_0,\dots,i_T)}\gamma_{(i_0,\dots,i_T)}^2 \leq \gamma^2, \tag{10}$$

where $\pi_{(i_0,\dots,i_T)}$ are the T-step probabilities given by (5).

Proof: The proof is similar to that of [22, Theorem 1], so we will only sketch it. If $(\mathscr{G},\mathbf{P},\mathbf{p})$ is almost surely uniformly exponentially stable, then $(\mathscr{G},\Theta(\mathbf{P},\mathbf{p}))$ is uniformly exponentially stable by Lemma 4. Moreover, if $(\mathscr{G},\mathbf{P},\mathbf{p})$ satisfies T-step average performance level γ, then we have

$$\begin{aligned}&\frac{1}{T+1}\sum_{t=t_0}^{t_0+T} \mathrm{E}\,\|\mathbf{z}(t)\|^2 \\ &= \sum_{(\theta(t_0),\dots,\theta(t_0+T))} \pi_{(\theta(t_0),\dots,\theta(t_0+T))}\frac{1}{T+1}\sum_{t=t_0}^{t_0+T} \mathrm{E}\,\|\mathbf{z}_\theta(t)\|^2.\end{aligned}$$

Let $\mathbf{Y}_{\theta,t}^{(\varepsilon,t_0)}$ satisfy the Lyapunov equation

$$\mathbf{A}_{\theta(t)}\mathbf{Y}_{\theta,t}^{(\varepsilon,t_0)}\mathbf{A}_{\theta(t)}^{\mathrm{T}} - \mathbf{Y}_{\theta,t+1}^{(t_0)} = -\mathbf{B}_{\theta(t)}\mathbf{B}_{\theta(t)}^{\mathrm{T}} - \varepsilon\mathbf{I}$$

for $\varepsilon > 0$, $\theta \in \Theta(\mathbf{P},\mathbf{p})$, and $0 \leq t_0 \leq t$, subject to $\mathbf{Y}_{\theta,t_0}^{(\varepsilon,t_0)} = \mathbf{0}$. It can be shown using (6)–(8) and proceeding as in the proof of [22, Theorem 1] that, whenever the Markovian jump system $(\mathscr{G},\mathbf{P},\mathbf{p})$ satisfies T-step average performance level γ and the corresponding switched system $(\mathscr{G},\Theta(\mathbf{P},\mathbf{p}))$ is uniformly exponentially stable, there exist a sufficiently small number $\varepsilon > 0$ and a sufficiently large integer $M > 0$ such that

$$\mathbf{A}_{\theta(t)}\mathbf{Y}_{\theta,t}^{(\varepsilon,t-M)}\mathbf{A}_{\theta(t)}^{\mathrm{T}} - \mathbf{Y}_{\theta,t+1}^{(\varepsilon,t-M+1)} < -\mathbf{B}_{\theta(t)}\mathbf{B}_{\theta(t)}^{\mathrm{T}}$$

and

$$\begin{aligned}&\sum_{(\theta(t_0),\dots,\theta(t_0+T))} \pi_{(\theta(t_0),\dots,\theta(t_0+T))} \\ &\times \frac{1}{T+1}\sum_{t=t_0}^{t_0+T} \mathrm{tr}\left(\mathbf{C}_{\theta(t)}\mathbf{Y}_{\theta,t}^{(\varepsilon,t-M)}\mathbf{C}_{\theta(t)}^{\mathrm{T}} + \mathbf{D}_{\theta(t)}\mathbf{D}_{\theta(t)}^{\mathrm{T}}\right) < \gamma^2\end{aligned}$$

whenever $t \geq M$ and $\theta \in \Theta$. Put

$$\mathbf{Y}_{(\theta(t-M),\dots,\theta(t-1))} = \begin{cases}\mathbf{Y}_{\theta,t}^{(\varepsilon,t-M)}, & t \geq M;\\ \mathbf{Y}_{\theta,t}^{(\varepsilon,0)}, & t < M\end{cases}$$

for all $t \geq 0$ and $\theta \in \Theta$. Then, using (6), one can show that (9a) and (9b) hold for some $\gamma_{(i_M,\dots,i_{M+T})} > 0$, $(i_M,\dots,i_{M+T}) \in \mathscr{W}_T(\Theta)$, such that (10) is satisfied. This proves necessity. Sufficiency follows easily from Lemma 4 and Lemma 7. ■

IV. SYNTHESIS RESULTS

A. Plants and Path-Dependent Controllers

With $\mathbf{A}_i \in \mathbb{R}^{n\times n}$, $\mathbf{B}_{1,i} \in \mathbb{R}^{n\times m_1}$, $\mathbf{B}_{2,i} \in \mathbb{R}^{n\times m_2}$, $\mathbf{C}_{1,i} \in \mathbb{R}^{l_1\times n}$, $\mathbf{D}_{11,i} \in \mathbb{R}^{l_1\times m_1}$, $\mathbf{D}_{12,i} \in \mathbb{R}^{l_1\times m_2}$, $\mathbf{C}_{2,i} \in \mathbb{R}^{l_2\times n}$, and $\mathbf{D}_{21,i} \in \mathbb{R}^{l_2\times m_1}$ for $i = 1, \ldots, N$, define

$$\mathscr{T} = \{(\mathbf{A}_i, \mathbf{B}_{1,i}, \mathbf{B}_{2,i}, \mathbf{C}_{1,i}, \mathbf{D}_{11,i}, \mathbf{D}_{12,i}, \mathbf{C}_{2,i}, \mathbf{D}_{21,i}) : \\ i = 1, \ldots, N\}. \quad (11)$$

We consider controlled plants of the form

$$\begin{aligned} \mathbf{x}(t+1) &= \mathbf{A}_{\theta(t)}\mathbf{x}(t) + \mathbf{B}_{1,\theta(t)}\mathbf{w}(t) + \mathbf{B}_{2,\theta(t)}\mathbf{u}(t), \\ \mathbf{z}(t) &= \mathbf{C}_{1,\theta(t)}\mathbf{x}(t) + \mathbf{D}_{11,\theta(t)}\mathbf{w}(t) + \mathbf{D}_{12,\theta(t)}\mathbf{u}(t), \quad (12) \\ \mathbf{y}(t) &= \mathbf{C}_{2,\theta(t)}\mathbf{x}(t) + \mathbf{D}_{21,\theta(t)}\mathbf{w}(t) \end{aligned}$$

for all $t \in \{0, 1, \ldots\}$, where $\mathbf{u} = (\mathbf{u}(0), \mathbf{u}(1), \ldots)$ is the control input sequence and $\mathbf{y} = (\mathbf{y}(0), \mathbf{y}(1), \ldots)$ the measured output sequence. For a nonempty $\Theta \subset \{1, \ldots, N\}^\infty$, the pair $(\mathscr{T}, \Theta)$ defines the *controlled switched linear system*; if $(\mathbf{P}, \mathbf{p})$ is a Markov chain, the triple $(\mathscr{T}, \mathbf{P}, \mathbf{p})$ defines the *controlled Markovian jump linear system.*

It is assumed that, in addition to the measured output $\mathbf{y}(t)$, the mode $\theta(t)$ is perfectly observed by the controller at each time t. Moreover, the controller is allowed to have a finite memory of past modes. For nonnegative integers L and nonempty subsets Θ of $\{1, \ldots, N\}^\infty$, let $\mathbf{A}_{K,(i_0,\ldots,i_L)} \in \mathbb{R}^{n_K\times n_K}$, $\mathbf{B}_{K,(i_0,\ldots,i_L)} \in \mathbb{R}^{n_K\times l_2}$, $\mathbf{C}_{K,(i_0,\ldots,i_L)} \in \mathbb{R}^{m_2\times n_K}$, $\mathbf{D}_{K,(i_0,\ldots,i_L)} \in \mathbb{R}^{m_2\times l_2}$ for $(i_0, \ldots, i_L) \in \mathscr{L}_L(\Theta)$. Let

$$\mathscr{K} = \{(\mathbf{A}_{K,(i_0,\ldots,i_L)}, \mathbf{B}_{K,(i_0,\ldots,i_L)}, \mathbf{C}_{K,(i_0,\ldots,i_L)}, \mathbf{D}_{K,(i_0,\ldots,i_L)}) : \\ (i_0, \ldots, i_L) \in \mathscr{L}_L(\Theta)\}. \quad (13)$$

Also, for $\theta \in \Theta$, with $\theta(-1) = \theta(-2) = \cdots = 0$, define

$$\begin{aligned} \theta_L(t) &= (\theta(t-L), \ldots, \theta(t)), \\ \Theta_L &= \{(\theta_L(0), \theta_L(1), \ldots) : \theta \in \Theta\}. \end{aligned}$$

Then the pair $(\mathscr{K}, \Theta_L)$ is identified with the *L-path-dependent (linear dynamic output feedback) controller* of order n_K represented by

$$\begin{aligned} \mathbf{x}_K(t+1) &= \mathbf{A}_{K,\theta_L(t)}\mathbf{x}_K(t) + \mathbf{B}_{K,\theta_L(t)}\mathbf{y}(t), \\ \mathbf{u}(t) &= \mathbf{C}_{K,\theta_L(t)}\mathbf{x}_K(t) + \mathbf{D}_{K,\theta_L(t)}\mathbf{y}(t) \end{aligned}$$

for all $t \in \{0, 1, \ldots\}$ and $\theta \in \Theta$. Note that the controller coefficients are constrained to depend solely on the current mode and past switching path of finite length. This ensures that the controller can be used under nondeterministic autonomous switching sequences and that closing the feedback loop preserves the finiteness of the number of modes.

B. Synthesis of Switched Linear Systems

Given a controlled switched linear system $(\mathscr{T}, \Theta)$ and an L-path-dependent linear dynamic output feedback controller $(\mathscr{K}, \Theta_L)$, let

$$\begin{aligned} \widetilde{\mathbf{A}}_{(i_0,\ldots,i_L)} &= \widehat{\mathbf{A}}_{i_L} + \widehat{\mathbf{B}}_{2,i_L}\mathbf{K}_{(i_0,\ldots,i_L)}\widehat{\mathbf{C}}_{2,i_L}, \\ \widetilde{\mathbf{B}}_{(i_0,\ldots,i_L)} &= \widehat{\mathbf{B}}_{1,i_L} + \widehat{\mathbf{B}}_{2,i_L}\mathbf{K}_{(i_0,\ldots,i_L)}\widehat{\mathbf{D}}_{21,i_L}, \\ \widetilde{\mathbf{C}}_{(i_0,\ldots,i_L)} &= \widehat{\mathbf{C}}_{1,i_L} + \widehat{\mathbf{D}}_{12,i_L}\mathbf{K}_{(i_0,\ldots,i_L)}\widehat{\mathbf{C}}_{2,i_L}, \\ \widetilde{\mathbf{D}}_{(i_0,\ldots,i_L)} &= \mathbf{D}_{11,i_L} + \widehat{\mathbf{D}}_{12,i_L}\mathbf{K}_{(i_0,\ldots,i_L)}\widehat{\mathbf{D}}_{21,i_L} \end{aligned}$$

with

$$\mathbf{K}_{(i_0,\ldots,i_L)} = \begin{bmatrix} \mathbf{A}_{K,(i_0,\ldots,i_L)} & \mathbf{B}_{K,(i_0,\ldots,i_L)} \\ \mathbf{C}_{K,(i_0,\ldots,i_L)} & \mathbf{D}_{K,(i_0,\ldots,i_L)} \end{bmatrix}$$

for $(i_0, \ldots, i_L) \in \mathscr{L}_L(\Theta)$, where

$$\widehat{\mathbf{A}}_i = \begin{bmatrix} \mathbf{A}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \widehat{\mathbf{B}}_{1,i} = \begin{bmatrix} \mathbf{B}_{1,i} \\ \mathbf{0} \end{bmatrix}, \quad \widehat{\mathbf{B}}_{2,i} = \begin{bmatrix} \mathbf{0} & \mathbf{B}_{2,i} \\ \mathbf{I} & \mathbf{0} \end{bmatrix},$$
$$\widehat{\mathbf{C}}_{1,i} = \begin{bmatrix} \mathbf{C}_{1,i} & \mathbf{0} \end{bmatrix}, \quad \widehat{\mathbf{D}}_{12,i} = \begin{bmatrix} \mathbf{0} & \mathbf{D}_{12,i} \end{bmatrix},$$
$$\widehat{\mathbf{C}}_{2,i} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{C}_{2,i} & \mathbf{0} \end{bmatrix}, \quad \widehat{\mathbf{D}}_{21,i} = \begin{bmatrix} \mathbf{0} \\ \mathbf{D}_{21,i} \end{bmatrix}$$

for $i \in \{1, \ldots, N\}$. Let

$$\mathscr{T}_{\mathscr{K}} = \{(\widetilde{\mathbf{A}}_{(i_0,\ldots,i_L)}, \widetilde{\mathbf{B}}_{(i_0,\ldots,i_L)}, \widetilde{\mathbf{C}}_{(i_0,\ldots,i_L)}, \widetilde{\mathbf{D}}_{(i_0,\ldots,i_L)}) : \\ (i_0, \ldots, i_L) \in \mathscr{L}_L(\Theta)\}.$$

If we define $\tilde{x}(t) = [x(t)^{\mathrm{T}}\ x_K(t)^{\mathrm{T}}]^{\mathrm{T}}$, then the *closed-loop switched linear system* $(\mathscr{T}_{\mathscr{K}}, \Theta_L)$ has the representation

$$\begin{aligned} \tilde{x}(t+1) &= \widetilde{\mathbf{A}}_{\theta_L(t)}\tilde{x}(t) + \widetilde{\mathbf{B}}_{\theta_L(t)}w(t), \\ z(t) &= \widetilde{\mathbf{C}}_{\theta_L(t)}\tilde{x}(t) + \widetilde{\mathbf{D}}_{\theta_L(t)}w(t) \end{aligned} \quad (14)$$

for each $\theta \in \Theta$. As argued in [22], the *closed-loop mode* at time t is given by the switching path $\theta_L(t) \in \mathscr{L}_L(\Theta)$.

Theorem 10: Let $\mathscr{T}$ be as in (11), and let $\Theta \subset \{1, \ldots, N\}^\infty$ be nonempty. Given nonnegative integers $n_K \geq n$ and T, there exist a path length L and an L-path-dependent controller $(\mathscr{K}, \Theta_L)$ of order n_K such that the closed-loop system $(\mathscr{T}_{\mathscr{K}}, \Theta_L)$ is uniformly exponentially stable and satisfies T-step path-by-path performance levels $\gamma_{(i_0,\ldots,i_T)} > 0$, $(i_0, \ldots, i_T) \in \mathscr{W}_T(\Theta)$, if and only if there exist an integer $M \geq L$, symmetric matrices $\mathbf{R}_{(j_0,\ldots,j_{M-1})}$, $\mathbf{S}_{(j_0,\ldots,j_{M-1})}$, $\mathbf{Z}_{(j_0,\ldots,j_M)}$, and rectangular matrices $\mathbf{W}_{(j_0,\ldots,j_M)}$ such that

$$\mathbf{H}_{(i_0,\ldots,i_M)} + \mathbf{F}_{i_M}^{\mathrm{T}}\mathbf{W}_{(i_0,\ldots,i_M)}\mathbf{G}_{i_M} + \mathbf{G}_{i_M}^{\mathrm{T}}\mathbf{W}_{(i_0,\ldots,i_M)}^{\mathrm{T}}\mathbf{F}_{i_M} < \mathbf{0}, \quad (15a)$$

$$\widehat{\mathbf{H}}_{(i_0,\ldots,i_M)} + \widehat{\mathbf{F}}_{i_M}^{\mathrm{T}}\mathbf{W}_{(i_0,\ldots,i_M)}\widehat{\mathbf{G}}_{i_M} + \widehat{\mathbf{G}}_{i_M}^{\mathrm{T}}\mathbf{W}_{(i_0,\ldots,i_M)}^{\mathrm{T}}\widehat{\mathbf{F}}_{i_M} < \mathbf{0} \quad (15b)$$

for all $(i_0, \ldots, i_M) \in \mathscr{L}_M(\Theta)$, and such that

$$\frac{1}{T+1}\sum_{t=M}^{M+T} \operatorname{tr} \mathbf{Z}_{(i_{t-M},\ldots,i_t)} < \gamma_{(i_M,\ldots,i_{M+T})}^2 \quad (15c)$$

for all $(i_0, \ldots, i_{M+T}) \in \mathscr{L}_{M+T}(\Theta)$ with $(i_M, \ldots, i_{M+T}) \in \mathscr{W}_T(\Theta)$, where

$$\mathbf{H}_{(i_0,\ldots,i_M)} = \begin{bmatrix} -\mathbf{S}_{(i_0,\ldots,i_{M-1})} & -\mathbf{I} & \mathbf{A}_{i_M}^{\mathrm{T}} & \mathbf{A}_{i_M}^{\mathrm{T}}\mathbf{S}_{(i_1,\ldots,i_M)} & \mathbf{0} \\ * & -\mathbf{R}_{(i_0,\ldots,i_{M-1})} & \mathbf{R}_{(i_0,\ldots,i_{M-1})}\mathbf{A}_{i_M}^{\mathrm{T}} & \mathbf{0} & \mathbf{0} \\ * & * & -\mathbf{R}_{(i_1,\ldots,i_M)} & -\mathbf{I} & \mathbf{B}_{1,i_M} \\ * & * & * & -\mathbf{S}_{(i_1,\ldots,i_M)} & \mathbf{S}_{(i_1,\ldots,i_M)}\mathbf{B}_{1,i_M} \\ * & * & * & * & -\mathbf{I} \end{bmatrix},$$

$$\mathbf{G}_{i_M} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{C}_{2,i_M} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D}_{21,i_M} \end{bmatrix},$$

$$\mathbf{F}_{i_M} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{2,i_M}^{\mathrm{T}} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

and

$$\widehat{\mathbf{H}}_{(i_0,\dots,i_M)} = \begin{bmatrix} -\mathbf{S}_{(i_0,\dots,i_{M-1})} & -\mathbf{I} & \mathbf{C}_{1,i_M}^{\mathrm{T}} & \mathbf{0} \\ * & -\mathbf{R}_{(i_0,\dots,i_{M-1})} & \mathbf{R}_{(i_0,\dots,i_{M-1})}\mathbf{C}_{1,i_M}^{\mathrm{T}} & \mathbf{0} \\ * & * & -\mathbf{Z}_{(i_0,\dots,i_M)} & \mathbf{D}_{11,i_M} \\ * & * & * & -\mathbf{I} \end{bmatrix},$$

$$\widehat{\mathbf{G}}_{i_M} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{C}_{2,i_M} & \mathbf{0} & \mathbf{0} & \mathbf{D}_{21,i_M} \end{bmatrix},$$

$$\widehat{\mathbf{F}}_{i_M} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_{12,i_M}^{\mathrm{T}} & \mathbf{0} \end{bmatrix}$$

for all $(i_0,\dots,i_M) \in \mathscr{L}_M(\Theta)$. In particular, if this condition holds, then the controller $(\mathscr{K},\Theta_L)$ can be taken to be M-path-dependent (i.e., $L = M$) and of order n (i.e., $n_k = n$).

Proof: The proof is based on Theorem 8 and the change-of-variable technique developed in [30], and it is identical to that of [22, Theorem 2] except that we now have multiple performance levels. ∎

As in [27], [30], once the linear matrix inequalities in Theorem 10 have been solved for some M, the coefficients $\mathbf{A}_{K,(i_0,\dots,i_M)}$, $\mathbf{B}_{K,(i_0,\dots,i_M)}$, $\mathbf{C}_{K,(i_0,\dots,i_M)}$, and $\mathbf{D}_{K,(i_0,\dots,i_M)}$ of an M-path-dependent controller are obtained via

$$\mathbf{W}_{(i_0,\dots,i_M)} = \begin{bmatrix} \mathbf{S}_{(i_1,\dots,i_M)}\mathbf{A}_{i_M}\mathbf{R}_{(i_0,\dots,i_{M-1})} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{U}_{(i_1,\dots,i_M)} & \mathbf{S}_{(i_1,\dots,i_M)}\mathbf{B}_{2,i_M} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{K,(i_0,\dots,i_M)} & \mathbf{B}_{K,(i_0,\dots,i_M)} \\ \mathbf{C}_{K,(i_0,\dots,i_M))} & \mathbf{D}_{K,(i_0,\dots,i_M)} \end{bmatrix} \times \begin{bmatrix} \mathbf{T}_{(i_0,\dots,i_{M-1})}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{C}_{2,i_M}\mathbf{R}_{(i_0,\dots,i_{M-1})} & \mathbf{I} \end{bmatrix}$$

for all $(i_0,\dots,i_M) \in \mathscr{L}_M(\Theta)$, where $\mathbf{T}_{(j_0,\dots,j_{M-1})}$ and $\mathbf{U}_{(j_0,\dots,j_{M-1})}$ are any nonsingular matrices such that

$$\mathbf{T}_{(j_0,\dots,j_{M-1})}\mathbf{U}_{(j_0,\dots,j_{M-1})}^{\mathrm{T}} = \mathbf{I} - \mathbf{R}_{(j_0,\dots,j_{M-1})}\mathbf{S}_{(j_0,\dots,j_{M-1})}$$

for $(j_0,\dots,j_{M-1}) \in \{1,\dots,N\}^M$.

C. Synthesis of Markovian Jump Linear Systems

If $(\mathbf{P},\mathbf{p})$ is a Markov chain and if L is a nonnegative integer, then, as in [27], we can define a state transition matrix $\mathbf{Q}_L(\mathbf{P},\mathbf{p}) = (q_{(i_0,\dots,i_L)(j_0,\dots,j_L)})$ and an initial distribution vector $\mathbf{q}_L(\mathbf{P},\mathbf{p}) = (q_{(i_0,\dots,i_L)})$ among the switching paths in $\mathscr{L}_L(\Theta(\mathbf{P},\mathbf{p}))$ as follows: $q_{(i_0,\dots,i_L)(j_0,\dots,j_L)} = p_{i_L j_L}$ if $(i_1,\dots,i_L) = (j_0,\dots,j_{L-1})$, and $q_{(i_0,\dots,i_L)(j_0,\dots,j_L)} = 0$ otherwise; also, $q_{(i_0,\dots,i_L)} = p_{i_L}$ if $(i_0,\dots,i_{L-1}) = (0,\dots,0)$, and $q_{(i_0,\dots,i_L)} = 0$ otherwise. The feedback interconnection of $(\mathscr{T},\mathbf{P},\mathbf{p})$ and $(\mathscr{K},\Theta(\mathbf{P},\mathbf{p})_L)$ leads to a *closed-loop Markovian jump linear system* $(\mathscr{T}_{\mathscr{K}},\mathbf{Q}_L(\mathbf{P},\mathbf{p}),\mathbf{q}_L(\mathbf{P},\mathbf{p}))$, where $(\mathbf{Q}_L(\mathbf{P},\mathbf{p}),\mathbf{q}_L(\mathbf{P},\mathbf{p}))$ defines the *closed-loop Markov chain*.

Suppose that $\mathbf{p}$ is $\mathbf{P}$-invariant. Given integers L, $T \geq 0$, define

$$\mathscr{N}_{L,T,k}(\Theta(\mathbf{P},\mathbf{p})) = \{(i_0,\dots,i_{L+T}) \in \mathscr{L}_{L+T}(\Theta(\mathbf{P},\mathbf{p})): (i_0,\dots,i_{k-1}) = (0,\dots,0),\, (i_k,\dots,i_{L+T}) \in \mathscr{W}_{L+T-k}(\Theta(\mathbf{P},\mathbf{p}))\}$$

for $k \in \{0,\dots,L\}$. Then, for $0 < k \leq L$, $\mathscr{N}_{L,T,k}(\Theta(\mathbf{P},\mathbf{p}))$ is the set of all admissible closed-loop switching paths $(\theta(k-L),\dots,\theta(k+T))$ that can occur with positive probability at time $t = L-k$; similarly, for $k = 0$, $\mathscr{N}_{L,T,0}(\Theta(\mathbf{P},\mathbf{p}))$ is the set of all admissible $(\theta_L(t-L),\dots,\theta(t+T))$ that can occur with positive probability at $t \geq L$. For each $k \in \{0,\dots,L\}$, the $(L+T)$-step probabilities $\pi_{(i_0,\dots,i_{L+T})}$ over $(i_0,\dots,i_{L+T}) \in \mathscr{N}_{L,T,k}(\Theta(\mathbf{P},\mathbf{p}))$ are given by

$$\pi_{(i_0,\dots,i_{L+T})} = p_{i_k} p_{i_k i_{k+1}} \cdots p_{i_{L+T-1} i_{L+T}}, \tag{16a}$$

$$\sum_{(i_0,\dots,i_{L+T}) \in \mathscr{N}_{L,T,k}(\Theta(\mathbf{P},\mathbf{p}))} \pi_{(i_0,\dots,i_{L+T})} = 1 \tag{16b}$$

for each $k \in \{0,\dots,L\}$.

Theorem 11: Let $\mathscr{T}$ be as in (11); let $(\mathbf{P},\mathbf{p})$ be a Markov chain, where $\mathbf{p}$ is $\mathbf{P}$-invariant. Given a nonnegative integers $n_K \geq n$ and T, there exist a path length L and an L-path-dependent controller $(\mathscr{K},\Theta(\mathbf{P},\mathbf{p})_L)$ of order n_K such that the closed-loop system $(\mathscr{T}_{\mathscr{K}},\mathbf{Q}_L(\mathbf{P},\mathbf{p}),\mathbf{q}_L(\mathbf{P},\mathbf{p}))$ is almost surely uniformly exponentially stable and satisfies a T-step average performance level $\gamma > 0$ if and only if there exists a nonnegative integer M such that (15a) and (15b) hold for all $(i_0,\dots,i_M) \in \mathscr{L}_M(\Theta(\mathbf{P},\mathbf{p}))$, and (15c) holds for all $(i_0,\dots,i_{M+T}) \in \mathscr{L}_{M+T}(\Theta(\mathbf{P},\mathbf{p}))$ with $(i_M,\dots,i_{M+T}) \in \mathscr{W}_T(\Theta(\mathbf{P},\mathbf{p}))$, so that

$$\sum_{(i_M,\dots,i_{M+T}) \in \mathscr{W}_T(\Theta(\mathbf{P},\mathbf{p}))} \pi_{(i_M,\dots,i_{M+T})} \gamma_{(i_M,\dots,i_{M+T})}^2 \leq \gamma^2, \tag{17}$$

where $\pi_{(i_M,\dots,i_{M+T})}$ are as in (5). Moreover, such a controller can be taken to be M-path-dependent and of order n.

Proof: We first show that, for any T, L, and $\mathscr{K}$, the closed-loop T-step probabilities are given by the $(L+T)$-step probabilities defined in (16); that is, the probability of the closed-loop switching path $(\theta_L(t),\dots,\theta_L(t+T))$ is equal to $\pi_{(\theta(t-L),\dots,\theta(t+T))}$ for all $t \in \{0,1,\dots\}$ and for all realizations θ of $(\mathbf{P},\mathbf{p})$ such that $(\theta(t-L),\dots,\theta(t+T))$ has positive probability. If $t = 0$, then $\theta_L(0) = (0,\dots,0,\theta(0))$, so $q_{\theta_L(0)} = p_{\theta(0)}$ under $(\mathbf{Q}_L(\mathbf{P},\mathbf{p}),\mathbf{q}_L(\mathbf{P},\mathbf{p}))$. If $t = 1$, then $\theta_L(1) = (0,\dots,0,\theta(0),\theta(1))$, so the probability of $\theta_L(1)$ is equal to $q_{\theta_L(0)} q_{\theta_L(0)\theta_L(1)} = p_{\theta(0)} p_{\theta(0)\theta(1)}$ for each realization θ of $(\mathbf{P},\mathbf{p})$. By induction, we establish that the probability of $(i_0,\dots,i_L) \in \mathscr{L}_L(\Theta(\mathbf{P},\mathbf{p}))$ is $p_{i_k} p_{i_k i_{k+1}} \cdots p_{i_{L-1} i_L}$ whenever

$$(i_0,\dots,i_L) = (\underbrace{0,\dots,0}_{k\text{ times}},i_k,\dots,i_L)$$

for some $k \in \{0,\dots,L\}$. Thus, whenever $(i_0,\dots,i_{L+T}) \in \mathscr{L}_{L+T}(\Theta(\mathbf{P},\mathbf{p}))$ is such that $(i_L,\dots,i_{L+T}) \in \mathscr{W}_T(\Theta(\mathbf{P},\mathbf{p}))$ and such that

$$(i_0,\dots,i_{L+T}) = (\underbrace{0,\dots,0}_{k\text{ times}},i_k,\dots,i_{L+T})$$

for some $k \in \{0,\ldots,L\}$, the $\mathbf{P}$-invariance of $\mathbf{p}$ yields that the closed-loop T-step probabilities $\pi_{(i_0,\ldots,i_{L+T})}$ satisfy (16). Finally, the $\mathbf{P}$-invariance of $\mathbf{p}$ implies that

$$\sum_{\{(i_k,\ldots,i_{L-1}) \colon (i_0,\ldots,i_{L+T}) \in \mathscr{N}_{L,T,k}(\Theta(\mathbf{P},\mathbf{p}))\}} \pi_{(i_0,\ldots,i_{L+T})} = \pi_{(i_L,\ldots,i_{L+T})}$$

for each $k \in \{0,\ldots,L\}$.

Now, in view of Theorem 9 and the fact that T-step probabilities are invariant under L-path-dependent feedback, if we replace Θ in Theorem 10 with $\Theta(\mathbf{P},\mathbf{p})$ and add (17) to the conditions in the theorem, then we obtain the desired synthesis result for $(\mathscr{T},\mathbf{P},\mathbf{p})$. ■

For switched systems, one way to jointly optimize the performance levels $\gamma_{(i_0,\ldots,i_T)} > 0$ over all length-T switching paths $(i_0,\ldots,i_T) \in \mathscr{W}_T(\Theta)$ is to minimize a convex combination of $\gamma_{(i_0,\ldots,i_T)}$. The path-by-path performance levels resulting from this optimization will be Pareto optimal; that is, no path-by-path performance levels $\tilde{\gamma}_{(i_0,\ldots,i_T)}$ satisfying $\tilde{\gamma}_{(i_0,\ldots,i_T)} \leq \gamma_{(i_0,\ldots,i_T)}$ for all $(i_0,\ldots,i_T) \in \mathscr{W}_T(\Theta)$ and $\tilde{\gamma}_{(j_0,\ldots,j_T)} < \gamma_{(j_0,\ldots,j_T)}$ for some $(j_0,\ldots,j_T) \in \mathscr{W}_T(\Theta)$ are achievable subject to closed-loop stability. Due to Theorem 11, such Pareto optimization leads to closed-loop stability and optimal average performance level for Markovian jump linear systems if the coefficients for the convex combination are the T-step probabilities.

V. CONCLUSIONS

A novel approach for optimal receding-horizon-type control of switched linear systems and Markovian jump linear systems was proposed. Salient features of this approach include that a sufficiently large control horizon is not required to guarantee stability, and that the synthesis condition for dynamic output feedback controllers is nonconservative. For performance optimization, this approach involves semidefinite programming over future switching paths of length T; on the other hand, to guarantee stability, the semidefinite program is subject to a set of Lyapunov inequalities over past switching paths of length M. As T and M approach infinity, the performance approaches that of the infinite-horizon LQG controllers developed in [27]. However, saturation in performance often occurs at a small M, and hence, in practice, M can be taken to remain constant over all T.

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