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Finite-Time Stabilization of Impulsive Dynamical Linear Systems

F. Amato†, R. Ambrosino‡, M. Ariola‡, C. Cosentino†, G. De Tommasi§

***Abstract*— This paper deals with the finite-time stability problem for a special class of hybrid systems, namely *impulsive dynamical linear systems* (IDLS). IDLS are systems that exhibit jumps in the state trajectory. Both analysis and design problems are tackled, both for time-dependent and state-dependent IDLS. The presented results require to solve feasibility problems involving Differential Linear Matrix Inequalities (DLMIs), which can be solved numerically in an efficient way, as illustrated by the proposed example.**

I. Introduction

The concept of finite-time stability (FTS) dates back to the Sixties, when it was introduced in the control literature [1], [2]. A system is said to be finite-time stable if, given a bound on the initial condition, its state does not exceed a certain threshold during a specified time interval. It is important to recall that FTS and Lyapunov Asymptotic Stability (LAS) are independent concepts; indeed a system can be FTS but not LAS, and vice versa.

In [3] sufficient conditions for FTS and finite-time stabilization of continuous-time linear time-invariant systems are provided; such conditions require the solution of a feasibility problem involving Linear Matrix Inequalities (LMIs). A different approach, which is reminiscent of optimal control techniques and it is also applicable to linear time-varying (LTV) systems, has been proposed in [4]. In the time-invariant case, the main result of [4] turns out to be less conservative than the condition provided in [3], but it is computationally more demanding, since the solution of a Differential Linear Matrix Inequality (DLMI) is required.

In this paper we consider the class of LTV systems with finite state jumps [5], which are linear continuous-time system whose state undergoes finite jump discontinuities at discrete instants of time. Such systems can be regarded as a special class of hybrid systems, namely *impulsive dynamical linear systems* (IDLS) [6], which can be either *time–dependent*, if the state jumps are time-driven, or *state–dependent*, where the state jumps occur when the trajectory reaches an assigned subset of the state space, the so-called *resetting set*. This work follows the spirit of [4] to derive the main results for FTS analysis and control of IDLS. The first contribution of the paper is a necessary and sufficient condition for FTS. It requires the computation of the state transition matrix of the given system, a numerically hard problem except for time-invariant systems. Therefore we also provide some sufficient conditions for FTS (one for time-dependent IDLS and one for state-dependent IDLS), which require the solution of two coupled differential–difference Lyapunov inequalities; the Lyapunov inequalities can be turned into differential–difference linear matrix inequalities (D/DLMIs) which can be efficiently solved with many existing software packages. Sufficient conditions for finite-time stabilization in the state and output feedback cases are derived as well.

The paper is organized as follows: in Section II both the class of impulsive dynamical linear systems and the definition of FTS are introduced. In Section III the analysis conditions are given both for time-dependent and state-dependent IDLS. Synthesis results are provided in Section IV so as to solve the problem of finite-time stabilization, either via output or state feedback. A numerical example is then discussed in order to demonstrate the effectiveness of the proposed approach. Eventually some conclusions are drawn.

Notations. The symbols $\mathbb{R}$ and $\mathbb{N}$ denote the set of the real and positive integer numbers respectively; accordingly, $\mathbb{R}^{n\times m}$ is the set of $n \times m$ matrices whose entries are real numbers. By $\mathcal{L}^2_{[0,T]}$ (l^2_r) we denote the set of square integrable (summable) vector-valued functions defined over the interval $[0,T]$ (over the set $\{1,2,\dots,r\}$).

II. Problem Statement

Let us consider the IDLS described by

$$\dot{x}(t) = A_c(t)x(t)\,, \quad x(0) = x_0\,, \quad \big(t, x(t)\big) \notin \mathcal{S} \tag{1a}$$

$$x(t^+) = A_d(t)x(t)\,, \quad \big(t, x(t)\big) \in \mathcal{S} \tag{1b}$$

where $A_c(\cdot)\,, A_d(\cdot) : t \in [0,+\infty) \mapsto \mathbb{R}^{n\times n}$, are continuous matrix-valued functions and $\mathcal{S} \subset \big[0,+\infty\big]\times\mathbb{R}^n$ is called the *resetting set* [6]. In particular (1a) describes the *continuous-time dynamic* of the IDLS, and (1b) is the *resetting law*. According to (1b), system (1) may exhibit a finite jump from $x(t_k)$ to $x(t_k^+) \neq x(t_k)$. For a particular trajectory $x(t)$, we let t_k, $k \in \mathbb{N}$, to denote the k-th instant of time at which $\big(t, x(t)\big)$ intersects $\mathcal{S}$, and we call t_k, $k \in \mathbb{N}$, *resetting times*. Furthermore we assume that when the trajectory $x(t)$ intersects the resetting set $\mathcal{S}$, it instantaneously exits $\mathcal{S}$. Depending on the definition of the resetting set $\mathcal{S}$, IDLS can be classified as follows [6]:

- **i)** *Time-dependent IDLS* – $\mathcal{S} = \mathcal{T} \times \mathbb{R}^n$, with $\mathcal{T} := \big\{t_1\,, t_2\,, \dots\big\}$, i.e. the resetting set is defined by a prescribed sequence of times, which are independent of the state $x(\cdot)$;

† F. Amato and C. Cosentino are with the School of Computer Science and Biomedical Engineering, Università degli Studi Magna Græcia di Catanzaro, Via T. Campanella 115, 88100 Catanzaro, Italy.

‡ R. Ambrosino and M. Ariola are with the Dipartimento per le Tecnologie, Università degli Studi di Napoli Parthenope, Centro Direzionale di Napoli, Isola C4, 80143 Napoli, Italy.

§ G. De Tommasi is with the Dipartimento di Informatica e Sistemistica, Università degli Studi di Napoli Federico II, Via Claudio 21, 80125 Napoli, Italy.

ii) *State-dependent IDLS* – $\mathcal{S} = \big[0,+\infty\big] \times \bigcup_k \mathcal{D}_k$, with $\mathcal{D}_k \subset \mathbb{R}^n$, i.e. the resetting set is defined by a region in the state space, which is independent of time.

The following assumption is introduced in order to prevent system (1) to exhibit Zeno behavior.

Assumption 1: Given an interval $[0,T]$, there are only a finite number of resetting times, either if (1) is time-dependent, or state-dependent. It follows that the resetting set to be considered in the time interval $[0,T]$ is given by

$$\mathcal{S} = \mathcal{T} \times \mathcal{D} \subset [0,T] \times \mathbb{R}^n\,, \quad \text{with } \mathcal{T} = \big\{t_1\,, t_2\,, \ldots\,, t_r\big\}\,. \quad \diamondsuit$$

In this paper we deal with the behavior of the system (1) within a finite interval $[0,T]$. The solution of system (1) in the considered interval is given by

$$x(t) = \Phi(t,0)x_0\,, \quad t \in [0,T],$$

where the matrix function $\Phi(t,\tau)$ is the state transition matrix of system (1). The transition matrix turns out to be piecewise continuous with possible right discontinuities at the time instants t_k, $k = 1,\ldots,r$. In the first interval, $\Phi(t,\tau)$ satisfies the following matrix differential equation

$$\frac{\partial}{\partial t}\Phi(t,0) = A_c(t)\Phi(t,0)\,,\, t \in [0,t_1]\,, \quad \Phi(0,0) = I\,;$$

in the following intervals we have

$$\frac{\partial}{\partial t}\Phi(t,t_k^+) = A_c(t)\Phi(t,t_k^+)\,,\, t \in]t_k,t_{k+1}]\,,\, k = 1,\ldots,r-1$$
$$\Phi(t_k^+,t_k^+) = A_d(t_k)\Phi(t_k,t_{k-1}^+)\,, \quad k = 1,\ldots,r-1\,,$$

where $t_0^+ = t_0 := 0$ (obviously at $t_0 = 0$ there is no discontinuity). Finally in the last interval we have

$$\frac{\partial}{\partial t}\Phi(t,t_r^+) = A_c(t)\Phi(t,t_r^+)\,, \quad t \in]t_r,T]$$
$$\Phi(t_r^+,t_r^+) = A_d(t_r)\Phi(t_r,t_{r-1}^+)\,.$$

We now extend the definition of finite-time stability [2] to the class of IDLS.

Definition 1 (FTS of impulsive systems): Given a positive scalar T, a positive definite matrix R, a positive definite matrix-valued function $\Gamma(\cdot)$ defined over $[0,T]$, with $\Gamma(0) < R$, the IDLS described by (1) is said to be finite-time stable with respect to $\big(T,R,\Gamma(\cdot)\big)$ if

$$x_0^T R x_0 \le 1 \Rightarrow x(t)^T\Gamma(t)x(t) < 1 \quad \forall\, t \in [0,T]\,. \qquad (2)$$

$\diamondsuit$

Remark 1: Definition 1 can be interpreted in terms of ellipsoidal domains. The set defined by $x_0^T R x_0 \le 1$ contains all the admissible initial states. The inequality $x(t)^T\Gamma(t)x(t) < 1$, instead, defines a time-varying ellipsoid that bounds the state trajectory over the interval $[0,T]$. $\diamondsuit$

Given a piecewise continuous vector-valued function $z(\cdot) \in \mathcal{L}^2_{[0,T]}$, with right discontinuities at the points $t_1\,,\ldots\,,t_r$, we can define three norms, as follows. The first is the classical $\mathcal{L}^2$-norm

$$\|z\|_{2,L} := \left[\int_0^T z^T(t)z(t)dt\right]^{1/2}\,.$$

Note that the function $z(\cdot)$ univocally defines the sequence $\{z(t_k)\}_{k=1,\ldots,r} \in l_r^2$ (remember that $z(t_k)$ represents the left limit of $z(\cdot)$ in t_k and that $z(\cdot)$ is assumed to be left continuous in t_k); therefore

$$\|z\|_{2,l} := \left[\sum_{k=1}^r z^T(t_k)z(t_k)\right]^{1/2}$$

is the classical l^2-norm of the sequence $\{z(t_k)\}$. Notice that $\|z\|_{2,l}$ turns out to be a semi-norm for the signal $z(\cdot)$.

Finally we can think of $z(\cdot)$ as the composition of two signals, one belonging to $\mathcal{L}^2$, and the other to l^2, therefore defining a "mixed norm" over $\mathcal{L}^2 \oplus l^2$

$$\|z\|_{2,m} := \left[\|z\|_{2,L}^2 + \|z\|_{2,l}^2\right]^{1/2}\,. \qquad (3)$$

It is simple to recognize that the mixed norm (3) is actually a norm for $\mathcal{L}^2 \oplus l^2$.

III. MAIN RESULTS

The following theorem provides a necessary and sufficient condition for FTS of system (1).

Theorem 1: System (1) is FTS wrt $(T,R,\Gamma(\cdot))$ *iff* for all $t \in [0,T]$

$$\Phi(t,0)^T\Gamma(t)\Phi(t,0) < R\,. \qquad (4)$$

Proof: Assume that (4) holds and let $x_0^T R x_0 \le 1$. Then

$$x(t)^T\Gamma(t)x(t) = x_0^T\Phi(t,0)^T\Gamma(t)\Phi(t,0)x_0 < x_0^T R x_0 < 1\,.$$

Therefore system (1) is FTS. Conversely, assume by contradiction that system (1) is FTS and that for some $\bar{t}$, $\bar{x}$

$$\bar{x}^T\Phi(\bar{t},0)^T\Gamma(\bar{t})\Phi(\bar{t},0)\bar{x} \ge \bar{x}^T R\bar{x}\,. \qquad (5)$$

Now let $x(0) = \lambda\bar{x}$, where λ is such that $x(0)^T Rx(0) = 1$. Then (5) implies that

$$x(\bar{t})^T\Gamma(\bar{t})x(\bar{t}) = x(0)^T\Phi(\bar{t},0)^T\Gamma(\bar{t})\Phi(\bar{t},0)x(0) \ge 1\,,$$

contradicting the assumption of FTS for system (1). ■

It is worth noticing that for state-dependent IDLS, the condition in Theorem 1 cannot be applied, since the resetting times are not known *a priori*. Condition (4) may be difficult to check also for time-dependent IDLS, unless we are in the time-invariant case, because it requires the computation of the transition matrix. For these reasons, in the next lemma we provide an alternative condition for FTS which involves two coupled differential–difference Lyapunov inequalities. This condition is further exploited so as to introduce conditions that can be checked numerically in a more efficient way by means of LMIs.

Lemma 1: Given system (1) and $t \in [0,T]$, the condition

$$x_0^T R x_0 \le 1 \Rightarrow x(t)^T\Gamma(t)x(t) < 1 \qquad (6)$$

is satisfied *iff* the following coupled differential–difference Lyapunov inequalities, with terminal and initial conditions, admit a piecewise continuously differentiable symmetric solution $P(\cdot)$:

$$\dot{P}(\tau) + A_c(\tau)^T P(\tau) + P(\tau)A_c(\tau) < 0\,, \qquad (7a)$$
$$\tau \in]0,t]\,, \quad \tau \notin \mathcal{T}$$
$$x(t_k)^T\big(A_d(t_k)^T P(t_k^+)A_d(t_k) - P(t_k)\big)x(t_k) \le 0\,, \qquad (7b)$$
$$t_k \in]0,t]\,, \quad \big(t_k,x(t_k)\big) \in \mathcal{S}$$
$$P(t) \ge \Gamma(t)\,, \qquad (7c)$$
$$P(0) < R\,. \qquad (7d)$$

Proof: Let $V(\tau,x) = x^T P(\tau)x$. Then, if $\tau \notin \mathcal{T}$, the derivative of V along the trajectories of system (1) yields

$$\dot{V}(\tau,x) = x^T\left(\dot{P}(\tau) + A_c(\tau)^T P(\tau) + P(\tau)A_c(\tau)\right)x\,,$$

which is negative definite by virtue of (7a). At the discontinuity points we have

$$V(t_k^+, x) - V(t_k, x) = x^T(t_k)\left(A_d(t_k)P(t_k^+)A_d(t_k) - P(t_k)\right)x(t_k)\,,$$

which is negative semidefinite in view of (7b). We can conclude that $V(\tau, x)$ is strictly decreasing along the trajectories of system (1); hence, given x_0 such that $x_0^T R x_0 \le 1$, we have

$$\begin{aligned} x(t)^T\Gamma(t)x(t) &\le x(t)^T P(t)x(t) \quad \text{by (7c)} \\ &< x(0)^T P(0)x(0) \\ &< x(0)^T R x(0) \le 1 \quad \text{by (7d)}\,. \end{aligned}$$

Conversely, let us assume that (6) holds. By continuity arguments there exists a sufficiently small $\epsilon > 0$ such that, letting $z = \epsilon x$, we have

$$x(0)^T R x(0) \le 1 \Rightarrow x(t)^T\Gamma(t)x(t) + \|z\|_{2,m}^2 < 1\,, \tag{8}$$

where $\|\cdot\|_{2,m}$ has been defined in (3). Let us define $P(\cdot)$ as the unique symmetric solution of

$$\dot P(\tau) + A_c(\tau)^T P(\tau) + P(\tau)A_c(\tau) + \epsilon^2 I = 0\,, \quad \tau \in]0,t]\,, \quad \tau \notin \mathcal{T} \tag{9a}$$

$$P(t) = \Gamma(t) \tag{9b}$$

$$P(t_k) = A_d(t_k)^T P(t_k^+)A_d(t_k) + \epsilon^2 I\,, \quad t_k \in]0,t]\,, \quad t_k \in \mathcal{T} \tag{9c}$$

and assume, by contradiction, that (7d) is not satisfied, i.e. for some $\bar x$

$$\bar x^T P(0)\bar x \ge \bar x^T R \bar x\,. \tag{10}$$

Now let $x_0 = \lambda \bar x$, where λ is such that

$$x_0^T R x_0 = 1\,. \tag{11}$$

Then (10) implies

$$x_0^T P(0) x_0 \ge 1\,. \tag{12}$$

First, consider the time interval $]t_k, t_{k+1}]$, with $t_k\,, t_{k+1} \in\ ]0,t] \cap \mathcal{T}$. From (9a) we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} x(\tau)^T P(\tau)x(\tau) = -\epsilon^2 x(\tau)^T x(\tau)\,, \tag{13}$$

and, integrating (13) from t_k^+ to t_{k+1}

$$x(t_{k+1})^T P(t_{k+1})x(t_{k+1}) - x(t_k^+)^T P(t_k^+)x(t_k^+) = -\epsilon^2 \|x_{]t_k,t_{k+1}]}\|_{2,L}^2\,. \tag{14}$$

If $t \notin \mathcal{T}$, we can write an analogous relation for the last interval $]t_{r_t}, t]$

$$x(t)^T P(t)x(t) - x(t_{r_t}^+)^T P(t_{r_t}^+)x(t_{r_t}^+) = -\epsilon^2 \|x_{]t_{r_t},t]}\|_{2,L}^2\,, \tag{15}$$

where

$$t_{r_t} = \max\{t_k \in \mathcal{T} \,:\, t_k < t\}\,.$$

From (9c) we obtain for the resetting times $t_k \in]0,t] \cap \mathcal{T}$,

$$x(t_k^+)^T P(t_k^+)x(t_k^+) - x(t_k)^T P(t_k)x(t_k) = -\epsilon^2 \|x(t_k)\|^2\,, \tag{16}$$

where $\|v\|$ denotes the Euclidean norm of the vector v.

Then, summing over all the intervals (14) and (16) with (15), and taking into account (12) and (9c), we have

$$\begin{aligned} x(t)^T\Gamma(t)x(t) = x(t)^T P(t)x(t) &= x_0^T P(0)x_0 - \epsilon^2\|x\|_{2,m}^2 \\ &\ge 1 - \|z\|_{2,m}^2\,. \end{aligned} \tag{17}$$

We can conclude that an x_0 has been found which satisfies (11) and such that (17) holds; this contradicts (8). Therefore (10) is not true, and (7d) holds true. Hence we have built a symmetric $P(\cdot)$ through (9) which satisfies (7a)–(7c) and (7d); this completes the proof. ■

Lemma 1 can be immediately applied to state an alternative (to Theorem 1) necessary and sufficient condition for FTS. Looking at Definition 1, in principle we should check that for any $t \in [0,T]$ the hypotheses of Lemma 1 are satisfied. In other words we should check the feasibility of infinitely many optimization problems involving two coupled differential–difference Lyapunov inequalities; this is obviously an impossible task. Even performing a fine gridding of the interval $[0,T]$, the problem would remain hard from a computational point of view. To this regard Theorem 1 remains more appealing. Moreover, as for Theorem 1, conditions (7) cannot be checked for state-dependent IDLS.

Although Lemma 1 is not useful *per se* to derive a reliable result to analyze the FTS of a given IDLS, it permits to prove in a simple way the following theorems which require to check the feasibility of only two coupled differential–difference Lyapunov inequalities. The conditions stated in the next theorems, however, are only sufficient thus they introduce a certain degree of conservatism in the analysis.

Theorem 2 (FTS for time-dependent IDLS): Assume that the coupled differential–difference Lyapunov inequalities with terminal and initial conditions

$$\dot P(t) + A_c(t)^T P(t) + P(t)A_c(t) < 0\,, \quad t \in]0,T]\,, \quad t \notin \mathcal{T} \tag{18a}$$

$$P(t_k) \ge A_d(t_k)^T P(t_k^+)A_d(t_k)\,, \quad t_k \in \mathcal{T} \tag{18b}$$

$$P(t) \ge \Gamma(t)\,, \quad \forall\, t \in [0,T]\,, \tag{18c}$$

$$P(0) < R\,, \tag{18d}$$

admit a piecewise continuously differentiable symmetric solution $P(\cdot)$; then the *time-dependent* IDLS (1) is FTS with respect to $\big(T, R, \Gamma(\cdot)\big)$.

Proof: It is straightforward to check that, if the IDLS (1) is time-dependent, then condition (18b) is equivalent to (7b), since $\mathcal{S} = \mathcal{T} \times \mathbb{R}^n$. It follows that a symmetric matrix-valued function $P(\cdot)$ satisfying conditions (18) also satisfies (7) for all $t \in [0,T]$. Therefore condition (2) is satisfied and system (1) is FTS. ■

Remark 2: The conservatism introduced by Theorem 2 has to be interpreted as follows. Given two time instants t_1 and t_2, $t_1 < t_2$, belonging to the interval $[0,T]$, the necessary and sufficient condition coming from Lemma 1 requires to find two symmetric matrix-valued functions $P_1(\cdot)$ and $P_2(\cdot)$ which satisfy the hypotheses of Lemma 1 for $t = t_1$ and $t = t_2$, respectively. Note that, concerning condition (7d), it is only required that at the terminal point of the respective definition interval, namely $[0,t_1]$ for $P_1(\cdot)$ and $[0,t_2]$ for $P_2(\cdot)$, the two functions satisfy $P_1(t_1) \ge \Gamma$ and $P_2(t_2) \ge \Gamma$, while $P_1(\cdot)$ and $P_2(\cdot)$ are not constrained to be the same function over the common interval $[0,t_1]$. Conversely, Theorem 2 requires to find one symmetric matrix-valued function $P(\cdot)$ which satisfies for the whole interval $[0,T]$ the condition $P(t) \ge \Gamma(t)$; this is the source of conservatism.

Remark 3: By using Schur complements arguments, inequality (18b) can be turned into the following difference linear matrix inequality

$$\begin{pmatrix} -P(t_k) & A_d^T(t_k)P(t_k^+) \\ P(t_k^+)A_d(t_k) & -P(t_k) \end{pmatrix} < 0\,. \qquad (19)$$

◊

Theorem 3 (FTS for state-dependent IDLS): Assume that the coupled differential–difference Lyapunov inequalities with terminal and initial conditions

$$\dot{P}(t) + A_c(t)^T P(t) + P(t)A_c(t) < 0\,, \quad t \in]0,T] \qquad (20a)$$
$$x^T(t)\big(A_d(t)^T P(t) A_d(t) - P(t)\big)x(t) \le 0\,, \qquad (20b)$$
$$t \in]0,T]\,, \quad x \in \bigcup_k \mathcal{D}_k$$
$$P(t) \ge \Gamma(t)\,, \quad \forall\, t \in [0,T]\,, \qquad (20c)$$
$$P(0) < R\,, \qquad (20d)$$

admit a continuously differentiable symmetric solution $P(\cdot)$; then the *state-dependent* IDLS (1) is FTS with respect to $\big(T, R, \Gamma(\cdot)\big)$.

Proof: The proof follows similar arguments to those used in the proof of Theorem 2. Note that, since for state-dependent IDLS the resetting times are not known *a priori*, conditions (7a) and (7b) have to be checked for all t in $[0,T]$, yielding conditions (20a) and (20b), respectively. ■

Remark 4: Condition (20b) can be turned into LMIs by using the *S*-procedure, as shown in [7]. ◊

Theorems 2 and 3 reduce the FTS analysis to a feasibility problem in the matrix variable $P(\cdot)$ involving two coupled differential–difference linear matrix inequalities (D/DLMIs), a LMI to be tested for all $t \in [0,T]$, and the LMI (18d). When the structure of the matrix $P(\cdot)$ is fixed *a priori*, for example piecewise affine (see the example in Section IV), the feasibility problem can be turned into a classical optimization problem involving LMIs [8].

IV. Finite-time stabilization

Problem 1 (Finite-time Control via Output Feedback): Consider the following IDLS

$$\dot{x}(t) = A_c(t)x(t) + B(t)u(t),\ x(0) = x_0\,, \qquad (21a)$$
$$\big(t, x(t)\big) \notin \mathcal{S}$$
$$x(t_k^+) = A_d(t_k)x(t_k)\,, \quad \big(t_k, x(t_k)\big) \in \mathcal{S} \qquad (21b)$$
$$y(t) = Cx(t) + Du(t), \quad t \ge 0 \qquad (21c)$$

where $u(t)$ is the control input, $y(t)$ is the output. Given a positive number T, two positive definite matrices R and R_K, two positive definite matrices $\Gamma(\cdot)$ and $\Gamma_K(\cdot)$ defined over $[0,T]$, with $\Gamma(0) < R$, $\Gamma_K(0) < R_K$. Find a dynamic output feedback controller in the form

$$\dot{x}_K(t) = A_K(t)x_K(t) + B_K(t)y(t) \qquad (22a)$$
$$\big(t, x(t)\big) \notin \mathcal{S}$$
$$x_K(t_k^+) = A_{d,K}(t_k)x_K(t_k) + B_{d,K}(t_k)y(t_k), \qquad (22b)$$
$$\big(t_k, x(t_k)\big) \in \mathcal{S}$$
$$u(t) = C_K(t)x_K(t) + D_K(t)y(t) \qquad (22c)$$

where $x_K(t)$ has the same dimension of $x(t)$, such that the closed loop system obtained by the interconnection of (21) and (22) is FTS with respect to $(T, blockdiag(R, R_K), blockdiag(\Gamma(\cdot), \Gamma_K(\cdot)))$.

The following lemma is useful for the solution to the problem.

Lemma 2 ([9]): Given symmetric matrices $S \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times n}$, there exist a symmetric matrix $V \in \mathbb{R}^{n\times n}$ and two nonsingular matrices $M \in \mathbb{R}^{n\times n}$, $N \in \mathbb{R}^{n\times n}$ such that[1]

$$P := \begin{pmatrix} S & M \\ M^T & V \end{pmatrix} > 0\,, \quad P^{-1} = \begin{pmatrix} Q & N \\ N^T & \star \end{pmatrix}\,,$$

iff

$$\begin{pmatrix} Q & I \\ I & S \end{pmatrix} > 0 \qquad (23)$$

◊

Theorem 4 (Output feedback for time-dependent IDLS): Problem 1 is solvable for time-dependent IDLS, if there exist piecewise continuously symmetric positive definite matrix-valued functions $Q(\cdot)$ and $S(\cdot)$, a nonsingular matrix-valued function $N(\cdot)$ and matrix-valued functions $\hat{A}_K(\cdot)$, $\hat{B}_K(\cdot)$, $\hat{C}_K(\cdot)$, $D_K(\cdot)$, $\hat{A}_{d,K}(\cdot)$, and $\hat{B}_{d,K}(\cdot)$ such that

$$\begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^T & \Theta_{22} \end{pmatrix} < 0, \qquad (24a)$$
$$t \in]0,T]\,, \quad t \notin \mathcal{T}$$
$$\begin{pmatrix} \Theta_{d,11} & \Theta_{d,12} \\ \Theta_{d,12}^T & \Theta_{d,22} \end{pmatrix} < 0\,, \quad t_k \in \mathcal{T} \qquad (24b)$$
$$\begin{pmatrix} Q & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ \Psi_{12}^T & \Psi_{22} & 0 & 0 \\ \Psi_{13}^T & 0 & I & 0 \\ \Psi_{14}^T & 0 & 0 & I \end{pmatrix} \ge 0\,, \quad t \in [0,T] \qquad (24c)$$
$$\begin{pmatrix} Q(0) & I \\ I & S(0) \end{pmatrix} \le \begin{pmatrix} \Delta_{11} & Q(0)R \\ RQ(0) & R \end{pmatrix}\,, \qquad (24d)$$

where[2]

$$\Theta_{11} = -\dot{Q} + A_c Q + Q A_c^T + B\hat{C}_K + \hat{C}_K^T B^T \qquad (25a)$$
$$\Theta_{12} = A_c + \hat{A}_K^T + BD_K C \qquad (25b)$$
$$\Theta_{22} = \dot{S} + SA_c + A_c^T S + \hat{B}_K C + C^T \hat{B}_K^T \qquad (25c)$$
$$\Theta_{d,11} = -\begin{pmatrix} Q & I \\ I & S \end{pmatrix} \qquad (25d)$$
$$\Theta_{d,12} = \begin{pmatrix} Q(t_k)A_d(t_k)^T & \hat{A}_{d,K}(t_k)^T \\ A_d(t_k)^T & A_d(t_k)^T S(t_k^+) + C(t_k)^T \hat{B}_{d,K}(t_k)^T \end{pmatrix} \qquad (25e)$$
$$\Theta_{d,22} = -\begin{pmatrix} Q(t_k^+) & I \\ I & S(t_k^+) \end{pmatrix} \qquad (25f)$$
$$\Psi_{12} = I - Q\Gamma \qquad (25g)$$
$$\Psi_{13} = Q\Gamma^{1/2} \qquad (25h)$$
$$\Psi_{14} = N\Gamma_K^{1/2} \qquad (25i)$$
$$\Psi_{22} = S - \Gamma \qquad (25j)$$
$$\Delta_{11} = Q(0)RQ(0) + N(0)R_K N(0)^T \qquad (25k)$$

[1] The symbol $\star$ denotes a “don’t care” block.

[2] In order to avoid awkward notation, when possible we discard the time dependence in both (24) and (25).

Proof: The connection of systems (21) and (22) reads

$$\begin{aligned}\dot{x}_{CL}(t) &= \begin{pmatrix} A_c(t)+B(t)D_K(t)C(t) & B(t)C_K(t) \\ B_K(t)C(t) & A_K(t)\end{pmatrix} x_{CL}(t) \\ &= A_{CL}(t)x_{CL}(t)\,, \quad t\geq 0\,, \quad t\notin\mathcal{T} \\ x_{CL}(t_k^+) &= \begin{pmatrix} A_d(t_k) & 0 \\ B_{d,K}(t_k)C(t_k) & A_{d,K}(t_k)\end{pmatrix} x_{CL}(t_k) \\ &= A_{d,CL}(t_k)x_{CL}(t_k)\,, \quad t_k\in\mathcal{T}\end{aligned}$$

where $x_{CL} = \begin{bmatrix} x^T & x_K^T\end{bmatrix}^T$.

According to Theorem 2 it follows that Problem 1 is solvable if there exist a piecewise continuously symmetric matrix-valued function $P(\cdot)$ such that

$$\dot{P}(t) + A_{CL}(t)^T P(t) + P(t)A_{CL}(t) < 0 \qquad t\in]0,T]\,, \quad t\notin\mathcal{T} \tag{26a}$$

$$\begin{pmatrix} -P(t_k) & A_{d,CL}(t_k)^T P(t_k^+) \\ P(t_k^+)A_{d,CL}(t_k) & -P(t_k^+)\end{pmatrix} < 0 \qquad t_k\in\mathcal{T} \tag{26b}$$

$$P(t) \geq blockdiag(\Gamma(\cdot),\Gamma_K(\cdot))\,, \quad \forall\, t\in[0,T]\,, \tag{26c}$$

$$P(0) < blockdiag(R, R_K). \tag{26d}$$

Now let us define the following matrix-valued functions

$$P(t) = \begin{pmatrix} S(t) & M(t) \\ M(t)^T & T(t)\end{pmatrix}, P(t)^{-1} = \begin{pmatrix} Q(t) & N(t) \\ N(t)^T & \star\end{pmatrix},$$

$$\Pi_1(t) = \begin{pmatrix} Q(t) & I \\ N(t)^T & 0\end{pmatrix}, \Pi_2(t) = \begin{pmatrix} I & S(t) \\ 0 & M(t)^T\end{pmatrix}.$$

Note that by definition

$$P(t)\Pi_1(t) = \Pi_2(t) \tag{27a}$$

$$S(t)Q(t) + M(t)N(t)^T = I \tag{27b}$$

$$\begin{aligned}Q(t)\dot{S}(t)Q(t) + N(t)\dot{M}(t)^T Q(t) + Q(t)\dot{M}(t)N(t)^T + \\ + N(t)\dot{T}(t)N(t)^T = -\dot{Q}(t)\end{aligned} \tag{27c}$$

By pre- and post-multiplying inequality (26a) by $\Pi_1(t)^T$ and $\Pi_1(t)$ respectively, condition (24a) follows once we let

$$\begin{aligned}\hat{A}_K(t) =& \dot{S}(t)Q(t) + \dot{M}(t)N(t)^T + M(t)A_K(t)N(t)^T + \\ &+ S(t)B(t)C_K(t)N(t)^T + M(t)B_K(t)C(t)Q(t) + \\ &+ S(t)\left(A_c(t) + B(t)D_K(t)C(t)\right)Q(t)\end{aligned} \tag{28a}$$

$$\hat{B}_K(t) = M(t)B_K(t) + S(t)B(t)D_K(t) \tag{28b}$$

$$\hat{C}_K(t) = C_K(t)N(t)^T + D_K(t)C(t)Q(t) \tag{28c}$$

By pre- and post-multiplying inequality (26b) by $blockdiag(\Pi_1(t_k)^T, \Pi_1(t_k^+)^T)$ and $blockdiag(\Pi_1(t_k), \Pi_1(t_k^+))$ respectively, condition (24b) follows once we let

$$\begin{aligned}\hat{A}_{d,K}(t_k) =& M(t_k^+)A_{d,K}(t_k)N(t_k)^T + \\ &+ M(t_k^+)B_{d,K}(t_k)C(t_k)Q(t_k) + \\ &+ S(t_k^+)A_d(t_k)Q(t_k)\end{aligned} \tag{29a}$$

$$\hat{B}_{d,K}(t_k) = M(t_k^+)B_{d,K}(t_k) \tag{29b}$$

By pre- and post-multiplying inequality (26c) and (26d) by $\Pi_1(t)^T$ and $\Pi_1(t)$ respectively, tacking into account (27) and Lemma 2, conditions (24c) and (24d) follow. Note that (24c) implies (23). ■

Remark 5: The statement of Theorem 4 requires to find a nonsingular $N(\cdot)$; this can be obtained by adding a further LMI constraint requiring positive definiteness of N for all $t\in[0,T]$. ♢

Remark 6 (Controller design): Assume now that the hypothesis of Theorem 4 are satisfied; in order to design the controller the following steps have to be followed:

1) Find $Q(\cdot)$, $S(\cdot)$, $N(\cdot)$, $\hat{A}_K(\cdot)$, $\hat{B}_K(\cdot)$, $\hat{C}_K(\cdot)$, $D_K(\cdot)$, $\hat{A}_{d,K}(\cdot)$ and $\hat{B}_{d,K}(\cdot)$ such that conditions (24) are satisfied.
2) Calculate the matrix function
$$M(t) = \left(I - S(t)Q(t)\right)N^{-T}(t)$$
and its derivative
$$\dot{M}(t) = -\left(\dot{S}(t)Q(t) + S(t)\dot{Q}(t) + M(t)\dot{N}(t)^T\right)N(t)^{-T}$$
3) Obtain $A_K(\cdot)$, $B_K(\cdot)$ and $C_K(\cdot)$ by inverting (28)
$$\begin{aligned}B_K(t) =& M(t)^{-1}\left(\hat{B}_K(t) - S(t)B(t)D_K(t)\right) \\ C_K(t) =& \left(\hat{C}_K(t) - D_K(t)C(t)Q(t)\right)N(t)^{-T} \\ A_K(t) =& M(t)^{-1}\Big(\hat{A}_K(t) - \dot{S}(t)Q(t) + \\ &- \dot{M}(t)N(t)^T - S(t)B(t)C_K(t)N(t)^T + \\ &- M(t)B_K(t)C(t)Q(t) + \\ &- S(t)\left(A_c(t) + B(t)D_K(t)C(t)\right)Q(t)\Big)N(t)^{-T}\end{aligned}$$
and $A_{d,K}(\cdot)$ and $B_{d,K}(\cdot)$ by inverting (29)
$$\begin{aligned}B_{d,K}(t_k) =& M(t_k^+)^{-1}\hat{B}_{d,K}(t_k) \\ A_{d,K}(t_k) =& M(t_k^+)^{-1}\Big(\hat{A}_{d,K}(t_k) + \\ &- \hat{B}_{d,K}(t_k)C(t_k)Q(t_k) + \\ &- S(t_k^+)A_d(t_k)Q(t_k)\Big)N(t_k)^{-T}\end{aligned}$$ ♢

When the state of system (21) is fully available, we can look for a state feedback finite-time stabilizing controller in the form $u(t) = K(t)x(t)$. In this case by pre- and post-multiplying (7a) by $P^{-1}(t) := Q(t)$, noticing that $\dot{Q}(t) = -Q(t)\dot{P}(t)Q(t)$, and letting $L(t) = K(t)Q(t)$, we readily obtain the following theorem.

Theorem 5 (State feedback for time-dependent IDLS): Problem 1 is solvable via state feedback control if there exist a piecewise continuously differentiable symmetric matrix-valued function $Q(\cdot)$ and a matrix-valued function $L(\cdot)$ such that

$$\begin{aligned}-\dot{Q}(t) + A_c(t)Q(t) + Q(t)A_c(t)^T + L(t)^T B(t)^T \\ + B(t)L(t) < 0\,, \quad t\in]0,T]\,, \quad t\notin\mathcal{T}\end{aligned} \tag{30a}$$

$$\begin{pmatrix} -Q(t_k^+) & A_d(t_k)Q(t_k) \\ Q(t_k)A_d(t_k)^T & -Q(t_k)\end{pmatrix} < 0\,, \quad t\in\mathcal{T} \tag{30b}$$

$$Q(t) \leq \Gamma^{-1}(t) \quad \forall\, t\in[0,T] \tag{30c}$$

$$Q(0) > R^{-1}\,. \tag{30d}$$

In this case a controller gain which solves Problem 1 via state feedback is $K(t) = L(t)Q^{-1}(t)$. ♢

Remark 7 (Feedback control of state-dependent IDLS): Since for state-dependent IDLS the resetting times are not know *a priori*, it follows that for this class of IDLS Problem 1 is solvable:

- via output feedback, if there exist matrix-valued functions $Q(\cdot)$, $S(\cdot)$, $N(\cdot)$, $\hat{A}_K(\cdot)$,$\hat{B}_K(\cdot)$, $\hat{C}_K(\cdot)$, $D_K(\cdot)$,

$\hat{A}_{d,K}(\cdot)$, and $\hat{B}_{d,K}(\cdot)$ such that conditions (24c) and (24d) hold, while both conditions (24a) and (24b) are satisfied for all t in $]0, T]$;

- via state feedback, if there exist two matrix-valued functions $Q(\cdot)$, $L(\cdot)$ such that conditions (30c) and (30d) hold, while conditions (30a) and (30b) are satisfied for all t in $]0, T]$. $\Diamond$

An example of controller design is now presented, so as to illustrate the effectiveness of the proposed procedure. Consider the second order time-dependent IDLS

$$A_c = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix},$$
$$D = 0, \quad A_d = \begin{pmatrix} 1.1 & 0 \\ 0 & 1.1 \end{pmatrix},$$

with $t_k = kT_s$, $T_s = 0.1\,s$. Theorem 4 is exploited to design an output feedback controller (22), of the same order of the system (21), which guarantees the FTS of the closed-loop time-dependent IDLS wrt $(T, blockdiag(R, R_K), blockdiag(\Gamma(\cdot), \Gamma_K(\cdot)))$, where

$$T = 1s, \quad \Gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 2.5 & 0 \\ 0 & 2.5 \end{pmatrix},$$
$$\Gamma_K = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad R_K = \begin{pmatrix} 2.5 & 0 \\ 0 & 2.5 \end{pmatrix}.$$

In order to recast the conditions provided in Theorem 4 in terms of LMIs, the matrix-valued functions $Q(\cdot)$, $S(\cdot)$, $N(\cdot)$, $\hat{A}_K(\cdot)$, $\hat{B}_K(\cdot)$, $\hat{C}_K(\cdot)$, $\hat{A}_{d,K}(\cdot)$ and $\hat{B}_{d,K}(\cdot)$ have been assumed piecewise affine, that is

$$Q(t) = \Omega_j^0 + \Omega_j^*(t - (j-1)T_s), \quad t \in [(j-1)T_s, jT_s], \quad j = 1, \dots, J+1.$$

where $J = \max\{j \in \mathbb{N} \ : j < T/T_s\}$, and Ω_j^0, Ω_j^* are the optimization variables. Exploiting the Matlab LMI toolbox [10], it is possible to find matrix functions $Q(\cdot)$, $S(\cdot)$, $N(\cdot)$, $\hat{A}_K(\cdot)$, $\hat{B}_K(\cdot)$, $\hat{C}_K(\cdot)$, $\hat{A}_{d,K}(\cdot)$ and $\hat{B}_{d,K}(\cdot)$ verifying the conditions of Theorem 4. Therefore, based on Remark 6, we can calculate the five matrix functions $A_K(\cdot)$, $B_K(\cdot)$ $A_{d,K}(\cdot)$, $B_{d,K}(\cdot)$ and $C_K(\cdot)$ and conclude that the closed loop system obtained by the interconnection of (21) and (22), is FTS wrt $(T, blockdiag(R, R_K), blockdiag(\Gamma(\cdot), \Gamma_K(\cdot)))$. As example, the evolution of the state of the plant, for $x_0 = \begin{pmatrix} 0.25 & 0 \end{pmatrix}^T$, and $x_{K_0} = \begin{pmatrix} 0 & 0 \end{pmatrix}^T$ is shown in Fig. 1, while the controller's state is shown in Fig. 2.

Conclusions

An extension of the finite-time stability concept to a class of impulsive dynamical linear systems has been presented in this paper. First a necessary and sufficient condition for FTS of this class of systems has been given. In order to check FTS for time-dependent and state-dependent IDLS by solving feasibility problems involving DLMIs, two further sufficient conditions have been presented. Sufficient condition for the existence and the design of both output and state feedback controller have been provided as well. The effectiveness of the results has been illustrated by a numerical example.

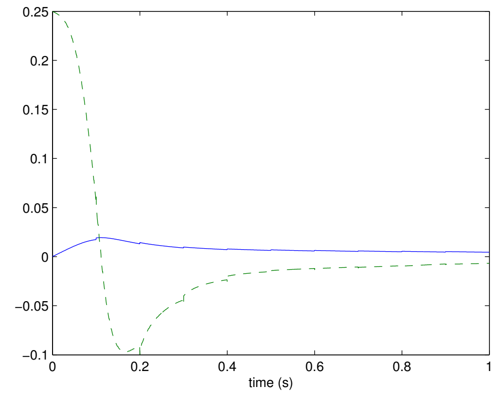

Fig. 1. Evolution of the state of system (21) subject to the output feedback controller. $x_1(t)$ (solid line) and $x_2(t)$ (dashed line).

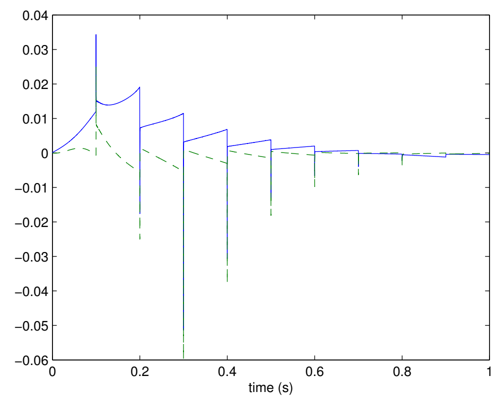

Fig. 2. Evolution of the state of the output feedback controller. $x_{K1}(t)$ (solid line) and $x_{K2}(t)$ (dashed line).

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