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Robust Generalized Asymptotic Regulation against Non-Stationary Sinusoidal Disturbances

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Abstract— **Attenuation of sinusoidal disturbances with uncertain and arbitrarily time-varying frequencies is considered in the form of a generalized asymptotic regulation problem. The disturbances are modeled as the outputs of a parameter-dependent unexcited exogenous system that evolves from nonzero initial conditions. The parameter dependence is assumed to be in such a form that the state of the exogenous system has constant norm at all times. Considering a partially parameter-dependent system, the problem is then formulated as the synthesis of a linear time-invariant controller with which the closed-loop respects a desired level of attenuation profile in steady-state and exhibits sufficiently fast transient response for all admissible parameter variations. The main result of the paper is a synthesis procedure based on a convex optimization problem, which is identified by a set of parameter-dependent linear matrix inequalities and can be rendered tractable through standard relaxation schemes. The order of the synthesized controller is equal to the order of the plant plus the order of the exogenous system.**

I. Introduction

Rejection of sinusoidal or periodic disturbances is a common problem in various engineering systems ranging from disk drives, [22], to CD players, [13], [4], helicopters, [2] and steel casting, [17]. Part of the recent interest concerning sinusoidal disturbance attenuation is on improving robustness against the variations in the frequencies and the periods of the disturbances [30], [15], [26], [20], [9], [27]. With the disturbances generated by a known autonomous and marginally stable exogenous system from unknown initial conditions, it is well-established in the theory of asymptotic regulation (see [21], [3]) when and how the sinusoidal disturbance rejection problems can be solved in a stationary setting. In this case, the solution essentially amounts to replicating in the feedback loop the dynamics of the exogenous system as required by the Internal Model Principle of [7]. The classical asymptotic regulation theory, however, does not offer an immediate solution when the disturbances have a non-stationary and uncertain nature. The classical focus has been on maintaining robustness against the uncertainty in the plant parameters and this is done by replicating as many times as required the dynamics of the exo-system generating the considered infinite-energy disturbances. The situation is rather different if the uncertainty is in the exo-system, since exact asymptotic regulation will be possible in such a case only under rather strong assumptions or -intuitively- with an infinite-dimensional controller. Another option that applies to a restrictive class of plants is a controller that is scheduled with the measurements of the uncertain parameters [10] if they are available at all.

A standard approach to non-stationary sinusoidal disturbance attenuation with a linear time-invariant controller would be based on $\mathscr{H}_\infty$ or $\mathscr{H}_2$ synthesis techniques in which the disturbances can be characterized as the outputs of stable band-pass filters that emphasize the frequency range of interest and that are excited by impulsive or white noise inputs. This is admittedly an ad-hoc approach and hence does not provide theoretical guarantees for situations in which the frequencies can change with time arbitrarily. Alternatively, one might consider designing a robust controller by first modeling the disturbances as the outputs of a critically-damped yet stable filter, which depends on uncertain parameters, the ranges of which reflect the frequency intervals of interest. Such an approach is presented in [5], where a convex solution is provided to robust controller synthesis for uncertain disturbance filters. Although this approach is recently extended to exploit any available bounds on the rates-of-variation of uncertain parameters [6], there are inherent limitations since the filters are required to be stable and the initial conditions are assumed to be zero. As a matter of fact, the usual $\mathscr{H}_\infty$ or $\mathscr{H}_2$ synthesis framework with stable disturbance filters is not an ideal setting for this problem. Although unstable weighting filters have already been considered in nominal synthesis (see e.g. [19]), the available approaches are appropriate for exact cancelation of disturbances rather than for attenuation thereof. On the other hand, a natural extension of the classical regulation theory in which a nonzero bound is required on the steady-state peak of the output is possible for sinusoidal disturbances. Inspired by [8], such a generalization has been considered in [11] on top of an $\mathscr{H}_\infty$ synthesis problem, where a structure is identified for the candidate controllers similarly to the previous works on exact regulation with additional performance objectives [25], [28]. The generalized version of the asymptotic regulation problem provides a convenient framework for handling uncertainty in the exo-system, since it is solvable with a finite-dimensional controller. In fact, robust controller synthesis has already been considered as a generalized asymptotic regulation problem in [12], where a

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potentially conservative solution is derived by adopting the generic controller structure derived for the nominal version of the problem. We also note the previous work by [14], which is based on replicating the nominal exo-system in the loop and guaranteeing a robust $\mathscr{H}_\infty$ norm constraint to decrease sensitivity to the variations in the frequencies.

In this paper, we study the non-stationary sinusoidal disturbance attenuation problem in the spirit of the classical regulation theory with the help of a marginally stable exo-system. The exo-system depends on uncertain and possibly time-varying parameters, which correspond to the variations in the frequencies of the sinusoidal disturbances. With inspirations from [8], we impose requirements on the steady-state disturbance attenuation level and the transient response, as stated precisely in Section II. The main result of the paper is a linear time-invariant controller synthesis procedure based on a convex optimization problem derived in Section III. The optimization problem is not per se tractable since it has infinitely many constraints to be satisfied, as is usually the case for robust control problems. Nevertheless, standard relaxation techniques allow one to render the problem tractable by replacing the semi-infinite constrains with a set of linear matrix inequalities (LMI) and design robust controllers as illustrated for a mass-spring damper system in Section IV. The problem considered and the solution proposed in this paper offer further opportunities as addressed briefly in the concluding remarks.

II. Problem Formulation

This paper is concerned with the attenuation of multi-sinusoidal disturbances with uncertain and possibly time-varying frequencies. These disturbances are viewed as the outputs of an unexcited system of the form

$$\dot{v} = A_{\rm e}(\delta)v;\ A_{\rm e}(\delta) = -A_{\rm e}(\delta)^T \in \mathbb{R}^{l\times l}, \tag{1}$$

where $\delta = [\,\delta_1 \ \cdots \ \delta_\eta\,]^T$ represents a vector of uncertain and possibly time-varying parameters and the state evolves from a nonzero initial condition $v(0)$. As a simple yet sufficiently representative example, let us consider

$$A_{\rm e} = \begin{bmatrix} 0 & -\varpi(t) \\ \varpi(t) & 0 \end{bmatrix}, \varpi(t) = (1+\delta(t))\omega_0, \tag{2}$$

where $\omega_0 \geq 0$ corresponds to a nominal frequency. With

$$\phi(t) = \int_0^t \varpi(\tau)d\tau = \omega_0 t + \omega_0 \int_0^t \delta(\tau)d\tau, \tag{3}$$

it is straightforward to verify for this example that

$$\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} \cos(\phi(t)) & -\sin(\phi(t)) \\ \sin(\phi(t)) & \cos(\phi(t)) \end{bmatrix} \begin{bmatrix} v_1(0) \\ v_2(0) \end{bmatrix}, \tag{4}$$

which reveals the motivations behind viewing the systems described by (1) as the generators of non-stationary sinusoidal disturbances. Systems that generate multi-sinusoidal disturbances can be obtained -for instance- by using block-diagonal system matrices with sub-blocks of the form given in (2). In fact, a larger class of disturbances can be considered with exo-systems for which there exists a positive-definite matrix P such that $A_{\rm e}^T(\delta)P + PA_{\rm e}(\delta) = 0$. There is, though, no need for formulating the problem for this general class of exo-systems as they can easily be subsumed to our framework through the state transformation $\nu = P^{1/2}v$, since $P^{1/2}A_{\rm e}(\delta)P^{-1/2}$ is then skew-symmetric.

The uncertain parameters in our setting basically reflect the deviations of the frequencies from their nominal values and are assumed to vary in time in a compact region $\mathscr{R} \subset \mathbb{R}^\eta$. The admissible parameter trajectories are hence identified as $\mathscr{T}_{\mathscr{R}} \triangleq \{\delta(\cdot) : [0,\infty) \to \mathscr{R}\}$. Note that, irrespective of the parameter trajectory, the state of the system in (1) evolves with a constant norm, i.e.

$$\|v(t)\|^2 \triangleq v(t)^T v(t) = \|v(0)\|^2, \forall t \geq 0. \tag{5}$$

This can easily be established as $d\|v(t)\|^2/dt = v(t)^T \mathfrak{He}(A_{\rm e}(\delta(t)))v(t) = 0$, where $\mathfrak{He} A_{\rm e} \triangleq A_{\rm e} + A_{\rm e}^T$. This is a property which we particularly rely on in the problem formulation.

The disturbance attenuation problem is formulated for a plant with dynamics

$$G: \left[\begin{array}{c} \dot{x} \\ \hline e \\ y \end{array}\right] = \left[\begin{array}{c|cc} A & B_{\rm r}(\delta) & B \\ \hline C_{\rm r}(\delta) & D_{\rm r}(\delta) & D_{\rm rc}(\delta) \\ C & D_{\rm cr} & 0 \end{array}\right] \left[\begin{array}{c} x \\ \hline v \\ u \end{array}\right], \tag{6}$$

where $x(t) \in \mathbb{R}^k$ denotes the state vector, while $u(t) \in \mathbb{R}^n$ is the vector of control inputs that are to be used to regulate the outputs $e(t) \in \mathbb{R}^r$ based on the measurements $y(t) \in \mathbb{R}^m$. The dynamics of the exo-system can be appended to the plant to obtain the dynamics of the extended plant as

$$\begin{aligned} \dot{\tilde{x}} &= \underbrace{\begin{bmatrix} A & B_{\rm r}(\delta) \\ 0 & A_{\rm e}(\delta) \end{bmatrix}}_{\tilde{A}(\delta)} \underbrace{\begin{bmatrix} x \\ v \end{bmatrix}}_{\tilde{x}} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\tilde{B}} u, \\ e &= \underbrace{\begin{bmatrix} C_{\rm r}(\delta) & D_{\rm r}(\delta) \end{bmatrix}}_{\tilde{C}_{\rm r}(\delta)} \tilde{x} + D_{\rm rc}(\delta)u, \\ y &= \underbrace{\begin{bmatrix} C & D_{\rm cr} \end{bmatrix}}_{\tilde{C}} \tilde{x}. \end{aligned} \tag{7}$$

With an LTI controller of the form

$$K: \left[\begin{array}{c} \dot{\xi} \\ \hline u \end{array}\right] = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] \left[\begin{array}{c} \xi \\ \hline y \end{array}\right], \tag{8}$$

the closed-loop dynamics are described by

$$\begin{aligned} \dot{\varkappa} &= \underbrace{\begin{bmatrix} A + BD_KC & BC_K \\ B_KC & A_K \end{bmatrix}}_{\mathscr{A}} \underbrace{\begin{bmatrix} x \\ \xi \end{bmatrix}}_{\varkappa} + \underbrace{\begin{bmatrix} B_{\rm r}(\delta) + BD_KD_{\rm cr} \\ B_KD_{\rm cr} \end{bmatrix}}_{\mathscr{B}_{\rm r}(\delta)} v, \\ e &= \underbrace{\begin{bmatrix} C_{\rm r}(\delta) + D_{\rm rc}(\delta)D_KC & D_{\rm rc}(\delta)C_K \end{bmatrix}}_{\mathscr{C}_{\rm r}(\delta)} \varkappa \\ &\quad + \underbrace{\begin{bmatrix} D_{\rm r}(\delta) + D_{\rm rc}(\delta)D_KD_{\rm cr} \end{bmatrix}}_{\mathscr{D}_{\rm r}(\delta)} v. \end{aligned} \tag{9}$$

The complete description of the closed-loop system can be obtained by addition of the dynamics of the exo-system as

$$\begin{aligned}\dot{\tilde{\varkappa}} &= \underbrace{\begin{bmatrix} \tilde{A}(\delta)+\tilde{B}D_K\tilde{C} & \tilde{B}C_K \\ B_K\tilde{C} & A_K \end{bmatrix}}_{\tilde{\mathscr{A}}(\delta)} \underbrace{\begin{bmatrix} \tilde{x} \\ \xi \end{bmatrix}}_{\tilde{\varkappa}}, \\ e &= \underbrace{\begin{bmatrix} \tilde{C}_{\mathrm{r}}(\delta)+D_{\mathrm{rc}}(\delta)D_K\tilde{C} & D_{\mathrm{rc}}(\delta)C_K \end{bmatrix}}_{\tilde{\mathscr{C}}_{\mathrm{r}}(\delta)}\tilde{\varkappa}.\end{aligned} \tag{10}$$

Within this setup, the problem that we consider is formulated as follows: Given the plant in (6), the exo-system in (1) and the uncertainty region $\mathscr{R}$ that satisfy:

A.1 $\mathscr{R}$ is compact and $\tilde{B}_{\mathrm{r}} \triangleq \begin{bmatrix} B_{\mathrm{r}} \\ A_{\mathrm{e}} \end{bmatrix}$ and C_{r}, $D_{\mathrm{r}}, D_{\mathrm{rc}}$ depend continuously on δ;

A.2 (A,B) is stabilizable ($\exists F: A+BF$ is Hurwitz);

A.3 (C,A) is detectable ($\exists L: A+LC$ is Hurwitz);

design a linear time-invariant controller K such that:

C.1. (**Internal Stability**) The feedback system formed by G and K is exponentially stable (i.e. $\mathscr{A}$ is Hurwitz);

C.2. (**Robust Generalized Asymptotic Regulation with a Decay Rate $\rho \in \mathbb{R}_+$ and an attenuation profile $\kappa: \mathscr{R} \to \mathbb{R}_+$**) There exists a $\varphi \in \mathbb{R}_+$ such that $\|e(t)\|^2 \leq \varphi\|\tilde{\varkappa}(0)\|^2 e^{-2\rho t} + (\kappa(\delta(t)))^2\|v(0)\|^2, \forall t \geq 0$, $\forall \delta(\cdot) \in \mathscr{T}_{\mathscr{R}}$.

Remark 1: The notion of generalized asymptotic regulation adopted in this paper is inspired by the recent work [8]. Although a standard extension from exact to almost regulation would not require the introduction of the decay-rate (since the regulation theory is primarily concerned with the steady-state behavior), we have done otherwise since the transient behavior becomes especially important in a non-stationary setting. The choice of the attenuation level $\kappa(\cdot)$ as parameter-dependent is to offer flexibility in design. If a fixed level of attenuation is desired for all parameter variations, one can simply set $\kappa = \kappa_0$. Nevertheless, it is quite intuitive to think of a (graceful) attenuation level degradation profile by considering a non-negative and non-decreasing function $\vartheta(\|\delta\|)$ with $\vartheta(0) = 0$ (e.g. $\vartheta(\|\delta\|) = \kappa_1\|\delta\|$ or $\vartheta(\|\delta\|) = \kappa_1\|\delta\|^2$, where $\kappa_1 > 0$) and try to achieve generalized asymptotic regulation down to levels $\kappa(\delta(t)) = \kappa_0 + \vartheta(\|\delta(t)\|)$. Since δ in our setting reflects the deviation of the parameters from their nominal values, such a profile corresponds to requiring κ_0-level attenuation for the nominal case and allowing for an increase in the attenuation level proportional to the amount of deviation from the nominal values of the parameters.

III. Robust Generalized Asymptotic Regulation

In this section, we develop a convex optimization based procedure for the synthesis of an LTI controller for robust generalized asymptotic regulation. The optimization problem is derived using the following matrix inequality condition for generalized asymptotic regulation, adapted from [8]:

Lemma 1: There exists a solution to the generalized asymptotic regulation problem formulated in Section II, if there exists a symmetric matrix $\tilde{\mathscr{X}} = \tilde{\mathscr{X}}^T$ that satisfies

$$\tilde{\mathscr{L}}_{\mathrm{s}}(\delta) = \mathfrak{He}\left(\tilde{\mathscr{X}}\tilde{\mathscr{A}}(\delta) + \rho\tilde{\mathscr{X}}\right) \preccurlyeq 0, \forall\delta \in \mathscr{R}, \tag{11}$$

$$\tilde{\mathscr{L}}_{\mathrm{r}}(\delta) = \begin{bmatrix} \tilde{\mathscr{X}} + \kappa(\delta)\tilde{\mathscr{I}}^T\tilde{\mathscr{I}} & \tilde{\mathscr{C}}_{\mathrm{r}}(\delta)^T \\ \tilde{\mathscr{C}}_{\mathrm{r}}(\delta) & \kappa(\delta)I \end{bmatrix} \succ 0, \forall\delta \in \mathscr{R}, \tag{12}$$

where $\tilde{\mathscr{I}} \triangleq \begin{bmatrix} \tilde{E} & 0 \end{bmatrix}$, with $\tilde{E} \triangleq \begin{bmatrix} 0_{l\times k} & I_l \end{bmatrix}$.

Proof: We first infer from (11) that the function $\mathscr{V}(\tilde{\varkappa}) \triangleq \tilde{\varkappa}^T\tilde{\mathscr{X}}\tilde{\varkappa}$ satisfies

$$d\left(e^{2\rho t}\mathscr{V}(\tilde{\varkappa}(t))\right)/dt = \tilde{\varkappa}(t)^T\tilde{\mathscr{L}}_{\mathrm{s}}(\delta(t))\tilde{\varkappa}(t) \leq 0,$$

along the trajectories of the extended closed-loop system for any admissible parameter trajectory. It hence decays with time according to

$$\mathscr{V}(\tilde{\varkappa}(t)) \leq \mathscr{V}(\tilde{\varkappa}(0))e^{-2\rho t} \leq \mu\|\tilde{\varkappa}(0)\|^2 e^{-2\rho t}, \forall t \geq 0,$$

where μ represents the maximum eigenvalue of $\tilde{\mathscr{X}}$. On the other hand, (12) implies the existence of a positive scalar ε for which $\tilde{\mathscr{X}} + \kappa(\delta)\tilde{\mathscr{I}}^T\tilde{\mathscr{I}} \succcurlyeq \varepsilon I, \forall\delta \in \mathscr{R}$. Hence, when $v(0) = 0$, we have $v(t) = 0$ as well as $\tilde{\mathscr{I}}\tilde{\varkappa}(t) = 0$, as a result of which the state of the closed-loop system obeys

$$\|\varkappa(t)\|^2 \leq \varepsilon^{-1}\mathscr{V}(\tilde{\varkappa}(t)) \leq \varepsilon^{-1}\mu\|\tilde{\varkappa}(0)\|^2 e^{-2\rho t}.$$

With the exponential stability thus being established, we now consider the system for general $v(0)$ and infer from (12) with $\psi(t)^T = \begin{bmatrix} \tilde{\varkappa}(t)^T & -(\kappa(\delta(t)))^{-1}e(t)^T \end{bmatrix}$ that

$$\begin{aligned}&\psi(t)^T\tilde{\mathscr{L}}_{\mathrm{r}}(\delta(t))\psi(t) \\ &= \mathscr{V}(\tilde{\varkappa}(t)) + \kappa(\delta(t))\|v(t)\|^2 - (\kappa(\delta(t)))^{-1}\|e(t)\|^2 \geq 0,\end{aligned}$$

which guarantees the generalized asymptotic regulation condition in C.2 with $\varphi = \mu\max_{\delta\in\mathscr{R}}\kappa(\delta)$ since $\|v(t)\| = \|v(0)\|$. ■

The solution to the problem considered in this paper is derived by first expressing conditions (11) and (12) equivalently as

$$\mathfrak{He}\left(\tilde{\mathscr{Y}}^T\tilde{\mathscr{X}}\tilde{\mathscr{A}}(\delta)\tilde{\mathscr{Y}} + \rho\tilde{\mathscr{Y}}^T\tilde{\mathscr{X}}\tilde{\mathscr{Y}}\right) \preccurlyeq 0, \forall\delta \in \mathscr{R}, \tag{13}$$

$$\begin{bmatrix} \tilde{\mathscr{Y}}^T\left(\tilde{\mathscr{X}} + \kappa(\delta)\tilde{\mathscr{I}}^T\tilde{\mathscr{I}}\right)\tilde{\mathscr{Y}} & \tilde{\mathscr{Y}}^T\tilde{\mathscr{C}}_{\mathrm{r}}(\delta)^T \\ \tilde{\mathscr{C}}_{\mathrm{r}}(\delta)\tilde{\mathscr{Y}} & \kappa(\delta)I \end{bmatrix} \succ 0, \forall\delta \in \mathscr{R}, \tag{14}$$

where $\mathscr{Y}$ is an invertible matrix, the choice of which forms the key step. In fact, the LMI approach to multi-objective controller synthesis for various performance objectives is based on suitable congruence transformations performed by a generic choice of $\tilde{\mathscr{Y}}$ in terms of certain sub-blocks of $\tilde{\mathscr{X}}$ and $\tilde{\mathscr{X}}^{-1}$ [18], [25]. The equivalent matrix inequality conditions obtained in this fashion are then rendered affine in all free matrix variables through a bijective transformation of the controller realization matrices into a set of new matrix variables. The standard approach of [18], [25], however, cannot lead us to a convex solution even in the nominal version of our problem (i.e. for $\delta = 0$), mainly due to the term $\kappa(\delta)\tilde{\mathscr{Y}}^T\tilde{\mathscr{I}}^T\tilde{\mathscr{I}}\tilde{\mathscr{Y}}$ in (14). As a matter of fact, there is an alternative approach that one can employ to solve the nominal generalized asymptotic regulation problem. This approach relies on identifying a generic structure for the candidate controllers in terms of a replica of the exo-system, two matrix variables that are required to satisfy an affine equality constraint as well as an LMI, and an accompanying controller which is to be designed to stabilize the overall system [11]. In fact, a solution has been obtained for the robust generalized asymptotic regulation problem in [12] by using a controller of this structure and imposing requirements on the involved variables that facilitate a tractable solution. Admittedly, this approach is potentially conservative and hence does not provide an ideal solution.

In the sequel, we first combine the transformations from [25], [23] similarly to [5], in order to obtain a suitable choice for $\tilde{\mathscr{Y}}$ and then introduce a new variable transformation that paves the way to a convex solution for our problem. Let us first express the realization matrices of the extended closed-loop in terms of the controller parameters as

$$\left[\begin{array}{c}\tilde{\mathscr{A}}(\delta)\\ \hline \tilde{\mathscr{C}}_{\mathrm{r}}(\delta)\end{array}\right]=\left[\begin{array}{cc}\tilde{A}(\delta) & 0\\ 0 & 0\\ \hline \tilde{C}_{\mathrm{r}}(\delta) & 0\end{array}\right]+\left[\begin{array}{cc}0 & \tilde{B}\\ I & 0\\ \hline 0 & D_{\mathrm{rc}}(\delta)\end{array}\right]\left[\begin{array}{cc}A_K & B_K\\ C_K & D_K\end{array}\right]\left[\begin{array}{cc}0 & I\\ \tilde{C} & 0\end{array}\right], \tag{15}$$

and partition the matrix $\tilde{\mathscr{X}}$ and -assuming that it exists- its inverse compatibly with this realization as

$$\tilde{\mathscr{X}} = \left[\begin{array}{cc}\tilde{X} & \tilde{U}\\ \tilde{U}^T & \tilde{H}\end{array}\right], \tilde{\mathscr{X}}^{-1}=\left[\begin{array}{cc}\tilde{Y} & \tilde{V}\\ \tilde{V}^T & \tilde{S}\end{array}\right] \tag{16}$$

We next introduce an invertible matrix $\tilde{T}$ as

$$\tilde{T}=\left[\begin{array}{c}\tilde{\Pi}\\ \hline P\tilde{E}\end{array}\right]=\left[\begin{array}{cc}I & \Pi\\ \hline 0 & P\end{array}\right], \tilde{T}^{-1}=\left[\tilde{I}^T \,\middle|\, \tilde{\Pi}_{\perp}^T P^{-1}\right]=\left[\begin{array}{c|c}I & -\Pi P^{-1}\\ 0 & P^{-1}\end{array}\right], \tag{17}$$

as partitioned compatibly with $\tilde{A}$ (see (7)), and assume that the matrix $\tilde{Y}$ is of the form

$$\tilde{Y}=\underbrace{\left[\begin{array}{cc}I & \Pi\\ 0 & P\end{array}\right]^{-1}}_{\tilde{T}^{-1}}\underbrace{\left[\begin{array}{cc}Y & 0\\ 0 & P\end{array}\right]}_{\hat{Y}}\underbrace{\left[\begin{array}{cc}I & \Pi\\ 0 & P\end{array}\right]^{-T}}_{\tilde{T}^{-T}}. \tag{18}$$

Note that there is no loss of generality in assuming $\tilde{Y}$ to be of this form, provided that $\tilde{Y}_{22}$ is invertible, since we can then simply obtain the matrices in (18) as $P=\tilde{Y}_{22}^{-1}$, $\Pi=-\tilde{Y}_{12}\tilde{Y}_{22}^{-1}$ and $Y=\tilde{Y}_{11}-\tilde{Y}_{12}\tilde{Y}_{22}^{-1}\tilde{Y}_{12}^T$, with $\tilde{Y}_{ij}$'s representing the corresponding sub-blocks of $\tilde{Y}$. Our solution to the problem is based on the choice of $\tilde{\mathscr{Y}}$ as

$$\tilde{\mathscr{Y}}=\left[\begin{array}{cc}\tilde{Y}\tilde{T}^T & I\\ \tilde{V}^T\tilde{T}^T & 0\end{array}\right], \tag{19}$$

with which we obtain by exploiting $\tilde{\mathscr{X}}\tilde{\mathscr{X}}^{-1}=I$ that

$$\tilde{\mathscr{Y}}^T\tilde{\mathscr{X}}=\left[\begin{array}{cc}\tilde{T} & 0\\ \tilde{X} & \tilde{U}\end{array}\right], \tilde{\mathscr{Y}}^T\tilde{\mathscr{X}}\tilde{\mathscr{Y}}=\left[\begin{array}{cc}\hat{Y} & \tilde{T}\\ \tilde{T}^T & \tilde{X}\end{array}\right], \tag{20}$$

$$\tilde{\mathscr{Y}}^T\tilde{\mathscr{I}}^T=\left[\begin{array}{c}\tilde{E}^T\\ \tilde{E}^T\end{array}\right], \tilde{\mathscr{Y}}^T\tilde{\mathscr{I}}^T\tilde{\mathscr{I}}\tilde{\mathscr{Y}}=\left[\begin{array}{cc}\tilde{E}^T\tilde{E} & \tilde{E}^T\tilde{E}\\ \tilde{E}^T\tilde{E} & \tilde{E}^T\tilde{E}\end{array}\right]. \tag{21}$$

Standard manipulations based on these expressions lead to

$$\left[\begin{array}{c}\tilde{\mathscr{Y}}^T\tilde{\mathscr{X}}\tilde{\mathscr{A}}(\delta)\tilde{\mathscr{Y}}\\ \hline \tilde{\mathscr{C}}_{\mathrm{r}}(\delta)\tilde{\mathscr{Y}}\end{array}\right]=\left[\begin{array}{cc}\tilde{T}\tilde{A}(\delta)\tilde{Y}\tilde{T}^T & \tilde{T}\tilde{A}(\delta)\\ \tilde{X}\tilde{A}(\delta)\tilde{Y}\tilde{T}^T & \tilde{X}\tilde{A}(\delta)\\ \hline \tilde{C}_{\mathrm{r}}(\delta)\tilde{Y}\tilde{T}^T & \tilde{C}_{\mathrm{r}}(\delta)\end{array}\right]$$
$$+\left[\begin{array}{cc}0 & \tilde{T}\tilde{B}\\ \tilde{U} & \tilde{X}\tilde{B}\\ \hline 0 & D_{\mathrm{rc}}(\delta)\end{array}\right]\left[\begin{array}{cc}A_K & B_K\\ C_K & D_K\end{array}\right]\left[\begin{array}{cc}\tilde{V}^T\tilde{T}^T & 0\\ \tilde{C}\tilde{Y}\tilde{T}^T & \tilde{C}\end{array}\right]. \tag{22}$$

The first important observation at this point is that the term $\tilde{T}\tilde{A}(\delta)\tilde{Y}\tilde{T}^T$ is affine in the matrix variables Y, P and Π, as is visible from

$$\tilde{T}\tilde{A}(\delta)\tilde{Y}\tilde{T}^T=\left[\begin{array}{cc}AY & \tilde{\Pi}\tilde{B}_{\mathrm{r}}(\delta)-A\Pi\\ 0 & PA_{\mathrm{e}}(\delta)\end{array}\right]. \tag{23}$$

The remaining bilinear term $\tilde{X}\tilde{A}(\delta)\tilde{Y}\tilde{T}^T$ can in fact be absorbed into the transformed controller parameters in a similar way to the usual technique, thanks to the fact that $\tilde{T}\tilde{B}=\tilde{B}$. A standard adaptation, however, will lead to a parameter dependence in the controller to be synthesized due to the parameter dependence of $\tilde{A}$. The second crucial observation is that this problem can in fact be avoided by decomposing $\tilde{X}\tilde{A}(\delta)\tilde{Y}\tilde{T}^T$ into two parts identified as

$$\tilde{X}\tilde{A}(\delta)\tilde{Y}\tilde{T}^T=\left[\begin{array}{cc}\tilde{X}\tilde{I}^T AY & -\tilde{X}\tilde{I}^T A\Pi\end{array}\right]+\left[\begin{array}{cc}0 & \tilde{X}\tilde{B}_{\mathrm{r}}(\delta)\end{array}\right]. \tag{24}$$

Absorbing the parameter-independent part into the transformed controller parameters as

$$\left[\begin{array}{ccc}\tilde{J} & -\tilde{\Psi} & \tilde{M}\\ N & -\Gamma & D\end{array}\right]\triangleq\left[\begin{array}{ccc}\tilde{X}\tilde{I}^T AY & -\tilde{X}\tilde{I}^T A\Pi & 0\\ 0 & 0 & 0\end{array}\right]$$
$$+\left[\begin{array}{cc}\tilde{U} & \tilde{X}\tilde{B}\\ 0 & I\end{array}\right]\left[\begin{array}{cc}A_K & B_K\\ C_K & D_K\end{array}\right]\left[\begin{array}{ccc}\tilde{V}^T\tilde{\Pi}^T & \tilde{V}^T\tilde{E}^T P & 0\\ \tilde{C}\tilde{Y}\tilde{\Pi}^T & \tilde{C}\tilde{Y}\tilde{E}^T P & I\end{array}\right], \tag{25}$$

we arrive at the following solution for the robust disturbance attenuation problem considered in this paper:

Theorem 1: There exists an LTI controller that solves the robust generalized asymptotic regulation problem formulated in Section II, if there exist $Y=Y^T\in\mathbb{R}^{k\times k}$, $\tilde{X}=\tilde{X}^T\in\mathbb{R}^{(k+l)\times(k+l)}$, $P=P^T\in\mathbb{R}^{l\times l}$, $\Pi\in\mathbb{R}^{k\times l}$, $\Gamma\in\mathbb{R}^{n\times l}$, $\tilde{J}\in\mathbb{R}^{(k+l)\times k}$, $\tilde{\Psi}\in\mathbb{R}^{(k+l)\times l}$, $\tilde{M}\in\mathbb{R}^{(k+l)\times m}$, $N\in\mathbb{R}^{n\times k}$ and $D\in\mathbb{R}^{n\times m}$ such that, for all $\delta\in\mathscr{R}$, we have

$$\mathfrak{He}\left[\begin{array}{ccc}AY+BN+\rho Y & \tilde{\Pi}\tilde{A}(\delta)+\rho\tilde{\Pi}+BD\tilde{C} & \tilde{\Pi}\tilde{B}_{\mathrm{r}}(\delta)-A\Pi-B\Gamma\\ \tilde{J}+\rho\tilde{\Pi}^T & \tilde{X}\tilde{A}(\delta)+\tilde{M}\tilde{C}+\rho\tilde{X} & \tilde{X}\tilde{B}_{\mathrm{r}}(\delta)-\tilde{\Psi}+\rho\tilde{E}^T P\\ 0 & PA_{\mathrm{e}}(\delta)\tilde{E}+\rho P\tilde{E} & PA_{\mathrm{e}}(\delta)+\rho P\end{array}\right]\preccurlyeq 0, \tag{26}$$

$$\left[\begin{array}{cccc}Y & \# & \# & \#\\ \tilde{\Pi}^T & \tilde{X}+\kappa(\delta)\tilde{E}^T\tilde{E} & \# & \#\\ 0 & P\tilde{E}+\kappa(\delta)\tilde{E} & P+\kappa(\delta)I & \#\\ C_{\mathrm{r}}(\delta)Y+D_{\mathrm{rc}}(\delta)N & \tilde{C}_{\mathrm{r}}(\delta)+D_{\mathrm{rc}}(\delta)D\tilde{C} & \Lambda(\Pi,\Gamma,\delta) & \kappa(\delta)I\end{array}\right]\succ 0, \tag{27}$$

where $\tilde{E}\triangleq[0\; I_l]$, $\tilde{\Pi}\triangleq[I\;\Pi]$, $\Lambda(\Pi,\Gamma,\delta)\triangleq D_{\mathrm{r}}(\delta)-C_{\mathrm{r}}(\delta)\Pi-D_{\mathrm{rc}}(\delta)\Gamma$ and #'s represent the terms that can be identified from symmetry. With $\tilde{I}^T$ and $\tilde{T}$ obtained as in (17) and $\tilde{W}=I-\tilde{X}\tilde{Y}$, a controller that solves the problem can then be constructed as

$$\left[\begin{array}{c|c}A_K & B_K\\ \hline C_K & D_K\end{array}\right]=\left[\begin{array}{cc}I & -\tilde{X}\tilde{B}\\ \hline 0 & I\end{array}\right]\left[\begin{array}{ccc}\tilde{J}-\tilde{X}\tilde{I}^T AY & \tilde{X}\tilde{I}^T A\Pi-\tilde{\Psi} & \tilde{M}\\ N & -\Gamma & D\end{array}\right]\left[\begin{array}{c|c}\tilde{T}^{-T}\tilde{W}^{-1} & 0\\ -\tilde{C}\tilde{Y}\tilde{W}^{-1} & I\end{array}\right]. \tag{28}$$

Proof: We first note that (26)-(27) are the inequalities obtained by expressing (13)-(14) in terms of the transformed controller parameters and interchanging certain row/column blocks. In order to finalize the proof, we only need to construct a positive-definite $\tilde{\mathscr{X}}$ and an invertible $\tilde{\mathscr{Y}}$, using the matrix variables that satisfy (26)-(27). For this, it suffices to choose an invertible $\tilde{U}$ and set $\tilde{V}^T=\tilde{U}^{-1}(I-\tilde{X}\tilde{Y})$. We then obtain $\tilde{S}=-\tilde{U}^{-1}\tilde{X}(I-\tilde{X}\tilde{Y})U^{-T}$, in terms of which $\tilde{H}=\tilde{S}^{-1}+\tilde{U}^T\tilde{X}^{-1}\tilde{U}$ becomes the suitable choice with which

$\mathscr{X}$ is to be constructed. Note that in this case $\mathscr{Y}^{-1}$ can be obtained explicitly as

$$\mathscr{Y}^{-1} = \begin{bmatrix} 0 & \tilde{T}^{-1}\tilde{V}^{-T} \\ I & -\tilde{Y}\tilde{V}^{-T} \end{bmatrix}.$$

A realization of K can then be obtained from the inverse of the transformation in (25), which reads for $\tilde{U} = I$ as given by (28). Different choices for $\tilde{U}$ in fact lead to different realizations of the same controller. We finally note that if any of P, $\tilde{X}$ or $I - \tilde{X}\tilde{Y}$ is not invertible, one can render them nonsingular by introducing slight perturbations that will not lead to the violation of the LMIs (together with a possible change in the value of ρ). ■

Remark 2: The necessary and sufficient conditions for the solvability of the nominal generalized asymptotic regulation problem (i.e. for $\delta = 0$) is obtained in [11] as the existence of Π and Γ for which $\Pi\tilde{B}_{\mathrm{r}}(0) - A\Pi - B\Gamma = 0$ and $\|\Lambda(\Pi,\Gamma,0)\| < \kappa(0)$. We briefly note that, when these conditions are satisfied, one can always find a solution to (26)-(27). More clearly, we can then choose $\tilde{J}$ and $\tilde{\Psi}$ to render the blocks they appear as zero. This allows us to select P arbitrarily close to zero and magnify any $\tilde{X}$ that satisfies (26) as $\tilde{X} \to \alpha\tilde{X}$ with a sufficiently large $\alpha > 0$ to facilitate the solution of (27). We also note that, the result of Theorem 1 is somewhat more preferable for the solution of the nominal problem since it provides LMI conditions without any equality constraints (cf. [11]).

Remark 3: For a multi-objective version of the problem in which n_{r} outputs e_i are to be regulated according to the profiles $\kappa_i(\cdot)$, Theorem 1 admits an immediate extension in which we have (26) accompanied by n_{r} constraints of the form (27) expressed in terms of the system matrices and $\kappa_i(\cdot)$ corresponding to the relevant outputs to be regulated. As an example, we consider here imposing a generalized regulation constraint on the control input u with a constant profile σ. With the expression $u = 0\cdot x + 0\cdot v + Iu$, we can easily adapt (12) to obtain

$$\begin{bmatrix} Y & \tilde{\Pi} & 0 & N^T \\ \tilde{\Pi}^T & \tilde{X} + \sigma\tilde{E}^T\tilde{E} & \tilde{E}^T P + \sigma\tilde{E}^T & \tilde{C}^T D^T \\ 0 & P\tilde{E} + \sigma\tilde{E} & P + \sigma I & -\Gamma^T \\ N & D\tilde{C} & -\Gamma & \sigma I \end{bmatrix} \succ 0, \qquad (29)$$

as the constraint to be added to (26) and (27) to also guarantee that the peak of the control input stays below σ in steady state. In fact, for a given regulation profile $\kappa(\cdot)$ that is known to be achievable, it would be quite reasonable to obtain a suitable controller by minimizing σ, so that the control effort is also kept as small as possible in the sense of steady-state peak minimization.

Remark 4: Conditions (26) and (27) read as infinitely many matrix inequalities and hence are not per se tractable. There are a variety of relaxations that can be applied to replace them with finitely many conditions (see [24] and the references therein). As far as the sinusoidal disturbance attenuation problem is concerned, the uncertainty can simply be described in the form of affine parameter dependence (i.e. $\tilde{B}_{\mathrm{r}}^1(\delta) = \tilde{B}_{\mathrm{r}}^1(\delta \otimes I)$ etc.). If, moreover, $\mathscr{R}$ is assumed to be a polytotic region identified by a set of vertices $\{\delta^1,\ldots,\delta^q\}$ (i.e. $\mathscr{R} = \{\sum_{j=1}^q \alpha_j\delta^j : \sum_{j=1}^q \alpha_j = 1, \alpha_j \geq 0\}$), conditions (26) and (27) are satisfied throughout $\mathscr{R}$ if and only if they are satisfied for each $\delta^j, j = 1,\ldots,q$.

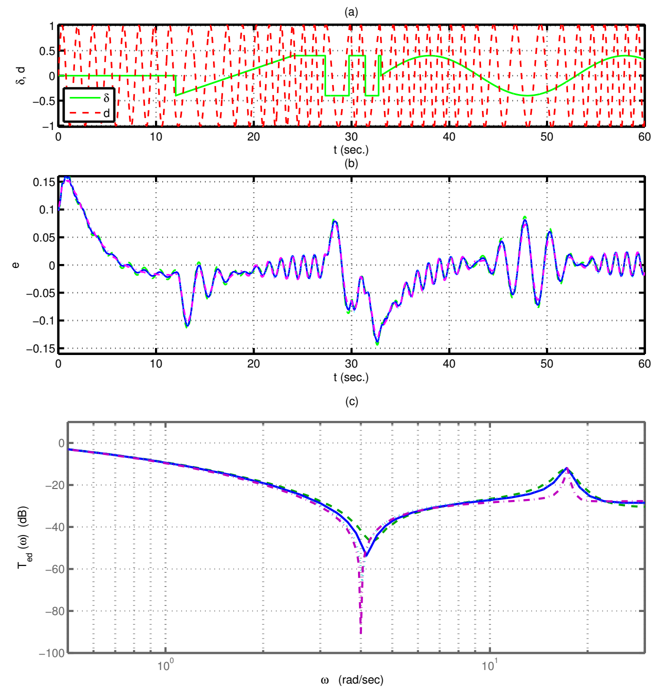

Fig. 1. Example designs and simulations: (a) Parameter trajectory and the disturbance; (b) Outputs, (c) Bode magnitude plots of T_{ed} for various designs (K_1:dashed, K_2:solid, K_3:dotted, K_4:dash-dotted)

IV. Illustrative Example

In this section, we consider a mass-spring damper system whose dynamics are described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \hline e \\ y \end{bmatrix} = \left[\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{k_1}{m_1} & -\frac{b}{m_1} & \frac{k_1}{m_1} & \frac{b}{m_1} & 0 & \frac{1}{m_1} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{k_1}{m_2} & \frac{b}{m_2} & -\frac{k_1+k_2}{m_2} & -\frac{b}{m_2} & \frac{k_2}{m_2} & -\frac{1}{m_2} \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \hline d \\ u \end{bmatrix},$$

where the disturbance affecting the system is assumed to be of the form $d(t) = \sin(\omega_0(1+\delta(t)))$, $\delta(t) \in [-\beta,\beta]$, as is the first state of (2) for the initial condition $v(0) = [0 \ \ -1]^T$. For a set of parameters given by $m_1 = 2, m_2 = 0.5, k_1 = 100, k_2 = 150, b = 10, \omega_0 = 4$, we synthesized four different controllers using the procedure of Theorem 1 for four different β values $\beta_1 = 0.3$, $\beta_2 = 0.2$, $\beta_3 = 0.1$ and $\beta_4 = 0.01$. By favor of the Yalmip ([16]) interface, we solved the optimization problems in MATLAB® with SeDuMi ([29]) and obtained the minimum (parameter-independent) κ levels for the four different parameter ranges respectively as $\kappa_1 = 0.3891$, $\kappa_2 = 0.2550$, $\kappa_3 = 0.1269$ and $\kappa_4 = 0.0125$. We observe significant improvement if compared to the previous work [12]. The transfer functions of the designed controllers are given by

$$K_1 = \frac{-11688.37(s+499.9)(s-66.3)(s+7.709)(s+2.807)(s^2+36.54s+981.5)}{(s+23.14)(s+2.687)(s^2-22.71s+1.544\cdot 10^4)(s^2+247.2s+8.574\cdot 10^4)},$$

$$K_2 = \frac{-23422.68(s+428.2)(s-47.45)(s+7.842)(s+2.621)(s^2+37.8s+1021)}{(s+23.41)(s+2.524)(s^2-152.1s+1.717\cdot 10^4)(s^2+375.1s+9.859\cdot 10^4)},$$

$$K_3 = \frac{-9952.33(s+440.9)(s-52.24)(s+7.754)(s+2.577)(s^2+37.84s+985.4)}{(s+22.12)(s+2.481)(s^2-87.15s+1.265\cdot 10^4)(s^2+244.3s+6.506\cdot 10^4)},$$
$$K_4 = \frac{-35464.00(s+306.7)(s-36.59)(s+8.113)(s+2.346)(s^2+41.86s+1110)}{(s-172.3)(s-98.96)(s+23.93)(s+2.278)(s^2+424.1s+9.053\cdot 10^4)}.$$

For a disturbance input as in Figure 1-a, the outputs obtained with these controllers (starting from $x(0) = [0.1\;0\;0.1\;0]^T$ and $\xi(0) = 0$) are presented in Figure 1-b. Note from Figure 1-a that the uncertain parameter first remains as zero for a while; then increases from its minimum value to its maximum value with constant rate; afterwards starts switching between 0.4 and -0.4 with increasing frequency; and finally shows a sinusoidal variation. Although the parameter range is much larger than the ranges considered to design the controllers, all of the controllers perform similarly well and the regulation performance does not degrade undesirably even when the parameter is close to its extreme values. The performance of the controllers for constant parameter trajectories can be analyzed based on the Bode magnitude plots of the transfer function from the disturbance to the error signal, which are displayed in Figure 1-c. The closed-loop exhibit band-pass behavior as is evident from the magnitude plots. Recall, however, that no theoretical guarantees can be inferred from the Bode plots in the case of time-variation.

V. Concluding Remarks

We have developed a novel procedure for the synthesis of an LTI controller that guarantees robust attenuation of non-stationary sinusoidal disturbances. One can consider extensions of the method based on enhanced (dilated) LMI characterizations [1]. The proposed method provides performance guarantees in contrast to ad-hoc loop-shaping procedures. In fact, one can develop alternative loop-shaping procedures based on the solution of the robust generalized asymptotic regulation problem, in which the parameter-dependent attenuation profile would be the key ingredient. The challenge then is to exploit any available bounds on the rates-of-variation of the uncertain parameters to reduce the potential conservatism in the design.

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