Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Adaptive Robust Control of a Class of Uncertain Nonlinear Systems with Unknown Sinusoidal Disturbances

Xiangbin Liu[†‡], Hongye Su[†], Bin Yao[§] and Jian Chu[†]

Abstract— **In this paper, nonlinear observers are incorporated into the discontinuous projection based adaptive robust control (ARC) to synthesize performance oriented controllers for a class of uncertain nonlinear systems with unknown sinusoidal disturbances. In addition to magnitudes and phases, frequencies of the sinusoidal disturbances need not to be known as well, so long as the overall order is known. A nonlinear observer is constructed to eliminate the effect of unknown sinusoidal disturbances to improve the steady-state output tracking performance – asymptotic output tracking is achieved when the system is subjected to unknown sinusoidal disturbance only. The discontinuous projection based adaptation law is used to obtain robust estimate of all unknown parameters. In addition, a dynamic normalization signal is introduced to construct adaptive robust control laws to effectively deal with various uncertainties for a guaranteed robust performance in general. Compared with the existing internal model principle based robust adaptive designs for unknown sinusoidal disturbances, the model uncertainties considered in the paper can be of unmatched. Furthermore, in the presence of other disturbances and uncertainties in addition to the sinusoidal disturbances, the proposed approach achieves a guaranteed output tracking robust performance in terms of both the transient and the steady-state, as opposed to the robust stability results of the existing internal model principle based designs.**

I. INTRODUCTION

One of the key ingredients in controller designs is how to effectively deal with uncertainties and disturbances to maximize the achievable closed-loop control performance. Adaptive control and robust control are two popular approaches among various control designs for uncertain nonlinear systems. In general, adaptive control uses certain parameter adaptation laws and certainty equivalence principle based control law designs to gradually eliminate the impact of unknown constant parameters – asymptotic output tracking can be achieved even in the presence of unknown system parameters [1]. However, the drawback of such an approach is its poor transient performance, which becomes an obstacle for its adoption in practical applications. Whereas, robust control aims at uncertainties without any structural information. By virtue of finding some known functions to bound the uncertain/disturbance signals, certain control laws can be synthesized to stabilize the uncertain systems without any attempt to learn from past control actions to gradually reduce the degree of uncertainties [2]. Hence, the resulting control performance is rather conservative.

The more information we have about uncertainties/disturbances, the better control performance we might be able to achieve. For the uncertainties or disturbances with known structural information, the observer/nonlinear observer with the estimate error dynamics being exp-ISS/ISpS can be used to estimate such signals for a better control performance [4]. For periodic signals with known period, traditional parameter adaptation or learning/repetitive controls can be used to relatively painlessly solve the problem [10]. For unbiased sinusoidal signals generated by exosystems with known order but unknown frequencies, the internal model principle can be used to estimate the sinusoidal signal for asymptotic output tracking [5], [11], [7]. However, these internal model principle based methods assume that the systems have no other uncertainties or disturbances at all other than the unknown sinusoidal signals and/or constant unknown parameters. In practice, systems may be subjected to both unknown sinusoidal signals and other types of uncertainties and disturbances. In [6], using the same modification techniques as in the traditional robust adaptive controls, *Nikiforov* also presented a robustified version of [5] to bounded disturbances. Output feedback versions of the problem are studied in [8].

In this paper, the nonlinear observer in [6] is incorporated into the discontinuous projection based adaptive robust control(ARC) [3] to synthesize performance oriented controllers for a class of uncertain nonlinear systems with unknown sinusoidal disturbances. In addition to magnitudes and phases, frequencies of the sinusoidal disturbances need not to be known as well, so long as the overall order is known. A nonlinear observer is constructed to eliminate the effect of unknown sinusoidal disturbances to improve the steady-state output tracking performance – asymptotic output tracking is achieved when the system is subjected to unknown sinusoidal disturbance only. The discontinuous projection based adaptation law is used to obtain robust estimate of all unknown parameters. In addition, a dynamic normalization signal is introduced to construct adaptive robust control laws to effectively deal with various uncertainties for a guaranteed robust performance in general. Compared with the existing internal model principle based robust adaptive designs [6], [11], [7], the model uncertainties considered in the paper can be of unmatched. Furthermore, in the presence of other uncertainties in addition to the sinusoidal disturbances, the proposed approach achieves a guaranteed output tracking

This work is supported by the National Creative Research Groups Science Foundation of P.R. China(NCRGSFC:60721062) and the National Basic Research Program of P.R. China(973 Program 2007CB714000).

Xiangbin Liu, Hongye Su and Jian Chu are with State Key Lab. of Industrial Control Technology, Institute of Cyber-Systems and Control, Zhejiang University, Hangzhou, 310027, P. R. China. Xiangbin Liu is also with Electronic Information Engineering College, Henan University of Science & Technology, Luoyang, 471003, P. R. China. xbnliu@yahoo.cn

Bin Yao is with School of Mechanical Engineering, Purdue University, West Lafayette, IN 47907,U.S.A. byao@purdue.edu

robust performance in terms of both the transient and the steady-state.

II. PROBLEM FORMULATION

Consider a class of the single-input-single-output uncertain nonlinear systems described by

$$\begin{array}{rcl} \dot{x}_i & = & x_{i+1}+\phi_i(\bar{x}_i)\mu+\Delta_i(x,\mu,u,t), 1\leq i\leq n-1, \\ \dot{x}_n & = & u+\phi_n(x)\mu+\Delta_n(x,\mu,u,t), \\ y & = & x_1 \end{array} \quad (1)$$

where $\bar{x}_i=[x_1,\cdots,x_i]^T$, $x=[x_1,\cdots,x_n]^T\in\mathbb{R}^n$ are system states. $u\in\mathbb{R}$ and $y\in\mathbb{R}$ are the control input and the output, respectively. $\phi_i(\bar{x}_i), i=1,\cdots,n$, is a known smooth nonlinear function. $\Delta_i(x,\mu,u,t), i=1,\cdots,n$, is a lumped unknown nonlinear function. μ represents the unknown sinusoidal disturbances, generated from an unknown exosystem given by

$$\begin{array}{rcl} \dot{\omega} & = & S\omega, \\ \mu & = & L^T\omega \end{array} \quad (2)$$

in which we know nothing about exosystem matrix S and vector L except the order or dimension of (2), i.e., $\omega\in\mathbb{R}^s$, where s is a known number.

It is worth pointing out that, as opposed to the matched sinusoidal disturbances assumed in [6], [7], [8], the uncertain nonlinearities $\Delta_i(x,\mu,u,t)$ and the unknown sinusoidal disturbances μ in (1) are unmatched. The following assumptions are made.

Assumption 1: The extent of parametric uncertainties and uncertain nonlinearities are known, i.e., $\forall i$,

$$\Delta_i\in\Omega_{\Delta i}\triangleq\{\Delta_i: \ \ |\Delta_i(x,\mu,u,t)|\leq\delta_i(\bar{x}_i)\} \quad (3)$$

where $\delta_i(\bar{x}_i)$ is a known function in which $|\cdot|$ denotes the Euclidean norm.

Assumption 2: The eigenvalues of S are with zero real parts and distinct, i.e., the exosystem is assumed to be neutrally stable, and $\{S,L^T\}$ of the unknown exosystem (2) observable. The signal μ is unaccessible to measurement.

Let $y_d(t)$ be the desired output trajectory, which is assumed to be known, bounded with derivatives up to nth orders. The control objective is to construct a control input u to make the output $y(t)$ track $y_d(t)$ with a prescribed accuracy in spite of various uncertainties. In addition, when $\Delta_i(x,\mu,u,t)=0, 1\leq i\leq n$, asymptotic output tracking should be achieved.

III. UNKNOWN SINUSOIDAL DISTURBANCE ESTIMATION

Motivated by [6] and [7], we use internal model principle to re-parameterize (2) for μ since $\{S,L^T\}$ in (2) are unknown. The following lemma, which is proved in [5], will be used in the subsequent ARC controller design.

Lemma 1: [1] For the sinusoidal disturbances generated by (2), consider the dynamic system

$$\dot{\eta}=M\eta+N\mu \quad (4)$$

where $\eta\in\mathbb{R}^s$ is the state and μ is the input, and the pair $\{M,N\}$ is controllable. Then for any $M\in\mathbb{R}^{s\times s}$ being *Hurwitz* matrix, there exists a unique constant vector $\psi\in\mathbb{R}^s$ such that the signal μ can be presented in the following form,

$$\mu=\psi^T(\eta+\varepsilon_\mu) \quad (5)$$

where the vector ε_μ satisfies the following equation,

$$\dot{\varepsilon}_\mu=M\varepsilon_\mu \quad (6)$$

with the initial condition $\varepsilon_\mu(0)=T\omega(0)-\eta(0)$, and the matrix T is the solution of *Sylvester* equation $TS-MT=NL^T$. $\diamondsuit$

Through Lemma 1, the signal μ can be obtained by estimating the unknown constant vector ψ.

Assumption 3: The extent of $\psi\in\mathbb{R}^s$ is known, i.e.,

$$\psi \ \in \ \Omega_\psi\triangleq\{\psi: \ \ \psi_{\min}\leq\psi\leq\psi_{\max}\} \quad (7)$$

where $\psi_{\min}$ and $\psi_{\max}$ are known vectors.

Assumption 4: There exists a vector of functions $h(x_1):\mathbb{R}\longrightarrow\mathbb{R}^s$, such that $\frac{dh(x_1)}{dx_1}\phi_1=N$.

Since μ and η are unaccessible to measurement in practice, we construct an observer based on internal model principle as

$$\dot{\xi}=M\xi+Mh(x_1)-\frac{dh(x_1)}{dx_1}x_2 \quad (8)$$

The biased error is defined by

$$\varepsilon_\eta=\eta-\xi-h(x_1) \quad (9)$$

Considering the Assumption 4, the derivative of ε_η is computed as

$$\begin{array}{rcl} \dot{\varepsilon}_\eta & = & M\eta+N\mu-[M\xi+Mh(x_1)-\frac{dh(x_1)}{dx_1}x_2] \\ & & -\frac{dh(x_1)}{dx_1}[x_2+\phi_1(x_1)\mu+\Delta_1(x,\mu,u,t)] \\ & = & M\varepsilon_\eta+\bar{\Delta} \end{array} \quad (10)$$

where $\bar{\Delta}=-\frac{dh(x_1)}{dx_1}\Delta_1(x,\mu,u,t)$. By virtue of (5), (9), (6) and (10), the unknown sinusoidal disturbances can also be represented by

$$\mu=\psi^T[\xi+h(x_1)+\varepsilon] \quad (11)$$

where the estimation error $\varepsilon=\varepsilon_\mu+\varepsilon_\eta$, ε is generated by

$$\dot{\varepsilon}=M\varepsilon+\bar{\Delta} \quad (12)$$

Remark 1: It should be noted that though the above observer design is based on that in [6] and the resulting observer error dynamics (12) are also similar in form, there are actually some fundamental differences between the two. Namely, the uncertain nonlinearity $\bar{\Delta}$ in (12) could be unbounded since Δ_1 is assumed to be bounded by a known function of state only, while the uncertainty Δ in Theorem 1 in [6] is assumed to be bounded.

Since M is a *Hurwitz* matrix, the unperturbed system of (12) is exponentially stable. Furthermore, let the matrix $P>0$ be the solution to the following *Lyapunov* equation

$$MP+PM^T=-I \quad (13)$$

[1] see [5], Lemma 3.1

Define a non-negative function as

$$V_\varepsilon(\varepsilon) = \varepsilon^T P \varepsilon \tag{14}$$

Then, we have

$$\gamma_1(|\varepsilon|) \le V_\varepsilon(\varepsilon) \le \gamma_2(|\varepsilon|), \quad \forall \varepsilon \in \mathbb{R}^s, \tag{15}$$

where $\gamma_1(|\varepsilon|) = \lambda_{\min}(P)|\varepsilon|^2$ and $\gamma_2(|\varepsilon|) = \lambda_{\max}(P)|\varepsilon|^2$. Noting Assumption 1, from (12), the derivative of $V_\varepsilon(\varepsilon)$ satisfies

$$\begin{aligned} \dot{V}_\varepsilon &\le -|\varepsilon|^2 + 2|\varepsilon|\lambda_{\max}(P)(|\bar{\Delta}|) \\ &\le -c\lambda_{\max}(P)|\varepsilon|^2 + \frac{\lambda_{\max}^2(P)}{1-c\lambda_{\max}(P)}(|\bar{\Delta}|)^2 \end{aligned} \tag{16}$$

where $0 < c < \frac{1}{\lambda_{\max}(P)}$. Viewing the Assumption 1, $|\bar{\Delta}| \le |\frac{dh(x_1)}{dx_1}|\delta_1(x_1)$. Thus there exist a known class $\mathscr{K}_\infty$ function $\gamma_\varepsilon(|x_1|)$ and a known positive constant d such that $\gamma_\varepsilon(|x_1|) + d \ge \frac{\lambda_{\max}^2(P)}{1-c\lambda_{\max}(P)}(|\frac{dh(x_1)}{dx_1}|\delta_1(x_1))^2$. Thus (16) becomes

$$\dot{V}_\varepsilon(\varepsilon) \le -cV_\varepsilon(\varepsilon) + \gamma_\varepsilon(|x_1|) + d \tag{17}$$

Lemma 2: [2] For any constants $\bar{c} \in (0, c)$, any initial condition $\varepsilon(0)$ and $r^0 = r(0) > 0$ and any function $\bar{\gamma}(x_1) \ge \gamma_\varepsilon(|x_1|)$, let r be a dynamic signal generated by

$$\dot{r} = -\bar{c}r + \bar{\gamma}(x_1) + d \tag{18}$$

If (17) hold, the for any initial condition $\varepsilon(0)$, there exists a finite $T^0 = T^0(\bar{c}, r^0, \varepsilon(0)) \ge 0$, a nonnegative function $D(t)$ defined for all $t \ge 0$ and such that $D(t) = 0$ for all $t \ge T^0$ and

$$V_\varepsilon \le r(t) + D(t) \tag{19}$$

for all $t \ge 0$ where the solutions are defined. it follows that

$$|\varepsilon(t)| \le \gamma_1^{-1}(2r(t)) + \gamma_1^{-1}(2D(t)) \tag{20}$$

IV. DISCONTINUOUS PROJECTION-BASED ARC BACKSTEPPING DESIGN

A. Parameter projection

In viewing (12), when $\Delta_1 = 0$ (i.e., in the absence of uncertain nonlinearity), the unknown sinusoidal signal μ can be recovered through the observer (8) if the constant vector ψ is known. As ψ is unknown, in the following, we use discontinuous projection type adaptation law [3] to estimate it on-line in order to recover the unknown sinusoidal disturbances μ. Specifically, let $\hat{\psi}$ denote the estimate of ψ and $\tilde{\psi}$ the estimation error (i.e. $\tilde{\psi} = \hat{\psi} - \psi$). Under Assumption 3, the following discontinuous projection type adaptation law can be used,

$$\dot{\hat{\psi}} = \mathrm{Proj}_{\hat{\psi}}(\Gamma\nu) \tag{21}$$

where $\Gamma > 0$ is a diagonal matrix, ν is an adaption function to be synthesized later. The projection mapping $Proj_{\hat{\psi}}(\bullet) = [Proj_{\hat{\psi}_1}(\bullet_1), \cdots, Proj_{\hat{\psi}_s}(\bullet_s)]^T$ is defined as

$$\mathrm{Proj}_{\hat{\psi}_i}(\bullet_i) = \begin{cases} 0 & \text{if } \hat{\psi}_i = \psi_{i\max} \text{ and } \bullet_i > 0 \\ 0 & \text{if } \hat{\psi}_i = \psi_{i\min} \text{ and } \bullet_i < 0 \\ \bullet_i & \text{otherwise} \end{cases} \tag{22}$$

[2]see [9], Lemma 3.1

It can be shown that for any adaptation function ν, the projection mapping used in (22) guarantees

$$\begin{array}{ll} \textbf{P1} & \hat{\psi} \in \Omega_\psi \triangleq \{\hat{\psi}: \ \psi_{\min} \le \hat{\psi} \le \psi_{\max}\} \\ \textbf{P2} & \tilde{\psi}^T[\Gamma^{-1}\mathrm{Proj}_{\hat{\psi}}(\Gamma\nu) - \nu] \le 0, \quad \forall \nu \end{array} \tag{23}$$

B. ARC Controller design

In this paper, the observer (8) is incorporated into the ARC backstepping design [3] to synthesize performance oriented robust controllers for the system (1) in the presence of both the unknown sinusoidal disturbances μ and the uncertain nonlinearities $\Delta_i, i = 1, \ldots, n$. The details are given below.

Step 1 For the control objective, we define output tracking error as $z_1 = y - y_d$. In view of the first equation in (1) and the unknown sinusoidal signal (11), the derivative of z_1 is

$$\begin{aligned} \dot{z}_1 &= x_2 + \phi_1(x_1)\psi^T[\xi + h(x_1) + \varepsilon] \\ &\quad + \Delta_1(x, \mu, u, t) - \dot{y}_d \end{aligned} \tag{24}$$

In (24), by viewing $x_2 = z_2 + \alpha_1$ as a virtual control. We choose the desired control function α_1 as

$$\begin{aligned} \alpha_1 &= \alpha_{1a} + \alpha_{1s}, \\ \alpha_{1s} &= \alpha_{1s1} + \alpha_{1s2} \end{aligned} \tag{25}$$

where α_{1a} is the adjustable model compensation given by

$$\alpha_{1a} = \dot{y}_d - \phi_1(x_1)\hat{\psi}^T[\xi + h(x_1)] \tag{26}$$

and α_{1s} is the robust control law having the following form,

$$\begin{aligned} &\alpha_{1s} = \alpha_{1s1} + \alpha_{1s2}, \\ &\alpha_{1s1} = -k_{1s}z_1 \end{aligned} \tag{27}$$

where $k_{1s} > 0$ is a nonlinear feedback gain which will be synthesized according to Theorem 1. α_{1s2} is a robust performance control term satisfying the following two conditions

$$\begin{aligned} &(i)\ z_1\{\alpha_{1s2} - \phi_1(x_1)\tilde{\psi}^T[\xi + h(x_1)] \\ &\qquad + \phi_1(x_1)\psi^T\varepsilon + \Delta_1(x, \mu, u, t)\} \le \varepsilon_1(1 + \rho^2), \\ &(ii)\ z_1\alpha_{1s2} \le 0 \end{aligned} \tag{28}$$

where $\varepsilon_1 > 0$ is a design parameter according to desired performance, and $\rho = \gamma_1^{-1}(2D(t))$. With the above control function, the derivative of output tracking error z_1 is

$$\begin{aligned} \dot{z}_1 &= z_2 - k_{1s}z_1 + \alpha_{1s2} - \phi_1(x_1)\tilde{\psi}^T[\xi + h(x_1)] \\ &\quad + \phi_1(x_1)\psi^T\varepsilon + \tilde{\Delta}_1(x, \mu, u, t) \end{aligned} \tag{29}$$

where $\tilde{\Delta}_1 = \Delta_1$.

Define a positive semi-definite(p.s.d.) function $V_1 = \frac{1}{2}z_1^2$, its time derivative satisfies

$$\begin{aligned} \dot{V}_1 &= z_1z_2 - k_{1s}z_1^2 + z_1\{\alpha_{1s2} - \phi_1(x_1)\tilde{\psi}^T[\xi + h(x_1)] \\ &\quad + \phi_1(x_1)\psi^T\varepsilon + \tilde{\Delta}_1(x, \mu, u, t)\} \end{aligned} \tag{30}$$

Define $\psi_{i\bar{m}} = \max\{|\psi_{i\min}|, |\psi_{i\max}|\}$, and $\psi_{\bar{M}} = [\psi_{1\bar{m}}, \cdots, \psi_{s\bar{m}}]^T$

Noting the Assumption 1 and 3 and the upper bounding functions of ε in (20), we have the following inequality,

$$\begin{aligned} &z_1\{\alpha_{1s2} - \phi_1(x_1)\tilde{\psi}^T[\xi + h(x_1)] \\ &+ \phi_1(x_1)\psi^T\varepsilon + \tilde{\Delta}_1(x, \mu, u, t)\} \\ &\le z_1\alpha_{1s2} + |z_1|\{|\phi_1(x_1)||\psi_M||\xi + h(x_1)| \\ &+ |\phi_1(x_1)||\psi_{\bar{M}}|[\gamma_1^{-1}(2r(t)) + \gamma_1^{-1}(2D(t))] + \tilde{\delta}_1\} \end{aligned} \tag{31}$$

where $\psi_M = \psi_{\max} - \psi_{\min}$.

Let h_1 be a known bounding function satisfying

$$\begin{aligned} h_1 \quad \geq \quad & |\phi_1(x_1)||\psi_M||\xi + h(x_1)| \\ & + |\phi_1(x_1)||\psi_{\bar{M}}|\gamma_1^{-1}(2r(t)) + \tilde{\delta}_1 \end{aligned} \tag{32}$$

then one example of α_{1s2} satisfying (28) is given by

$$\alpha_{1s2} = -\frac{1}{4\varepsilon_1}[h_1^2 + |\phi_1(x_1)|^2|\psi_{\bar{M}}|^2]z_1 \tag{33}$$

By substituting (33) into (31) and using the completion of squares technique, we have (i) of the two conditions (28).

Step i $(2 \leq i \leq (n-1))$ In i step, let α_i be the desired control function for the virtual control input x_{i+1} such that x_i tracks its desired ARC control law α_{i-1} at step $i-1$. Denote the tracking error at step i as $z_i = x_i - \alpha_{i-1}$ and recursively define the following functions,

$$\begin{aligned} & V_i = V_{i-1} + \tfrac{1}{2}z_i^2 \\ & \tilde{\Delta}_i = -\textstyle\sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k}\Delta_k + \Delta_i \\ & \bar{\phi}_i = -\textstyle\sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k}\phi_k + \phi_i \end{aligned} \tag{34}$$

Noting (1) and (11), the derivative of z_i is

$$\begin{aligned} \dot{z}_i \quad = \quad & x_{i+1} + \bar{\phi}_i(\bar{x}_i)\psi^T[\xi + h(x_1) + \varepsilon] + \tilde{\Delta}_i(x,\mu,u,t) \\ & - \textstyle\sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k}x_{k+1} - \frac{\partial \alpha_{i-1}}{\partial t} - \frac{\partial \alpha_{i-1}}{\partial r(t)}\dot{r}(t) \\ & - \frac{\partial \alpha_{i-1}}{\partial \xi}\dot{\xi} - \frac{\partial \alpha_{i-1}}{\partial \hat{\psi}}\dot{\hat{\psi}} \end{aligned} \tag{35}$$

Let $x_{i+1} = z_{i+1} + \alpha_i$ and the desired control function α_i as

$$\begin{aligned} \alpha_i \quad &= \quad \alpha_{ia} + \alpha_{is1} + \alpha_{is2}, \\ \alpha_{ia} \quad &= \quad -z_{i-1} - \bar{\phi}_i(\bar{x}_i)\hat{\psi}^T[\xi + h(x_1)] + \textstyle\sum_{k=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial x_k}x_{k+1} \\ & \quad\quad + \frac{\partial \alpha_{i-1}}{\partial t} + \frac{\partial \alpha_{i-1}}{\partial \xi}\dot{\xi} + \frac{\partial \alpha_{i-1}}{\partial r(t)}\dot{r}(t), \\ \alpha_{is1} \quad &= \quad -k_{is}z_i, k_{is} > 0 \end{aligned} \tag{36}$$

k_{is} which will be designed according to Theorem 1 and α_{is2} satisfies the following conditions,

$$\begin{aligned} & (i)z_i\{\alpha_{is2} - \bar{\phi}_i(\bar{x}_i)\tilde{\psi}^T[\xi + h(x_1)] \\ & \quad + \bar{\phi}_i(\bar{x}_i)\psi^T\varepsilon + \tilde{\Delta}_i(x,\mu,u,t)\} \leq \varepsilon_i(1+\rho^2), \\ & (ii)z_i\alpha_{is2} \leq 0 \end{aligned} \tag{37}$$

where $\varepsilon_i > 0$ is design parameter according to desired performance.

Noting (35) and (36), the derivative of augmented positive semi-definition(p.s.d.) function $V_i = V_{i-1} + \frac{1}{2}z_i^2$ satisfies

$$\begin{aligned} \dot{V}_i \quad = \quad & z_i z_{i+1} + \textstyle\sum_{k=1}^{i}\{-k_{ks}z_k^2 \\ & + z_k(\alpha_{ks2} - \bar{\phi}_k(\bar{x}_k)\tilde{\psi}^T[\xi + h(x_1)] \\ & + \bar{\phi}_k(\bar{x}_k)\psi^T\varepsilon + \tilde{\Delta}_k(x,\mu,u,t)) - z_k\frac{\partial \alpha_{k-1}}{\partial \hat{\psi}}\dot{\hat{\psi}}\} \end{aligned} \tag{38}$$

In view of Assumption 1 and 3, (20) and $|\tilde{\Delta}_i| \leq \tilde{\delta}_i \triangleq \sum_{k=1}^{i-1}|\frac{\partial \alpha_{i-1}}{\partial x_k}|\delta_k + \delta_i$, we have the following inequality

$$\begin{aligned} & z_i\{\alpha_{is2} - \bar{\phi}_i(\bar{x}_i)\tilde{\psi}^T[\xi + h(x_1)] + \bar{\phi}_i(\bar{x}_i)\psi^T\varepsilon + \tilde{\Delta}_i(x,\mu,u,t)\} \\ & \leq z_i\alpha_{is2} + |z_i||\bar{\phi}_i(\bar{x}_i)||\psi_M||[\xi + h(x_1)]| \\ & \quad + |z_i||\bar{\phi}_i(\bar{x}_i)||\psi_{\bar{M}}||[\gamma_1^{-1}(2r(t)) + \gamma_1^{-1}(2D(t))] + |z_i|\tilde{\delta}_i \end{aligned} \tag{39}$$

Let h_i be any smooth function satisfying

$$h_i \geq |\bar{\phi}_i(\bar{x}_i)||\psi_M||\xi + h(x_1)| + |\bar{\phi}_i(\bar{x}_i)||\psi_{\bar{M}}|\gamma_1^{-1}(2r(t)) + \tilde{\delta}_i \tag{40}$$

then one example of α_{is2} satisfying (37) can be designed as

$$\alpha_{is2} = -\frac{1}{4\varepsilon_i}[h_i^2 + |\bar{\phi}_i(\bar{x}_i)|^2|\psi_{\bar{M}}|^2]z_i \tag{41}$$

By substituting (41) into (39) and using the completion square technique, we have (i) of the two conditions (37).

Step n Let $u = x_{n+1}$, and noting (1) and (11), the derivative of $z_n = x_n - \alpha_{n-1}$ is

$$\begin{aligned} \dot{z}_n \quad = \quad & u + \bar{\phi}_n(x)\psi^T(\xi + h(x_1) + \varepsilon) - \textstyle\sum_{k=1}^{n-1}\frac{\partial \alpha_{n-1}}{\partial x_k}x_{k+1} \\ & + \Delta_n(x,\mu,u,t) - \textstyle\sum_{k=1}^{n-1}\frac{\partial \alpha_{n-1}}{\partial x_k}\Delta_k - \frac{\partial \alpha_{n-1}}{\partial t} \\ & - \frac{\partial \alpha_{n-1}}{\partial r(t)}\dot{r}(t) - \frac{\partial \alpha_{n-1}}{\partial \xi}\dot{\xi} - \frac{\partial \alpha_{n-1}}{\partial \hat{\psi}}\dot{\hat{\psi}} \end{aligned} \tag{42}$$

Let $u = \alpha_n$ and α_n is given as

$$\begin{aligned} \alpha_n \quad &= \quad \alpha_{na} + \alpha_{ns}, \\ \alpha_{na} \quad &= \quad -z_{n-1} - \bar{\phi}_n\hat{\psi}^T(\xi + h(x_1)) + \textstyle\sum_{k=1}^{n-1}\frac{\partial \alpha_{n-1}}{\partial x_k}x_{k+1} \\ & \quad\quad + \frac{\partial \alpha_{n-1}}{\partial t} + \frac{\partial \alpha_{n-1}}{\partial r(t)}\dot{r}(t) + \frac{\partial \alpha_{n-1}}{\partial \xi}\dot{\xi}, \\ \alpha_{ns} \quad &= \quad \alpha_{ns1} + \alpha_{ns2}, \\ \alpha_{ns1} \quad &= \quad -k_{ns}z_n, \\ k_{ns} \quad &\geq \quad 0, \quad c_n \geq 0 \end{aligned} \tag{43}$$

where k_{ns} will be designed according to Theorem 1.

α_{ns2} satisfies the following two conditions,

$$\begin{aligned} & (i)z_n\{\alpha_{ns2} - \bar{\phi}_n(\bar{x}_n)\tilde{\psi}^T[\xi + h(x_1)] \\ & \quad + \bar{\phi}_n(\bar{x}_n)\psi^T\varepsilon + \tilde{\Delta}_n(x,\mu,u,t)\} \leq \varepsilon_n(1+\rho^2), \\ & (ii)z_n\alpha_{ns2} \leq 0 \end{aligned} \tag{44}$$

where $\varepsilon_n > 0$ is design parameter according to desired performance.

Substituting (43) into (42), we have

$$\begin{aligned} \dot{z}_n \quad = \quad & -z_{n-1} - k_{ns}z_n + \alpha_{ns2} - \bar{\phi}_n\tilde{\psi}^T[\xi + h(x_1)] \\ & + \bar{\phi}_n\psi^T\varepsilon + \tilde{\Delta}_n(x,\mu,u,t) - \frac{\partial \alpha_{i-1}}{\partial \hat{\psi}}\dot{\hat{\psi}} \end{aligned} \tag{45}$$

From (45) and (43), the derivative of augmented positive semi-definition(p.s.d.) function $V_n = V_{n-1} + \frac{1}{2}z_n^2$ is

$$\begin{aligned} \dot{V}_n \quad = \quad & \textstyle\sum_{k=1}^{n}\{-k_{ks}z_k^2 + z_k[\alpha_{ks2} - \bar{\phi}_k(x)\tilde{\psi}^T(\xi + h(x_1)) \\ & + \bar{\phi}_k(x)\psi^T\varepsilon + \tilde{\Delta}_k(x,\mu,u,t)] - \frac{\partial \alpha_{k-1}}{\partial \hat{\psi}}\dot{\hat{\psi}}z_k\} \end{aligned} \tag{46}$$

From Assumption 1, 3 and the estimation error in (20), the following inequality is achieved,

$$\begin{aligned} & [\alpha_{ns2} - \bar{\phi}_n\tilde{\psi}^T(\xi + h(x_1)) + \bar{\phi}_n\psi^T\varepsilon + \tilde{\Delta}_n(x,\mu,u,t)]z_n \\ & \leq z_n\alpha_{ns2} + |z_n||\bar{\phi}_n||\psi_M||(\xi + h(x_1))| \\ & \quad + |z_n||\bar{\phi}_n||\psi_{\bar{M}}|[\gamma_1^{-1}(2r(t)) + \gamma_1^{-1}(2D(t))] + |z_n|\tilde{\delta}_n \end{aligned} \tag{47}$$

Let h_n be bounded functions Satisfying

$$h_n \geq |\bar{\phi}_n||\psi_M||\xi + h(x_1)| + |\bar{\phi}_n||\psi_{\bar{M}}|\gamma_1^{-1}(2r(t)) + \tilde{\delta}_n \tag{48}$$

and the examples of α_{ns2} can be chosen as

$$\alpha_{ns2} = -\frac{1}{4\varepsilon_n}\{h_n^2 + |\bar{\phi}_n|^2|\psi_{\bar{M}}|^2\}z_n \tag{49}$$

Substituting (49) into (47) and using completing squares technique, we have (i) of the two conditions (44).

Theorem 1: Consider the system (1) subjected to the Assumptions 1−4, and the adaptive robust controller which consists of the control law (43) and the adaptation law (21), in which the ν is chosen as

$$\nu = \sum_{k=1}^{n} \bar{\phi}_k(\bar{x}_k)[\xi + h(x_1)]z_k \quad (50)$$

and the observer (8). Let $k_{is} \geq g_i + |\frac{\partial \alpha_{i-1}}{\partial \hat{\psi}} C_{\psi i}|^2 + |C_{\phi i}\Gamma\bar{\phi}_i[\xi + h(x_1)]|^2 + c_\psi|\bar{\phi}_i(\bar{x}_i)|^2$, where $g_i > 0, c_\psi > 0$, $C_{\phi i}$ and $C_{\psi i}$ are positive-definite constant diagonal matrices. The $c_{\psi ji}$ and $c_{\phi ki}$ are the ith diagonal elements of the diagonal matrices $C_{\psi j}$ and $C_{\phi k}$, respectively. If the controller parameters $C_{\psi j}$ and $C_{\phi k}$ are chosen such that $c_{\phi ki}^2 \geq \frac{n}{4}\sum_{j=2}^{n} 1/c_{\psi ji}^2$, we have

A. The control input and all internal signals are bounded. V_n is bounded by

$$\begin{aligned} V_n(t) \leq\ & V_n(0)e^{-\lambda_n t} + \frac{\varepsilon_{sum}}{\lambda_n}[1 - e^{-\lambda_n t}] \\ & + \frac{\varepsilon_{sum}}{\lambda_n}\int_0^t e^{-\lambda_n(t-\tau)}\rho^2 d\tau \end{aligned} \quad (51)$$

where $\lambda_n = 2\ \min\{g_1, \cdots, g_n\}$, $\varepsilon_{sum} = \sum_{j=1}^{j=n}\varepsilon_j$. Noting that $\rho = 0$ for $t \geq T^0$, V_n is ultimately bounded by

$$V_n(\infty) \leq \frac{\varepsilon_{sum}}{\lambda_n} \quad (52)$$

B. If after a finite time t_f, $\Delta_i = 0$, i.e., in the presence of unknown sinusoidal disturbances only, then, in addition to results in A, asymptotic output tracking (or zero final tracking error) is also achieved. ♢.

Proof: A: Note that $z_{n+1} = 0$, from (46), (38) and (30), we have

$$\begin{aligned} \dot{V}_n =\ & \sum_{k=1}^{n}\{-k_{ks}z_k^2 + z_k[\alpha_{ks2} - \bar{\phi}_k(x)\tilde{\psi}^T(\xi + h(x_1)) \\ & + \bar{\phi}_k(x)\psi^T\varepsilon + \tilde{\Delta}_k(x,\mu,u,t)] - \frac{\partial\alpha_{k-1}}{\partial\hat{\psi}}\dot{\hat{\psi}}z_k\} \end{aligned} \quad (53)$$

By completion of squares

$$\begin{aligned} -\sum_{j=2}^{n} z_j \frac{\partial\alpha_{j-1}}{\partial\hat{\psi}}\dot{\hat{\psi}} \leq\ & \sum_{j=2}^{n}|z_j||\frac{\partial\alpha_{j-1}}{\partial\hat{\psi}}C_{\psi j}C_{\psi j}^{-1}\dot{\hat{\psi}}| \\ \leq\ & \sum_{j=2}^{n}(|\frac{\partial\alpha_{j-1}}{\partial\hat{\psi}}C_{\psi j}|^2 z_j^2 + \frac{1}{4}|C_{\psi j}^{-1}\dot{\hat{\psi}}|^2) \end{aligned} \quad (54)$$

Noting that $C_{\psi j}^{-1}$ and Γ are diagonal matrices, from (21) and (22), we have

$$\begin{aligned} & \sum_{j=2}^{n}|C_{\psi j}^{-1}\dot{\hat{\psi}}|^2 = \sum_{j=2}^{n}|C_{\psi j}^{-1}\mathrm{Proj}_{\hat{\psi}}(\Gamma\nu)|^2 \\ & \leq n\sum_{j=2}^{n}(\sum_{k=1}^{n}|C_{\psi j}^{-1}\Gamma\tilde{\phi}_k(\bar{x}_k)[\xi + h(x_1)]|^2 z_k^2) \end{aligned} \quad (55)$$

If $C_{\psi j}$ and $C_{\phi k}$ satisfied the conditions in the theorem, from (54) and (55),

$$\begin{aligned} & -\sum_{j=2}^{n} z_j \frac{\partial\alpha_{j-1}}{\partial\hat{\psi}}\dot{\hat{\psi}} \\ & \leq \sum_{j=2}^{n}|\frac{\partial\alpha_{j-1}}{\partial\hat{\psi}}C_{\psi j}|^2 {z_j}^2 \\ & \quad + \sum_{k=1}^{n}|C_{\phi k}\Gamma\bar{\phi}_k(\bar{x}_k)[\xi + h(x_1)]|^2 z_k^2 \end{aligned} \quad (56)$$

From (56) and the (i) of (28), (37) and (44), which readily leads to (51). Then, the boundedness of $z = [z_1, \cdots, z_n]^T$ can be proved. Since y_d is assumed to be a bounded signal with bounded derivatives up to nth order, together with Assumption 1, the backstepping designs [3] can prove that all internal signals are globally uniformly bounded . The boundedness of state x and (17) guarantee η, ε, r and ξ bounded. Then, all the intermediate control functions α_i are bounded. From (42) and (43), u is bounded.

B: Omitted, reader can refer to [4]. ■.

V. SIMULATION

To illustrate the proposed ARC scheme, this section considers the following specific system

$$\begin{aligned} \dot{x}_1 &= x_2 + \frac{1}{1+x_1^2}\mu + \Delta_1, \\ \dot{x}_2 &= u \end{aligned} \quad (57)$$

where μ is an unknown sinusoidal disturbance generated by

$$\begin{aligned} \dot{\omega} &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\omega, \\ \mu &= [1\ 0]\omega \end{aligned} \quad (58)$$

and Δ_1 is an unknown bounded uncertainty. The μ is shown in unmatched form which can not be applied by *Nikiforov*'s method in [6]. We take $\Delta_1(t) = 0.6\sin(2t)$ which bound is $\delta_1 = 0.8$.

According to Lemma 1, Let

$$M = \begin{bmatrix} -2 & 0 \\ 0 & -6 \end{bmatrix} \quad (59)$$

and the nonlinear vector function $h(x_1) = [x_1 + \frac{x_1^3}{3}, 2x_1 + \frac{2x_1^3}{3}]^T$ which satisfies the Assumption 4.

μ can be replaced by (11) where ξ is generated by (8) and vector ψ is $[-1.25\ 4.625]^T$ and the bounds describing its corresponding elements are $\psi_1 \in [-2,-1]$ and $\psi_2 \in [4,5]$.

The estimation error dynamics is

$$\dot{\varepsilon} = M\varepsilon - \frac{dh(x_1)}{dx_1}\Delta_1 \quad (60)$$

Let $V_\varepsilon(\varepsilon) = \varepsilon^T\varepsilon$ and $\gamma_1(|\varepsilon|) = \gamma_2(|\varepsilon|) = \varepsilon^T\varepsilon$. From (60), the derivative of the V_ε is computed as

$$\dot{V}_\varepsilon \leq -2V_\varepsilon + 0.64(x_1^4 + 2x_1^2) + 1 \quad (61)$$

It is thus clear that (16) is satisfied with $c_0 = 2$, $\gamma_3(|x_1|) = 0.64(x_1^4 + 2x_1^2)$ and $d_0 = 1$.

According to Lemma 2, a dynamic normalization signal r is generated by (5):

$$\dot{r} = -r + 0.64(x_1^4 + 2x_1^2) + 1, r(0) = r_0 \quad (62)$$

Define a nonnegative function

$$D = \max\{0, V_\varepsilon(\varepsilon(0))exp(-2t) - r(0)exp(-t)\} \quad (63)$$

It is easy to verify that there exist a finite T_0^0, and such that $D = 0$ for all $t > T_0^0$ and

$$V_\varepsilon \leq r + D \quad (64)$$

which agrees with Lemma 1.

From definition of V_ε and (64), then ε is bounded by

$$|\varepsilon| \leq (r+D)^{\frac{1}{2}} \leq (2r)^{\frac{1}{2}} + (2D)^{\frac{1}{2}} \quad (65)$$

which agrees with (20).

Step 1: We can construct the desired ARC controller for the first measured state equation according to section 4 as (25),(26),(27), where $g_1 > 0$, $c_\psi > 0$ and $\bar{\phi}_1 = \phi_1 = \frac{1}{1+x_1^2}$. Noting (32), we choose $h_1 \geq |\bar{\phi}_1||\psi_M||\xi + h(x_1)| + |\bar{\phi}_1||\psi_{\bar{M}}|\gamma_1^{-1}(2r) + \delta_1$, where $\gamma_1^{-1}(2r) = (2r)^{\frac{1}{2}}$,

Step 2: The desired ARC controller u for the simulation nonlinear system is synthesized as (43), where $g_2 > 0$ and $\bar{\phi}_2 = -\frac{\partial \alpha_1}{\partial x_1}\phi_1$. Noting (48), we choose $h_2 \geq |\bar{\phi}_2||\psi_M||\xi + h(x_1)| + |\bar{\phi}_2||\psi_{\bar{M}}|\gamma_1^{-1}(2r_\varepsilon) + \tilde{\delta}_2$. where $\tilde{\delta}_2 = |\frac{\partial \alpha_1}{\partial x_1}|\delta_1$.

The parameter adaptation law is given by

$$\begin{array}{rcl} \dot{\hat{\psi}} & = & \mathrm{Proj}_{\hat{\psi}}(\Gamma\nu), \\ \nu & = & \bar{\phi}_1[\xi + h(x_1)]z_1 + \bar{\phi}_2[\xi + h(x_1)]z_2 \end{array} \quad (66)$$

The design parameters and initial conditions for the proposed ARC are as follows:

$$\begin{array}{l} r(0) = 0.2, x_1(0) = x_2(0) = 0, \hat{\psi}_1 = -1.5, c_\psi = 0.1, \\ \hat{\psi}_2 = 3.5, \xi_1 = \xi_2 = 0, g_1 = g_2 = 10, \varepsilon_1 = 30, \varepsilon_2 = 300, \\ C_{\phi 1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C_{\phi 2} = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix}, \\ C_{\psi 2} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \Gamma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \quad (67)$$

Let desired trajectory $x_d = 0.5(1 - \cos(1.4\pi t))$. In order to show the properties of the proposed ARC scheme, we will consider the following two conditions to demonstrate them.

Case i: Without uncertain nonlinearities

The simulation is first run for the actual system with sinusoidal disturbances but without uncertain nonlinearities, i.e., letting $\Delta_1 = 0$ in (57), which corresponds to the ideal working conditions. It is seen from the simulation results shown in Figure 1 that the proposed extended ARC scheme has good transient performance and final tracking accuracy – output tracking error is small during the entire transient period and converges to zero eventually. The unknown sinusoidal disturbance estimate gradually converges to its true value as well. The control input also is reasonable.

Case ii: With uncertain nonlinearities

The simulation is also run for the actual system subjected to the sinusoidal disturbances and uncertain nonlinearities as well, i.e., the system (57) with $\Delta_1 = 0.6\sin(2t)$. The initial conditions for dynamic normalization signals are chosen as $r(0) = 0.2$. The tracking error plot for the ARC scheme is shown in Figures 2. The closed-loop system is stable and the output tracking error is still kept small during the entire transient period.

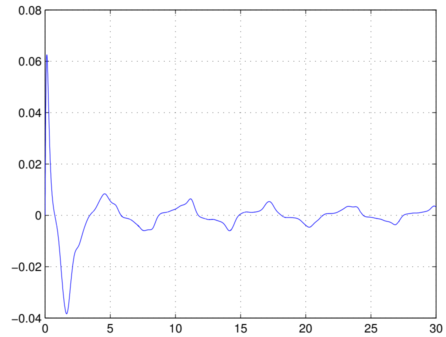

Fig. 1. Tracking errors in case i

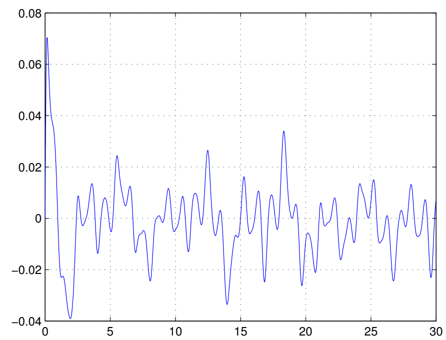

Fig. 2. Tracking errors in case ii

VI. CONCLUSIONS

In this paper, nonlinear observers are incorporated into the discontinuous projection based adaptive robust control (ARC) to synthesize performance oriented controllers for a class of uncertain nonlinear systems with unknown sinusoidal disturbances. A nonlinear adaptive robust observer is constructed to recover the unmeasured sinusoidal disturbances so that suitable model compensation can be constructed for a much improved steady-state output tracking performance – asymptotic output tracking is achieved in the presence of parametric uncertainties and unknown sinusoidal disturbances. The discontinuous projection based adaptation law is used to obtain robust estimate of all unknown parameters. A dynamic normalization signal is constructed so that certain robust feedback can be used to dominate the estimate error effect for a guaranteed transient and steady-state output tracking performance in general.

REFERENCES

[1] M. Krstic, I. Kanellakopoulos, and P.V. Kokotovic. *Nonlinear and adaptive control design*, New York: Wiley, 1995

[2] M.J. Corless and G. Leitmann. Continuous state feedback guaranteeing uniform ultimate boundedness for uncertain dynamic system. *IEEE Transaction on automatic control*, 26(10), 1139-1144, 1981.

[3] B. Yao. High performance adaptive robust control of nonlinear systems: a general framework and new schemes. *Proceedings of IEEE Conference on Decision and Control*, 2489-2494, 1997

[4] B. Yao and L. Xu. Observer-based adaptive robust control of a class of nonlinear systems with dynamic uncertainties. *Int. J. Robust Nonlinear Control*, 11(4) , 335-356, 2001.

[5] V.O. Nikiforov. Adaptive servomechanism controller an implicit reference model. *Int. J. Control* 68(2), 277-286,1997.

[6] V.O. Nikiforov. Nonlinear servocompensation of unknown external disturbances. *Automatica*, 37(10), 1647-1653, 2001.

[7] Z. Ding. Asymptotic rejection of unknown sinusoidal disturbances in nonlinear systems of unknown disturbances for nonlinear output. *Automatica*, 43(1), 174-177, 2007.

[8] R. Marino, G.L. Santosuosso, & P. Tomei. Robust adaptive compensation of biased sinusoidal disturbances with unknown frequency. *Automatica*, 39(10), 1755-1761, 2003.

[9] Z.P. Jiang and L. Praly. Design of robust adaptive controllers for nonlinear systems with dynamic uncertainties. *Automatica*, 34(7), 825-840, 1998.

[10] M.X. Sun and S.S. Ge. Adaptive Repetitive Control for a Class of Nonlinearly Parametrized Systems. *IEEE Transaction on automatic control*, 51(10), 1684-1688, 2006

[11] A. Serrani, A. Isidori and L. Marconi. Semiglobal nonlinear output regulation with adaptive internal mdel. *IEEE Transaction on automatic control*, 46(8), 1178-1194, 2001