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Stabilization of discrete–time quantized linear systems: an H_∞/ℓ_1 approach

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***Abstract*— A generalized small–gain theorem, suitable for the analysis of practical stability, is proved in the framework of ℓ_1 control. The result is combined with a practical stabilization technique based on a generalized small–gain theorem in H_∞. The resulting mixed H_∞/ℓ_1 approach allows us to provide systematic tools for the control synthesis and the closed loop analysis in the practical stabilization of linear systems under assigned input quantization. A numerical example is reported.**

I. INTRODUCTION

Quantized input systems are dynamical systems controlled by discrete variables. Quantization is a characteristic arising in many control applications as well as it is the origin of a number of significant theoretical issues. A renewed interest in quantized systems has been spurred by the pioneering paper [4]. Much attention of the scientific community has been addressed to the problem of control under communication constraints (see [14] and references therein). Other works deal with the digital implementation of controllers and, more generally, with the control of systems with discrete sensors and/or actuators [13], [20], [16].

As clarified in [4], the stabilization problem for systems under quantized control has to be formulated in terms of *practical stability* properties. Basically, the goal consists of designing a controller, taking values in a quantized set, so that the closed loop trajectories are eventually confined within sufficiently small neighborhoods of the equilibrium.

We are interested in the practical stabilization of linear systems under *arbitrarily assigned* input quantization (i.e., the control set $\mathcal{U}$ is fixed and, besides being quantized, there is not any further assumption on its structure). A plant actuated by a static quantized controller can be modelled as the feedback interconnection of an ideal (i.e., non–quantized) closed loop dynamics with a static nonlinearity taking the quantization effect into account. In this way, the control synthesis for stabilization can be carried out by designing a controller having robustness properties with respect to quantization. This approach is not new, see e.g., [13], [9], [3]. However, classical robust control techniques are often tailored to asymptotic, instead of practical, stabilization. In [15], a control synthesis method is proposed, which is based on a generalized small–gain theorem in H_∞ and provides systematic tools to solve the practical stabilization problem. In the present paper, those results are supplemented with a generalized small–gain theorem in the framework of ℓ_1 control, which enables us to obtain a less conservative steady–state analysis of the closed loop dynamics. Thus, a mixed H_∞/ℓ_1 approach to the stabilization problem is provided which joins the powerful control synthesis tools offered by the H_∞ theory with the effectiveness of the ℓ_1 analysis. A mixed H_∞/ℓ_1 control synthesis is also proposed. The latter formulation appears to be particularly promising to deal with the special class of *positive systems* [8]: in this case, in fact, it is shown that the H_∞ and the ℓ_1 norms coincide. Moreover, the interpretation in terms of ℓ_1 control is offered of recently published results on the practical stabilization of quantized systems [17].

Notation: The i–th component of $x \in \mathbb{R}^n$ is x_i; x' is the transpose of x. The (i,j)–th entry of $M \in \mathbb{R}^{h\times k}$, is $M_{i,j}$.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. Norm of signals and systems

A *signal* is a function $\vec{v} : \mathbb{N} \to \mathbb{R}^{h\times k}$; $v(t)$ denotes its value at time t. Let the space of bounded signals in $\mathbb{R}^p$ be

$$\ell_\infty(\mathbb{R}^p) := \left\{\vec{v} : \mathbb{N} \to \mathbb{R}^p \mid \sup_{t\in\mathbb{N}} \|v(t)\|_\infty < +\infty\right\},$$

where the infinity norm of a constant vector or matrix, say $M \in \mathbb{R}^{h\times k}$, is given by $\|M\|_\infty = \max_{i=1,\dots,h} \sum_{j=1}^k |M_{i,j}|$. The space $\ell_\infty(\mathbb{R}^p)$ is endowed with the norm

$$\|\vec{v}\,\|_\infty := \sup_{t\in\mathbb{N}} \|v(t)\|_\infty .$$

Consider a discrete–time linear system

$$\Sigma(A,B,C) := \begin{cases} x(t+1) = Ax(t) + Be(t) \\ y(t) = Cx(t) \\ x \in \mathbb{R}^n,\ e \in \mathbb{R}^m,\ y \in \mathbb{R}^q,\ t \in \mathbb{N}, \end{cases} \tag{1}$$

let $\vec{g}$ be its impulse response, namely

$$g(t) = \begin{cases} 0 & \text{if } t = 0 \\ CA^{t-1}B & \text{if } t \ge 1, \end{cases}$$

and $G(z) := \sum_{t=0}^{+\infty} g(t)z^{-t} = C(zI - A)^{-1}B$ be the system transfer matrix. System (1) is ***BIBO–stable*** iff $\forall\, \vec{e} \in \ell_\infty(\mathbb{R}^m)$ one has $\vec{g} * \vec{e} \in \ell_\infty(\mathbb{R}^q)$, where $(\vec{g} * \vec{e})(t) := \sum_{\tau=0}^{t-1} g(t-\tau)e(\tau)$. In this case, the linear operator

$$\begin{array}{rccc} \mathcal{G}: & \ell_\infty(\mathbb{R}^m) & \to & \ell_\infty(\mathbb{R}^q) \\ & \vec{e} & \mapsto & \vec{g} * \vec{e} \end{array}$$

is bounded and its induced operator norm is such that

$$\|\mathcal{G}\|_\infty := \sup_{\vec{e}\in\ell_\infty(\mathbb{R}^m)\setminus\{\vec{0}\}} \frac{\|\vec{g} * \vec{e}\|_\infty}{\|\vec{e}\|_\infty} \overset{(a)}{=} \max_{i=1,\dots,q} \sum_{j=1}^{m} \sum_{\tau=0}^{+\infty} |g_{i,j}(\tau)|$$

(see [5]). The operator $\mathcal{G}$ is referred to as the *input/output* operator associated to system (1) and $\|\mathcal{G}\|_\infty$ is called the

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ℓ_∞–*gain* of the system. Thanks to equality (a), system (1) is BIBO–stable if and only if $\vec{g} \in \ell_1(\mathbb{R}^{q\times m})$, where

$$\ell_1(\mathbb{R}^{q\times m}) := \{\vec{g} : \mathbb{N} \to \mathbb{R}^{q\times m} \mid \textstyle\sum_{\tau=0}^{+\infty} \|g(\tau)\|_\infty < +\infty\}.$$

This is the reason why the ℓ_∞–gain of a BIBO–stable system is also referred to as the ℓ_1–norm of the system.

Remark 2.1: Script symbols denote input/output operators. The norm $\|\mathcal{G}\|_\infty$ should not be confused with $\|G\|_\infty$: the latter is the H_∞–norm of the transfer matrix $G(z)$ and, actually, is the ℓ_2–gain of the system [12].

B. Problem formulation

Definition 2.1: A set $\mathcal{U} \subset \mathbb{R}^m$ is said to be *quantized* iff it is closed and discrete (i.e., all its points are isolated).

This is equivalent to say that any bounded subset of $\mathbb{R}^m$ contains only a finite number of elements of $\mathcal{U}$.

We are interested in the stabilization problem for quantized input systems of the type:

$$\begin{cases} x(t+1) = Ax(t) + Bu(t) \\ x \in \mathbb{R}^n,\ u \in \mathcal{U} \subset \mathbb{R}^m,\ t \in \mathbb{N}, \end{cases} \tag{2}$$

where the pair (A, B) is supposed to be stabilizable and $\mathcal{U}$ is an assigned quantized set containing 0.

If the system is open loop unstable and $\mathcal{U}$ is quantized, then neither stabilization nor confinement of the trajectories within arbitrarily small neighborhoods of the origin can be achieved [4]. We hence consider *practical stability* notions:

Definition 2.2: Consider a dynamical system of the type

$$x(t+1) = f\big(x(t)\big), \quad x \in \mathbb{R}^n. \tag{3}$$

$\imath$) A set $\Omega \subseteq \mathbb{R}^n$ is said to be *positively invariant* for system (3) iff $\forall\, x(t) \in \Omega,\ x(t+1) \in \Omega$.

Let Ω, X_0 and X_1 be bounded neighborhoods of the origin in $\mathbb{R}^n$ such that $\Omega \subseteq X_1$ and $X_0 \subseteq X_1$:

$\imath\imath$) system (3) is (X_0, X_1, Ω)–*stable* iff $\forall\, x(0) \in X_0$, $x(t) \in X_1\ \ \forall\, t \ge 0$ and $\exists\, \bar{t} \in \mathbb{N}$ such that $\forall\, t \ge \bar{t},\ x(t) \in \Omega$;

$\imath\imath\imath$) system (3) is (X_0, Ω)–*stable* iff both X_0 and Ω are positively invariant and $\forall\, x(0) \in X_0\ \ \exists\, \bar{t} \in \mathbb{N}$ such that $\forall\, t \ge \bar{t},\ x(t) \in \Omega$.

Remark 2.2: [(X_0, X_0, Ω)–stability vs (X_0, Ω)–stability] If system (3) is (X_0, Ω)–stable, then it is (X_0, X_0, Ω)–stable. In general, the contrary is not true because the set Ω is not guaranteed to be positively invariant.

We consider static state feedback laws of the type $u(x) = q_{\mathcal{U}}(Kx)$, where $K \in \mathbb{R}^{m\times n}$ and the input quantizer $q_{\mathcal{U}} : \mathbb{R}^m \to \mathcal{U}$ are to be designed (whilst $\mathcal{U}$ is assigned). We provide systematic tools to find K and $q_{\mathcal{U}}$ so that practical stabilization is ensured, and to analyze the practical stability properties of the resulting closed loop dynamics.

For a given control law $u(x) = q_{\mathcal{U}}(Kx)$, with the *quantization error* $q_e : \mathbb{R}^m \to \mathbb{R}^m$ defined by $q_e(y) := q_{\mathcal{U}}(y) - y$, the closed–loop dynamics is

$$x(t+1) = (A+BK)x(t) + Bq_e\big(Kx(t)\big) = \tag{4a}$$

$$= Fx(t) + B\psi\big(x(t)\big), \tag{4b}$$

where $F := A + BK$ and $\psi := q_e \circ K$. Accordingly, system (4) can be alternately seen as the feedback interconnection of the linear system $\Sigma(A+BK, B, K)$ with the nonlinearity q_e, or of system $\Sigma(F, B, I)$ with ψ. In both cases, small–gain conditions [12] can be used for the analysis of the system. In [15], a generalized version of the small–gain theorem is proposed in the framework of H_∞ theory, which is based on a generalized notion of gain and, consistently, it is suitable to deal with the control synthesis for practical stabilization. Here, that result is combined with analysis tools based on a small–gain theorem involving generalized ℓ_∞–gains and ℓ_1 theory.

Note that, since the control values do not accumulate towards 0, the quantization error does not vanish for $t \to +\infty$ and it has to be treated as a signal in ℓ_∞ by means of ℓ_1 theory. The H_∞ space, instead, is isomorphic to the space of the operators between signals in ℓ_2, hence vanishing for $t \to +\infty$. For this reason, while H_∞ theory provides suitable control synthesis tools to ensure convergence properties, the right approach to the steady–state analysis is the ℓ_1 theory. Indeed, the ℓ_1 based analysis allows us to prove the convergence of the trajectories to a *smaller* neighborhood of the equilibrium than that built on the H_∞ theory. This result has a counterpart in the *minimality* properties holding for invariant hypercubes and recently proved in [17].

III. Small–gain in ℓ_1 for practical stability

For a given $\psi : \mathbb{R}^n \to \mathbb{R}^m$, let us introduce some quantities which are useful for our analysis. For $\Omega \subseteq \mathbb{R}^n$, let

$$\mathscr{E}(\Omega) := \sup_{x\in\Omega} \|\psi(x)\|_\infty .$$

Along the paper, the function ψ is supposed to be *regular*, namely if $\Omega \subset \mathbb{R}^n$ is bounded, then $\mathscr{E}(\Omega) < +\infty$.

Consider the closed hypercube of edge length Δ:

$$Q_n(\Delta) := \left[-\tfrac{\Delta}{2}, \tfrac{\Delta}{2}\right]^n, \quad \Delta \ge 0 .$$

In this case, we use the notation

$$\mathscr{E}(\Delta) := \sup_{x\in Q_n(\Delta)} \|\psi(x)\|_\infty .$$

The function $\mathscr{E}(\Delta)$ is non–decreasing, we can hence define the right continuous function $\mathscr{E}^+(\Delta) := \lim_{\epsilon\to 0^+} \mathscr{E}(\Delta+\epsilon)$.

Definition 3.1: For $\Delta > 0$, let the *generalized* ℓ_∞*–gain* of the function ψ be defined by[1]

$$\gamma_e(\Delta) := \frac{\mathscr{E}^+(\Delta)}{\Delta/2} .$$

Theorem 3.1: [Small–gain in ℓ_1] Consider the system

$$x(t+1) = Fx(t) + B\psi\big(x(t)\big), \tag{5}$$

where $F \in \mathbb{R}^{n\times n}$ is a Schur matrix and $\psi : \mathbb{R}^n \to \mathbb{R}^m$. Assume that the system is (X_0, X_1, Ω)–stable. Denote by $\mathcal{G}^{(I)}$ the input/state operator associated to system $\Sigma(F, B, I)$ and let $\Delta_1 := 2\|\mathcal{G}^{(I)}\|_\infty \mathscr{E}(\Omega)$. Then,

$\imath$) $\forall\, \Delta > \Delta_1$, system (5) is $\big(X_0, X_1, Q_n(\Delta)\big)$–stable.

Let $\gamma_e(\Delta)$ be the generalized ℓ_∞–gain of the function ψ:

$\imath\imath$) if

$$\|\mathcal{G}^{(I)}\|_\infty \cdot \gamma_e(\Delta_1) < 1 ,$$

then the following is well–defined

$$\Delta_{\inf} := \begin{cases} \max\left\{\Delta < \Delta_1 \mid \|\mathcal{G}^{(I)}\|_\infty \cdot \gamma_e(\Delta) = 1\right\} & \text{if} \\ \qquad \left\{\Delta < \Delta_1 \mid \|\mathcal{G}^{(I)}\|_\infty \cdot \gamma_e(\Delta) = 1\right\} \neq \emptyset & \\ 0 \quad \text{otherwise}, & \end{cases} \tag{6}$$

and $\forall\, \Delta_\star > \Delta_{\inf}$, system (5) is $\big(X_0, X_1, Q_n(\Delta_\star)\big)$–stable.

[1] $\mathscr{E}^+(\Delta)$ is divided by $\Delta/2$ because $x \in Q_n(\Delta) \Leftrightarrow \|x\|_\infty \le \Delta/2$.

The proof of the Theorem is based on the following

Lemma 3.1: Consider $x(0) \in \mathbb{R}^n$ and its evolution under system (5). If $S \subseteq \mathbb{R}^n$ is such that $\mathscr{E}(S) < +\infty$ and $\exists\, \hat{t} \geq 0$ so that $\forall\, t \geq \hat{t},\ x(t) \in S$, then $\forall\, \Delta > 2\|\mathcal{G}^{(I)}\|_\infty \mathscr{E}(S)$, $\exists\, t_1 \geq 0$ such that $\forall\, t \geq t_1,\ x(t) \in Q_n(\Delta)$.

Proof: To prove the result it is sufficient to show that

$$\limsup_{t\to+\infty} \|x(t)\|_\infty \leq \|\mathcal{G}^{(I)}\|_\infty \mathscr{E}(S).$$

Denote by σ the *shift* operator, where $\sigma\vec{v}$ is defined by $\sigma v(t) := v(t+1)$, and by σ^τ its τ–th iteration. For $x(0) \in \mathbb{R}^n$ and $\vec{e}$ defined by $e(t) := \psi\big(x(t)\big)$, it holds that

$$\forall\, t \geq 0 \text{ and } \forall\, k \geq 0,\ \ x(t+k) = F^k x(t) + (\vec{g}^{\,(I)} * \sigma^t \vec{e})(k),$$

where $\vec{g}^{\,(I)}$ is the impulse response associated to system $\Sigma(F, B, I)$. Since $\forall\, t \geq \hat{t},\ x(t) \in S$, then $\forall\, t \geq \hat{t}$, $\|e(t)\|_\infty \leq \mathscr{E}(S)$ or, equivalently, $\|\sigma^t \vec{e}\|_\infty \leq \mathscr{E}(S)$. Thus,

$$\begin{aligned}&\limsup_{t\to+\infty} \|x(t)\|_\infty = \limsup_{k\to+\infty} \|x(\hat{t}+k)\|_\infty \leq \\ &\leq \limsup_{k\to+\infty} \big(\|F^k x(\hat{t})\|_\infty + \|(\vec{g}^{\,(I)} * \sigma^{\hat{t}}\vec{e})(k)\|_\infty\big) \leq \\ &\overset{(b)}{\leq} \|(\vec{g}^{\,(I)} * \sigma^{\hat{t}}\vec{e})\|_\infty \leq \|\mathcal{G}^{(I)}\|_\infty \|\sigma^{\hat{t}}\vec{e}\|_\infty \leq \|\mathcal{G}^{(I)}\|_\infty \mathscr{E}(S),\end{aligned}$$

where (b) holds because $\lim_{k\to+\infty} \|F^k x(\hat{t})\|_\infty = 0$. ■

Proof: [of Theorem 3.1] To prove part $\imath$, apply Lemma 3.1 with $S = \Omega$. The iteration of the same argument allows one to prove also the $\big(X_0, X_1, Q_n(\Delta_\star)\big)$–stability. All the technical details can be found in [19]. ■

Theorem 3.1 can be used to supplement the practical stability analysis of a dynamics that has been proved to be practically stable through some other technique. Since system (5) is both (X_0, X_1, Ω)–stable and $\big(X_0, X_1, Q_n(\Delta_\star)\big)$–stable, then it is $\big(X_0, X_1, \Omega \cap Q_n(\Delta_\star)\big)$–stable: thus the theorem enables one to prove the convergence of the trajectories to within a *smaller* neighborhood of the equilibrium.

A. Single–input reachable systems

In Theorem 3.1, system (5) is assumed to be (X_0, X_1, Ω)–stable and, by a small–gain condition in ℓ_1, the $\big(X_0, X_1, Q_n(\Delta_\star)\big)$–stability is deduced. A stronger result can be proved for single–input reachable systems: taking advantage of the canonical controller form, the positive invariance of hypercubes can be derived by a small–gain condition in ℓ_1 without a priori stability assumptions. Moreover, the stronger notion of (X_0, Ω)–stability is ensured.

Proposition 3.1: [Small–gain in ℓ_1: single–input systems] Consider system (5) where $F \in \mathbb{R}^{n\times n}$ is a Schur matrix and $\psi : \mathbb{R}^n \to \mathbb{R}$. Denote by $\gamma_e(\Delta)$ the generalized ℓ_∞–gain of the function ψ. If the pair (F, B) is in the controller form, with $z^n - f_n z^{n-1} - \cdots - f_2 z - f_1$ being the characteristic polynomial of F, and $f := \sum_{i=1}^n |f_i| < 1$, then:

$\imath$) $\forall\, \Delta > 0$ such that $\frac{\gamma_e(\Delta)}{1-f} \leq 1$, $Q_n(\Delta)$ is positively invariant;

$\imath\imath$) if $\Delta_0 > 0$ is such that

$$\frac{\gamma_e(\Delta_0)}{1-f} < 1, \tag{7}$$

then the following is well–defined

$$\Delta_{\inf} := \begin{cases} \max\left\{\Delta < \Delta_0 \,|\, \frac{\gamma_e(\Delta)}{1-f} = 1\right\} & \text{if} \\ \qquad \left\{\Delta < \Delta_0 \,|\, \frac{\gamma_e(\Delta)}{1-f} = 1\right\} \neq \emptyset \\ 0 \quad \text{otherwise}, \end{cases}$$

and $\forall\, \Delta_\star > \Delta_{\inf}$, the system is $\big(Q_n(\Delta_0), Q_n(\Delta_\star)\big)$–stable.

Proof: See [19]. ■

Remark 3.1: The ℓ_∞–gain of the input/state operator $\mathcal{G}^{(I)}$ associated to system (5) does not appear in condition (7). Nevertheless, $\frac{\gamma_e(\Delta_0)}{1-f} < 1$ is a small–gain condition in ℓ_1 because it can be shown that $\|\mathcal{G}^{(I)}\|_\infty \leq \frac{1}{1-f}$ (see [18]).

IV. A MIXED H_∞/ℓ_1 APPROACH TO THE STABILIZATION OF QUANTIZED INPUT LINEAR SYSTEMS

Let us illustrate how the combination of H_∞ theory with the proposed results on the small–gain theorem in ℓ_1 allows one to deal with the practical stabilization of system (2). To this end, let us recall a generalized notion of gain in ℓ_2. By $\|\cdot\|_2$, we denote either the Euclidean norm of a vector or the corresponding induced matrix norm.

Definition 4.1: [15] Let $\varrho_0 > 0$ and $\gamma_e \geq 0$. A map $\varphi : \mathbb{R}^p \to \mathbb{R}^m$ is said to have *ϱ_0–external gain* γ_e iff $\forall\, y \in \mathbb{R}^p$ such that $\|y\|_2 > \varrho_0$, it holds that $\|\varphi(y)\|_2 \leq \gamma_e \|y\|_2$.

Theorem 4.1: [Mixed H_∞/ℓ_1 closed loop analysis (control synthesis in H_∞)] Consider system (2), let $q_{\mathcal{U}} : \mathbb{R}^m \to \mathcal{U}$ be such that the corresponding quantization error q_e satisfies the following conditions:

$$\begin{cases} q_e \text{ has } \varrho_0\text{–external gain } \gamma_e \\ \text{if } \|y\|_2 \leq \varrho_0,\ \text{then } \|q_e(y)\|_2 \leq E_0. \end{cases} \tag{8}$$

For $\gamma_\infty \leq \frac{1}{\gamma_e}$, suppose that $K \in \mathbb{R}^{m\times n}$ is found so that

$$\begin{cases} F := A + BK \quad \text{is Schur} & \text{(9a)} \\ \|G_K\|_\infty < \gamma_\infty, & \text{(9b)} \end{cases}$$

where, according to (4a), $G_K(z) := K(zI - A - BK)^{-1}B$ is the transfer function of system $\Sigma(A+BK, B, K)$. Then:

$\imath$) A matrix $\mathbb{R}^{n\times n} \ni P > 0$ and a constant $r_{\rm i}^2 > 0$ can be explicitly determined such that $\forall\, r_1^2 \geq r_2^2 > r_{\rm i}^2$ the closed loop system (4) is $(\mathcal{E}_{P,r_1^2}, \mathcal{E}_{P,r_2^2})$–stable, with $\mathcal{E}_{P,r^2} := \{x \in \mathbb{R}^n \,|\, x'Px \leq r^2\}$. It holds that:

$$r_{\rm i}^2 = R^2\big(\lambda_{\max}(P - S) + \lambda_{\min}(S)\big), \tag{10}$$

where P is the unique positive definite solution of the following discrete–time algebraic Riccati equation

$$X = F'XF + F'XB(\gamma^2 I - B'XB)^{-1}B'XF + C'C + Q, \tag{11}$$

with $\gamma > \|G_K\|_\infty$ so that $\gamma\cdot\gamma_e < 1$ and $\mathbb{R}^{n\times n} \ni Q > 0$ is any matrix such that $\|G_K\|_\infty + \|Q^{1/2}(zI - F)^{-1}B\|_\infty < \gamma$;

$$\begin{gathered} S = F'PB(\gamma^2 I - B'PB)^{-1}B'PF + C'C + Q; \\ R = \frac{E_0}{\lambda_{\min}(S)}\,\alpha(P); \\ \alpha(P) = \|F'PB\|_2 + \sqrt{\|F'PB\|_2^2 + \lambda_{\min}(S)\|B'PB\|_2}. \end{gathered}$$

$\imath\imath$) Consider the operator $\mathcal{G}^{(I)}$ associated to system (4b) and $\psi = q_e \circ K$. For $r_2^2 \geq r_{\rm i}^2$, let $\mathscr{E}(r_2^2) := \mathscr{E}(\mathcal{E}_{P,r_2^2})$ and

$$\Delta_1 := \inf_{r_2^2 > r_{\rm i}^2} 2\|\mathcal{G}^{(I)}\|_\infty \mathscr{E}(r_2^2). \tag{12}$$

Then, $\forall\, r_1^2 > r_{\rm i}^2$ and $\forall\, \Delta > \Delta_1$, system (4) is $\big(\mathcal{E}_{P,r_1^2}, \mathcal{E}_{P,r_1^2}, Q_n(\Delta)\big)$–stable.

$\imath\imath\imath$) Let $\gamma_e(\Delta)$ be the generalized ℓ_∞–gain of ψ: if

$$\|\mathcal{G}^{(I)}\|_\infty \cdot \gamma_e(\Delta_1) < 1, \tag{13}$$

then $\forall\,\Delta_\star > \Delta_{\inf}$, system (4) is $\big(\mathcal{E}_{P,r_1^2}, \mathcal{E}_{P,r_1^2}, Q_n(\Delta_\star)\big)$–stable, where $\Delta_{\inf}$ is defined in equation (6).

Proof: Part $\imath$: see [15]. Parts $\imath\imath$ and $\imath\imath\imath$ directly follow by Theorem 3.1.$\imath$ and Theorem 3.1.$\imath\imath$, respectively. ■

Remark 4.1: [Mixed H_∞/ℓ_1 control synthesis] The control synthesis stage of Theorem 4.1 can be modified so that, not only the small–gain condition (9b) in H_∞ is met, but also the ℓ_∞–gain of the closed loop input/state operator is minimized (so that the size of the final hypercube $Q_n(\Delta_\star)$ is reduced). Namely, Theorem 4.1 can be restated with problem (9) replaced by the following *mixed H_∞/ℓ_1 control* problem: given $\gamma_\infty \le \frac{1}{\gamma_e}$, find

$$K = \operatorname*{argmin}_{\substack{X \in \mathbb{R}^{m\times n} \text{ such that} \\ \begin{cases} A+BX \text{ is Schur} \\ \|G_X\|_\infty < \gamma_\infty \end{cases}}} \|\mathcal{G}_X^{(I)}\|_\infty \,, \tag{14}$$

where $G_X(z) = X(zI - A - BX)^{-1}B$ and $\mathcal{G}_X^{(I)}$ is the input/state operator of system $\Sigma(A+BX, B, I)$.

To apply Theorem 4.1, the following problems must be faced:

1) Design the input quantizer $q_{\mathcal{U}}$ and analyze the corresponding quantization error in terms of properties (8);
2) Solve problem (9) (or problem (14) if the mixed H_∞/ℓ_1 control synthesis approach is taken);
3) Evaluate the ℓ_∞–gain of the input/state operator $\mathcal{G}^{(I)}$ associated to the closed loop dynamics;
4) Determine $\mathscr{E}(r^2)$ and the generalized ℓ_∞–gain of ψ.

Problem 1) For a given input quantizer $q_{\mathcal{U}}$, the analysis of q_e consists of two steps: first, in order to determine a ϱ_0–external gain (for fixed positive values of ϱ_0) one has to find an upper bound for $\sup_{\|y\|_2 > \varrho_0} \frac{\|q_e(y)\|_2}{\|y\|_2}$; secondly, in order to evaluate E_0, one has to find an upper bound for $\sup_{\|y\|_2 \le \varrho_0} \|q_e(y)\|_2$. This study, at least theoretically, can be done for any input quantizer $q_{\mathcal{U}} : \mathbb{R}^m \to \mathcal{U}$.

Example 1: [The logarithmic quantization of $\mathbb{R}$] Let $u_0 > 0$ and $\theta > 1$. A *logarithmic quantization of $\mathbb{R}$ with parameters* (u_0, θ) is a map $q_{\mathcal{U}} : \mathbb{R} \to \mathcal{U}$, where

$$\mathcal{U} = \{0\} \cup \{\pm\, u_0 \theta^h \,|\, h \in \mathbb{N}\}$$

and $\forall\, y \in \mathbb{R}$, $q_{\mathcal{U}}(y)$ is an element of $\mathcal{U}$ minimizing the distance from y (i.e., $q_{\mathcal{U}}$ is a *nearest neighbor quantizer*). The corresponding quantization error $q_e(y) = q_{\mathcal{U}}(y) - y$ is such that conditions (8) are satisfied with

$$\varrho_0 = \tfrac{u_0(\theta+1)}{2\theta}\,, \quad \gamma_e = \tfrac{\theta-1}{\theta+1} \quad \text{and} \quad E_0 = \tfrac{u_0}{2} \tag{15}$$

(it follows by elementary computations, see [19]). ♣

The analysis of other types of quantizers, including multi–input ones, is reported in [15], [19].

Problem 2) The one in (9) is an instance of the *state feedback* H_∞ control problem (see [11], [6]) known as the "actuator disturbance" case [2]. The following is a particularization to our case of a solution of the general state feedback problem:

Lemma 4.1: If A is *unmixed* (i.e., the eigenvalues of A are so that $|\lambda(A)| \ne 1$), then $\exists\, K \in \mathbb{R}^{m\times n}$ such that $A + BK$ is Schur and $\|G_K\|_\infty < \gamma_\infty$ if and only if there exists $\mathbb{R}^{n\times n} \ni P^* \ge 0$ such that the following conditions hold:

$$P^* = A'\Big(P^* - \tfrac{\gamma_\infty^2-1}{\gamma_\infty^2} P^* B\big(I + \tfrac{\gamma_\infty^2-1}{\gamma_\infty^2} B'P^*B\big)^{-1} B'P^*\Big) A \tag{16a}$$

$$\Big(I - BB'\big(I + \tfrac{\gamma_\infty^2-1}{\gamma_\infty^2} P^*BB'\big)^{-1} P^*\Big) A \quad \text{is Schur} \tag{16b}$$

$$\gamma_\infty^2 I - B'P^*B > 0\,. \tag{16c}$$

A feasible choice for K is the central H_∞ controller:

$$K_c(\gamma_\infty) := -B'\big(I + \tfrac{\gamma_\infty^2-1}{\gamma_\infty^2} P^*BB'\big)^{-1} P^*A\,. \tag{17}$$

Proof: See e.g., [21]. ■

In [11], also the case where A is not unmixed is treated.

Problem 3) It is a standard analysis problem in the ℓ_1 functional space: efficient numerical algorithms to evaluate $\|\mathcal{G}^{(I)}\|_\infty$ are available (see [1], [10]) and a simple analytical approach has been recently proposed in [18].

Problem 4) It is essentially a geometric study that, in principle, can be carried out for any ψ. However, for general input quantizers and large dimension of the input space, this analysis may be quite involved.

Example 2: [Analysis of ψ: logarithmic quantization of $\mathbb{R}$] Let $q_{\mathcal{U}} : \mathbb{R} \to \mathcal{U}$ be a logarithmic quantization of $\mathbb{R}$ with parameters (u_0, θ) and q_e be the corresponding quantization error. Let $K \in \mathbb{R}^{1\times n}$ and $\psi := q_e \circ K : \mathbb{R}^n \to \mathbb{R}$.

$\imath$) For $\mathbb{R}^{n\times n} \ni P > 0$, the function $\mathscr{E}(r_2^2) = \mathscr{E}(\mathcal{E}_{P,r_2^2})$ is continuous and, with $\mu_1 := \sqrt{r_2 K P^{-1} K'}$, one has

$$\mathscr{E}(r_2^2) = \begin{cases} \mu_1 & \text{if} \quad \mu_1 < \frac{u_0}{2} \\ \max\Big\{ \frac{u_0}{2}\,,\ \gamma_e \frac{u_0(\theta+1)}{2} \theta^{n(\mu_1)}\,, \\ \qquad |u_0\theta^{n(\mu_1)+1} - \mu_1| \Big\} & \text{otherwise}\,, \end{cases} \tag{18}$$

where $\gamma_e := \frac{\theta-1}{\theta+1}$ and $n(\mu) := \Big\lfloor \log_\theta \frac{2\mu}{u_0(\theta+1)} \Big\rfloor$.

$\imath\imath$) For $\Delta \ge 0$, the function $\mathscr{E}(\Delta) = \mathscr{E}\big(Q_n(\Delta)\big)$ is continuous and, with $\mu_2 := \|K\|_\infty \frac{\Delta}{2}$, it holds that

$$\mathscr{E}(\Delta) = \begin{cases} \mu_2 & \text{if} \quad \mu_2 < \frac{u_0}{2} \\ \max\Big\{ \frac{u_0}{2}\,,\ \gamma_e \frac{u_0(\theta+1)}{2} \theta^{n(\mu_2)}\,, \\ \qquad |u_0\theta^{n(\mu_2)+1} - \mu_2| \Big\} & \text{otherwise}\,. \end{cases} \tag{19}$$

The easy proofs of these facts can be found in [19]. ♣

Problem 2b) As for the variation of the control synthesis stage proposed in Remark 4.1, there is some literature on mixed H_∞/ℓ_1 control problems (see [7], [22]). We are currently investigating the special type of problem proposed in equation (14). In Theorem 4.2 below, a relation is proved between the H_∞–norm and the ℓ_∞–gain of *externally positive* SISO systems which, to the best of our knowledge, has not been pointed out before. This result provides a useful tool to deal with problem (14) for this special class of systems.

Definition 4.2: [8] System (1) is said to be *externally positive* iff its impulse response $\vec{g}$ is such that $\forall\, i = 1, \dots, q$, $\forall\, j = 1, \dots, m$ and $\forall\, t \in \mathbb{N}$, $g_{i,j}(t) \ge 0$.

Theorem 4.2: [Equivalence of H_∞ and ℓ_1 norms for positive SISO systems] If system (1) is BIBO–stable, externally positive, $e \in \mathbb{R}$ and $y \in \mathbb{R}$, then $\|G\|_\infty = \|\mathcal{G}\|_\infty = |G(1)|$.

Proof: It holds that $\|G\|_\infty \le \|\mathcal{G}\|_\infty$, in fact:

$$\|G\|_\infty = \max_{\theta \in [0,2\pi[} |G(e^{i\theta})| = \max_{\theta \in [0,2\pi[} \Big| \textstyle\sum_{t=0}^{+\infty} g(t) \cdot \frac{1}{e^{i\theta t}} \Big| \le$$
$$\le \max_{\theta \in [0,2\pi[} \textstyle\sum_{t=0}^{+\infty} |g(t)| \cdot \Big|\frac{1}{e^{i\theta t}}\Big| = \|\mathcal{G}\|_\infty\,.$$

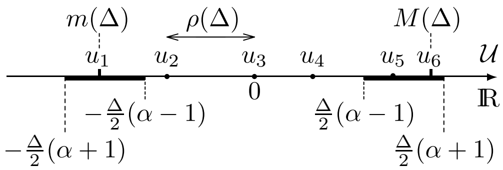

Fig. 1. $\mathcal{U}(\Delta) = \{u_1, u_2, u_3, u_4, u_5, u_6\}$: the thick segments are the intervals where $m(\Delta)$ and $M(\Delta)$ satisfy inequalities (20b–c).

For an externally positive system, $\|\mathcal{G}\|_\infty = |G(1)|$ (see [18]): since $|G(1)| \le \|G\|_\infty$, the thesis follows. ■

In [19], also the way to choose the parameters γ and $\mathbb{R}^{n\times n} \ni Q > 0$ involved in the definition of $r_{\rm i}^2$ (see equations (10) and (11)) is discussed. All the issues on the application of Theorem 4.1 are exemplified in Section V and a comparison between the two proposed approaches (i.e., H_∞ *vs* mixed H_∞/ℓ_1 control synthesis) is presented.

A. Single–input reachable systems

Both the proposed stabilization methods rely on the control synthesis in H_∞. For single–input reachable systems, a solution is presented which is entirely built on ℓ_1 theory. Not only this result provides an interpretation in terms of ℓ_1 control of the stabilization technique presented in [16], but it also extends that approach to a wider class of controllers.

Consider system (2) where $m = 1$ and the pair (A, B) is reachable. Assume that the system is in controller form with $z^n - a_n z^{n-1} - \cdots - a_2 z - a_1$ being the characteristic polynomial of A. If $K \in \mathbb{R}^{1\times n}$ is such that the characteristic polynomial $z^n - f_n z^{n-1} - \cdots - f_2 z - f_1$ of the matrix $F := A + BK$ satisfies $f := \sum_{i=1}^n |f_i| < 1$, then the practical stability properties of the closed loop system with $u(x) = q_{\mathcal{U}}(Kx)$ can be analyzed through Proposition 3.1. A drawback of this approach is that, in the small–gain condition $\frac{\gamma_e(\Delta)}{1-f} < 1$, both the term $\frac{1}{1-f}$ related with the ℓ_∞–gain of the linear system $\Sigma(F, B, I)$ and the parameter $\gamma_e(\Delta)$ taking the quantization error into account are depending on K. If $q_{\mathcal{U}}$ is a nearest neighbor quantizer, it is possible to obtain a practical stabilization result where the small–gain condition $\frac{\gamma_e(\Delta)}{1-f} < 1$ is replaced by a similar condition but the dependence on K is restricted to the linear part of the dynamics. Thus, similarly to Theorem 4.1, the analysis of the nonlinearity due to quantization can be carried out apart from the design of the control gain K.

To state the result, it is useful to introduce the following notation (see also [16]). Assume that $\alpha := \sum_{i=1}^n |a_i| \ge 1$ (if $\alpha < 1$, the matrix A is Schur) and, for $\Delta > 0$, let:

$$\begin{aligned} \mathcal{U}(\Delta) &:= \mathcal{U} \cap \left[-\tfrac{\Delta}{2}(\alpha+1) \,,\, \tfrac{\Delta}{2}(\alpha+1) \right] \\ m(\Delta) &:= \min \mathcal{U}(\Delta), \quad M(\Delta) := \max \mathcal{U}(\Delta) \end{aligned}$$

and

$$\rho(\Delta) := \begin{cases} \sup\left\{ b - a \,\middle|\,]a, b[\subseteq [m(\Delta), M(\Delta)] \text{ and } \right. \\ \qquad \left.]a, b[\cap \mathcal{U}(\Delta) = \emptyset \right\} \quad \text{if } \#\mathcal{U}(\Delta) > 1 \\ +\infty \quad \text{otherwise} \end{cases}$$

be the *dispersion* of $\mathcal{U}(\Delta)$ (see Fig. 1).

Theorem 4.3: [(X_0, Ω)–stabilization of single–input systems] Consider system (2) with $u \in \mathbb{R}$. Assume that it is represented in the controller form coordinates and $\alpha = \sum_{i=1}^n |a_i| \ge 1$. Let $K \in \mathbb{R}^{1\times n}$ be such that $F := A + BK$ satisfies $f := \sum_{i=1}^n |f_i| < 1$. Consider $\Delta_0 > 0$ such that

$$\begin{cases} \frac{\rho(\Delta_0)/\Delta_0}{1-f} < 1 & \text{(20a)} \\ m(\Delta_0) < -\frac{\Delta_0}{2}(\alpha-1) & \text{(20b)} \\ M(\Delta_0) > \frac{\Delta_0}{2}(\alpha-1)\,, & \text{(20c)} \end{cases}$$

$\mathcal{U} = \mathcal{U}(\Delta_0)$ holds[2] and let $q_{\mathcal{U}} : \mathbb{R} \to \mathcal{U}(\Delta_0)$ be a nearest neighbor quantizer. Then the following is well–defined

$$\Delta_{\inf}(f) := \max\left\{ \Delta < \Delta_0 \,|\, \rho(\Delta) = (1-f)\Delta \right\}$$

and $\forall\, \Delta_\star > \Delta_{\inf}(f)$, the control law $u(x) = q_{\mathcal{U}}(Kx)$ is $\big(Q_n(\Delta_0), Q_n(\Delta_\star)\big)$–stabilizing.

Proof: It is based on Theorem 3.1: see [19]. ■

In [19], it is also shown that the small–gain condition $\frac{\gamma_e(\Delta_0)}{1-f} < 1$ implies conditions (20). Thus, the range of applicability of Theorem 4.3 is wider than that of the technique which may be directly derived from Proposition 3.1.

V. Example

Consider the following system:

$$\begin{cases} x(t+1) = \begin{pmatrix} 0 & 1 \\ -1 & 5/2 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u(t) \\ u \in \mathcal{U} \subset \mathbb{R}\,, \end{cases} \tag{21}$$

where $\mathcal{U}$ is a logarithmically quantized set with parameters $(u_0, \theta) = (1\,,\,2)$. Let us illustrate the two proposed methods for the practical stabilization (i.e., by means of Theorem 4.1 and the mixed H_∞/ℓ_1 approach described in Remark 4.1). The pair (A, B) is not reachable but it is stabilizable, the eigenvalues of A are $\lambda_1(A) = 1/2$ and $\lambda_2(A) = 2$.

Let $K = (\,K_1\ K_2\,) \in \mathbb{R}^{1\times 2}$ be such that $A + BK$ is Schur: the transfer matrix $G_K^{(I)}(z)$ of system $\Sigma(A + BK, B, I)$ is

$$G_K^{(I)}(z) = \begin{pmatrix} \frac{1}{z-(2+K_1+2K_2)} \\ \frac{2}{z-(2+K_1+2K_2)} \end{pmatrix}$$

and (see [18])

$$\|\mathcal{G}_K^{(I)}\|_\infty = \frac{2}{1 - |2 + K_1 + 2K_2|}\,. \tag{22}$$

I *(Input quantization)*– Consider a nearest neighbor quantizer $q_{\mathcal{U}}$: by equation (15), the quantization error q_e satisfies conditions (8) with $\varrho_0 = 3/4$, $\gamma_e = 1/3$ and $E_0 = 1/2$.

II *(Control synthesis in H_∞)*– Apply Lemma 4.1 with $\gamma_\infty \le 1/\gamma_e = 3$. For single–input systems, problem (9) is feasible if and only if $\gamma_\infty > \Pi_{\lambda_{\rm u}(A)\in\mathcal{S}_{\rm u}(A)} |\lambda_{\rm u}(A)| = 2$, where $\mathcal{S}_{\rm u}(A)$ is the set of the unstable eigenvalues of A (see [9]). Solving system (16) with $\gamma_\infty = 2.01$, by equation (17) one obtains $K := K_{\rm c}(2.01) = (\ 0.6645 \quad -1.3289\ )$.

III *(Closed loop analysis: H_∞ stage)*– It holds that $\|G_K\|_\infty = 2.0066$. With $\gamma = 2.999$ and $Q = 0.194 \cdot I$, equation (11) is solved by

$$P = \begin{pmatrix} 0.7990 & -0.9630 \\ -0.9630 & 1.9992 \end{pmatrix}.$$

By equation (10), $r_{\rm i}^2 = 4.0579$. Therefore, $\forall\, r_1^2 \ge r_2^2 > 4.0579$, system (21) controlled with $u(x) = q_{\mathcal{U}}(Kx)$ is

[2] This is a technical assumption that can be always satisfied simply by saturating the controller and neglecting the inputs out of $\mathcal{U}(\Delta_0)$.

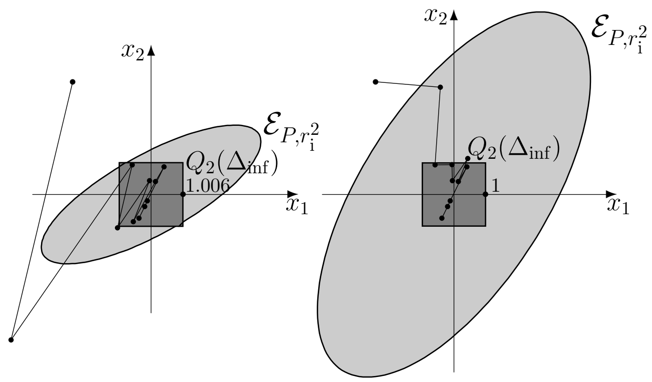

Fig. 2. Comparison between the final invariant ellipsoid $\mathcal{E}_{P,r_i^2}$ and the final hypercube $Q_2(\Delta_{\inf})$, and representation of the trajectory starting from $x(0) = (-2.48 \;\; 3.57)$, in the two cases of the example in Section V: on the left, synthesis in H_∞; on the right, mixed H_∞/ℓ_1 control synthesis.

$\big(\mathcal{E}_{P,r_1^2}, \mathcal{E}_{P,r_2^2}\big)$–stable.

IV *(Closed loop analysis: ℓ_1 stage)*– Let us compute Δ_1 (see equation (12)). By equation (22), one has $\|\mathcal{G}_K^{(I)}\|_\infty = 2.0133$. As for $\mathscr{E}(r_2^2)$, we are in the right framework of Example 2: since $\mathscr{E}(r_2^2)$ is continuous and non–decreasing, then $\inf_{r_2^2 > r_i^2} \mathscr{E}(r_2^2) = \mathscr{E}(r_i^2)$. Hence, by equation (18), $\mathscr{E}(r_i^2) = 1/2$ and $\Delta_1 = 2\|\mathcal{G}_K^{(I)}\|_\infty \mathscr{E}(r_i^2) = 2.0133$.

Part iii of Theorem 4.1 cannot be applied because the small–gain condition (13) is not satisfied. In fact: since $\mathscr{E}(\Delta)$ is continuous (see Example 2), then $\mathscr{E}^+(\Delta) = \mathscr{E}(\Delta)$ and $\gamma_e(\Delta) = \frac{\mathscr{E}(\Delta)}{\Delta/2}$. By equation (19), one computes $\mathscr{E}(\Delta_1) = 1/2$ and $\gamma_e(\Delta_1) = 1/\Delta_1$, so that $\|\mathcal{G}_K^{(I)}\|_\infty \cdot \gamma_e(\Delta_1) = 1$.

V *(Final result)*– With $u(x) = q_{\mathcal{U}}(Kx)$, $\forall\, r_1^2 > 4.0579$ and $\forall\, \Delta > 2.0133$, the closed loop dynamics is $\big(\mathcal{E}_{P,r_1^2}, \mathcal{E}_{P,r_1^2}, Q_2(\Delta)\big)$–stable.

Let us consider the variation proposed in Remark 4.1:

IIb *(Mixed H_∞/ℓ_1 control synthesis)*– Thanks to equation (22), problem (14) can be simplified to the following equivalent form: given $\gamma_\infty \le \frac{1}{\gamma_e}$, find

$$K = \underset{\substack{X \in \mathbb{R}^{1\times 2} \text{ such that} \\ \begin{cases} A + BX \text{ is Schur} \\ \|G_X^{(\mathrm{siso})}\|_\infty < \gamma_\infty \end{cases}}}{\operatorname{argmin}} \|\mathcal{G}_X^{(\mathrm{siso})}\|_\infty, \qquad (23)$$

where $G_X^{(\mathrm{siso})}(z) := \frac{2}{z-(2+X_1+2X_2)}$. By Theorem 4.2, it holds that $\|G_X^{(\mathrm{siso})}\|_\infty \le \|\mathcal{G}_X^{(\mathrm{siso})}\|_\infty$. Hence, $\forall\, \gamma_\infty > 2$, the constraint $\|G_X^{(\mathrm{siso})}\|_\infty < \gamma_\infty$ can be removed and, according to equation (22), a solution to problem (23) is given by $K = (\,0 \;\; -1\,)$ which yields $\|\mathcal{G}_K^{(\mathrm{siso})}\|_\infty = \|G_K^{(\mathrm{siso})}\|_\infty = 2$.

III– By proceeding as in the previous case we find that, with

$$P = \begin{pmatrix} 3.7165 & -1.7587 \\ -1.7587 & 2.0785 \end{pmatrix}$$

and $r_i^2 = 41.7306$, $\forall\, r_1^2 \ge r_2^2 > r_i^2$, the closed loop system with $u(x) = q_{\mathcal{U}}(Kx)$ is $\big(\mathcal{E}_{P,r_1^2}, \mathcal{E}_{P,r_2^2}\big)$–stable.

IV– Similarly to the previous case, one computes $\Delta_1 = 2\|\mathcal{G}_K^{(I)}\|_\infty \mathscr{E}(r_i^2) = 7.1457$. The small–gain condition (13) is satisfied, in fact $\|\mathcal{G}_K^{(I)}\|_\infty \cdot \gamma_e(\Delta_1) = 0.5598$. To determine $\Delta_{\inf}$ (see equation (6)), we can follow a recursive procedure: indeed the sequence defined by $\Delta_{k+1} = 2\|\mathcal{G}_K^{(I)}\|_\infty \mathscr{E}(\Delta_k)$ converges to $\Delta_{\inf} = 2$ (see [19]).

V *(Final result)*– With $u(x) = q_{\mathcal{U}}(Kx)$, $\forall\, r_1^2 > 41.7306$ and $\forall\, \Delta_\star > 2$, the closed loop dynamics is $\big(\mathcal{E}_{P,r_1^2}, \mathcal{E}_{P,r_1^2}, Q_2(\Delta_\star)\big)$–stable.

In both cases, Fig. 2 allows one to appreciate the contribution brought by the application of the ℓ_1 theory and the non–conservativeness of the obtained results. Note that $Q_2(\Delta_\star)$ is not positively invariant but, eventually, the trajectories are guaranteed to remain confined therein (see Remark 2.2).

VI. Conclusion

We have introduced a mixed H_∞/ℓ_1 approach for the stabilization of quantized input linear systems. Among the future directions of research, we find interesting to study the peculiarities of the proposed technique as for positive systems.

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