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Strong Solutions and Maximal Solutions of Generalized Algebraic Riccati Equations

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***Abstract*— In this paper, we present a comparison theorem for the solutions of two generalized algebraic Riccati equations (GAREs) coming from two different systems. We show that the so-called strong solutions, whose related matrix pencils have all their finite eigenvalues in the closed left half plane, are maximal. The results obtained generalize the existing monotonicity results of algebraic Riccati equations. As an application of the above results, we provide a parameterization of all strong solutions of the GARE related to the singular spectral factorization of a proper transfer function with finite and infinite imaginary axis zeros.**

I. INTRODUCTION

We consider the following generalized algebraic Riccati equation (GARE), which arises from an infinite-horizon linear-quadratic optimal control problem of descriptor systems ([13], [17], and [21]) or from the spectral factorization of a proper transfer function with finite and infinite imaginary axis zeros in [12],

$$\begin{cases} E^T X = X^T E, \\ A^T X + X^T A - X^T R X + Q = 0, \end{cases} \tag{1}$$

where E, A, R and Q are $n \times n$ real matrices with $R = R^T \geq 0$, $Q = Q^T$, and $-sE + A$ is a regular pencil, i.e., $|-sE+A|$ is not identically zero. E is usually not invertible.

Note that for an infinite-horizon linear-quadratic optimal control problem of descriptor systems, [13], [17], and [21] proposed the notion of *admissible solution*, that is, the real solution such that $-sE + A - RX$ is regular, impulsive-free and its finite eigenvalues are in the open left half plane. With the admissible solution, the optimal control problem satisfying certain assumptions has been solved. Note that for the case of E being an identity matrix, the admissible solution is nothing but the stabilizing solution for an algebraic Riccati equation.

On the other hand, for the spectral factorization of a proper transfer function with finite and infinite imaginary axis zeros, [12] studied the so-called *strong solutions* X_+, that is, the real solution such that $-sE + A - RX_+$ are regular and all their finite eigenvalues are in the *closed* left half plane. [12] proposes a method to find a strong solution of the GARE via solving a generalized eigenvector problem.

In this paper, we clarify some properties of the strong solution of (1). Before we proceed further, we will briefly review some related results in the literature. Consider the case of E being an identity matrix for (1). In this case, $X = X^T$ must hold and GARE (1) is reduced to a usual ARE which has been studied extensively, to name a few, see [5], [9], [11], [19], and the references therein. Note that the notion of strong solution was first presented in [2] for a discrete ARE. Several existence conditions and properties such as uniqueness for the strong solution were provided in [3] and [4]. Under the assumption of controllability of (A, R), [9] presents the existence and uniqueness theorems for an ARE. Under assumption of stabilizability of (A, R), [5] shows the existence and properties of maximal solutions; [19] obtains a comparison theorem for the solutions of two AREs and the monotonicity results of AREs, which shows that any solution of the ARE yielding $A - RX$ with eigenvalues in the closed left half plane, that is, the strong solution, is maximal; and [11] gives some improvements on [5] and [19].

The strong solution for GARE (1) has not been investigated much yet in comparison with its state-space counterpart. For example, the uniqueness of the strong solution has not been reported. In this paper, under the assumption of the stabilizability of (E, A, R) (see Section II for the definition), we investigate the properties of uniqueness and maximum of the strong solutions of GARE (1). To this end, we develop a comparison theorem for the solutions of two GAREs coming from two different systems. We show that the strong solutions are maximal in the way defined later. Thus the monotonicity of maximal solutions of GARE (1) is proved. The results obtained in this paper contain those of [19] as a special case of $E = I$. As an application of the above results to the strong solutions of the GARE in [12], we give a direct answer to the question of the uniqueness without needing to explore the algorithm involving the generalized eigenvalue problem in [12]; we also provide the parameterization of all strong solutions for the GARE.

For the infinite-horizon linear-quadratic optimal control problem of descriptor systems, Q in (1) is usually assumed to be $Q \geq 0$, and the solutions studied there are admissible ones. In this paper, Q is just assumed to be symmetric, and $-sE + A - RX$ is allowed to have eigenvalues on the finite imaginary axis and to have impulsive modes. These are essence differences. See Remark 1 for further discussion.

The following notations are used in this paper. The open left complex plane, open right half complex plane and open complex plane are denoted by $\mathbb{C}_-$, $\mathbb{C}_+$, and $\mathbb{C}$, respectively.

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The $j\omega-$axis and $j\omega-$axis with infinity are denoted by Ω and Ω_e, respectively. $\mathbb{R}^n$ denotes the real space of dimension n. $\mathbb{R}^{m\times r}$ denotes the set of all $m \times r$ constant real matrices. I_r denotes the identity matrix of size $r \times r$. $\operatorname{Im} A$ and $\operatorname{Ker} A$ denote the image space and the null space of matrix A, respectively. $|A|$ denotes the determinant of A. $\sigma_f(-sE+A)$ denotes the set of finite eigenvalues of regular pencil $-sE+A$. We denote $G^{\sim}(s) := G^T(-s)$ and we use the notation

$$C(sI-A)^{-1}B+D := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

II. BASIC KNOWLEDGE OF DESCRIPTOR SYSTEMS

In this section, we will review some basic knowledge about the descriptor systems [1] and [16].

The finite eigenvalues of a regular pencil $-sE+A$ of size $n \times n$ (which are the roots of $|-sE+A| = 0$) are called the finite dynamic modes of $-sE+A$. The infinite eigenvalues of $-sE+A$ are defined to be the zero eigenvalues of $-sA+E$. The infinite eigenvalues corresponding to grade-one infinite generalized eigenvectors, v_i^1, that satisfy $Ev_i^1 = 0$ are called the nondynamic modes of $-sE+A$. The infinite eigenvalues corresponding to the grade-k ($k \geq 2$) infinite generalized eigenvectors, v_i^k, that satisfy $Ev_i^k = Av_i^{k-1}$ are called the impulsive modes of $-sE+A$. Let $r = \operatorname{rank} E$ and $m = \deg|-sE+A|$. Then, $-sE+A$ has m finite dynamic modes, $r-m$ impulsive modes, and $n-r$ nondynamic modes. $-sE+A$ is said to be *impulsive free* if it has no impulsive modes, i.e., $r = m$.

The following statement will be found useful in this paper: $-sE+A$ is impulse-free if and only if there exist nonsingular matrices M_1 and V_1 such that

$$M_1^{-1}(-sE+A)V_1 = \begin{bmatrix} -sI+A_f & 0 \\ 0 & I \end{bmatrix}. \tag{2}$$

For regular $-sE+A$, triple (E,A,R) is said to be stabilizable [1] if

$$\operatorname{rank} \begin{bmatrix} -sE+A & R \end{bmatrix} = n, \quad \forall s \in \mathbb{C}_+ \cup \Omega_e, \tag{3}$$

which at $s = \infty$ means that

$$\operatorname{Im} E + A\operatorname{Ker} E + \operatorname{Im} R = \mathbb{R}^n. \tag{4}$$

III. COMPARISON THEOREM, STRONG SOLUTION AND MAXIMAL SOLUTION

First, we present the following inertia result with its proof given in Appendix A.

Theorem 1: Suppose E, A, R, and Q are in $\mathbb{R}^{n\times n}$ such that $-sE+A$ is regular, $\lambda_f(-sE+A) \in \mathbb{C}_- \cup \Omega$, (E,A,R) is stabilizable, and $R = R^T \geq 0$, $Q = Q^T \geq 0$. If X is a real solution of

$$\begin{cases} E^TX = X^TE, \\ A^TX + X^TA + X^TRX + Q = 0, \end{cases} \tag{5}$$

then

$$E^TX \geq 0. \tag{6}$$

Note that the sign of the quadratic term of GARE (5) is opposite to that in GARE (1). In addition, in Theorem 1, $Q \geq 0$ and $\lambda_f(-sE+A) \in \mathbb{C}_- \cup \Omega$ are also assumed for GARE (5).

Next, using Theorem 1, we can obtain a comparison theorem for two different GAREs.

Theorem 2: Suppose E, A, R, and Q are in $\mathbb{R}^{n\times n}$ such that $-sE+A$ is regular, (E,A,R) is stabilizable, and $R = \mathbb{R}^T \geq 0$, $Q = Q^T$. Let $X_+ \in \mathbb{R}^{n\times n}$ be a strong solution of GARE (1) such that $-sE+A-RX_+$ is regular and

$$\lambda_f(-sE+A-RX_+) \subset \mathbb{C}_- \cup \Omega. \tag{7}$$

Let X_1 be any real solution of another GARE

$$\begin{cases} E^TX_1 = X_1^TE, \\ A_1^TX_1 + X_1^TA_1 - X_1^TR_1X_1 + Q_1 = 0, \end{cases} \tag{8}$$

where A_1, $R_1 = R_1^T$, $Q_1 = Q_1^T$ are in $\mathbb{R}^{n\times n}$. Define

$$H := \begin{bmatrix} A & -R \\ -Q & -A^T \end{bmatrix}, \quad H_1 := \begin{bmatrix} A_1 & -R_1 \\ -Q_1 & -A_1^T \end{bmatrix}, \tag{9}$$

$$J := \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}. \tag{10}$$

If

$$J(H-H_1) = \begin{bmatrix} Q-Q_1 & (A-A_1)^T \\ A-A_1 & R_1-R \end{bmatrix} \geq 0, \tag{11}$$

then

$$E^TX_+ \geq E^TX_1. \tag{12}$$

Proof: Let $\Delta := X_+ - X_1$. Take the difference of (1) with X_+ and (8), we obtain

$$\begin{cases} E^T\Delta = \Delta^TE, \\ (A-RX_+)^T\Delta + \Delta^T(A-RX_+) + \Delta^TR\Delta \\ \quad + \begin{bmatrix} I & X_1^T \end{bmatrix} J(H-H_1) \begin{bmatrix} I \\ X_1 \end{bmatrix} = 0. \end{cases} \tag{13}$$

Since $E^T\Delta = \Delta^TE$ holds obviously, we just show the second equation in (13). From (1) and eqn:garea, we obtain

$$A^T\Delta + \Delta^TA - X_+^TRX_+ + X_1^TRX_1 + \tilde{Q} = 0,$$

where

$$\tilde{Q} := A^TX_1 + X_1^TA - X_1^TRX_1 + Q. \tag{14}$$

Using $X_1^TRX_1 = (X_+-\Delta)^TR(X_+-\Delta) = X_+^TRX_+ - \Delta^TRX_+ - X_+^TR\Delta + \Delta^TR\Delta$ and

$$\begin{aligned} \tilde{Q} &= \begin{bmatrix} I & X_1^T \end{bmatrix} JH \begin{bmatrix} I \\ X_1 \end{bmatrix} \\ &= \begin{bmatrix} I & X_1^T \end{bmatrix} J(H-H_1) \begin{bmatrix} I \\ X_1 \end{bmatrix}, \end{aligned} \tag{15}$$

[1]The definition of (E,A,R) is stabilizable in this paper is equivalent to that (E,A,R) is finite dynamics stabilizable and impulsive controllable in [15].

we prove the second equation in (13).

Since X_+ is a strong solution of GARE (1), then $-sE + A - RX_+$ is regular and $\sigma_f(-sE + A - RX_+) \in \mathbb{C}_- \cup \Omega$. Moreover, since (E, A, R) is stabilizable, so is $(E, A - RX_+, R)$. From (11), since the constant matrix of (13) is positive semidefinite, together with $R \geq 0$, applying Theorem 1 for GARE (13) shows $E^T\Delta \geq 0$. This completes the proof of Theorem 2. ∎

The following result is a direct consequence of Theorem 2 about the uniqueness and maximum of the strong solutions of GARE (1). Here, we say that a real solution X_m is *maximal* if $E^TX_m \geq E^TX$ for any other real solution of GARE (1).

Theorem 3: Suppose E, A, R and Q are in $\mathbb{R}^{n\times n}$ such that $-sE + A$ is regular, (E, A, R) is stabilizable, and $R = R^T \geq 0$, $Q = Q^T$. Let X_+ be a strong solution of GARE (1). Then E^TX_+ is maximal among all the real solutions of GARE (1), i.e., X_+ is maximal. Moreover, E^TX_+ is unique.

Proof: Let $H = H_1$ in (9). According to Theorem 2, E^TX_+ is maximal among all the real solutions of GARE (1). Moreover, E^TX_+ is unique. Indeed, let X_+ and $\widehat{X}_+$ be two strong solutions of GARE (1), then $E^TX_+ \geq E^T\widehat{X}_+$ and $E^T\widehat{X}_+ \geq E^TX_+$, it follows that $E^TX_+ = E^T\widehat{X}_+$. ∎

IV. APPLICATION AND EXAMPLE

A. Application to the GARE related to Singular Factorization

We present an application of the comparison theorem to the GARE related to the spectral factorization of a proper transfer function with infinite and finite $j\omega$-axis zeros. Such factorization and its related GARE are useful to solve the H_∞ control problem with finite and infinite imaginary axis zeros [20].

Consider the $m \times p$ proper system

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \tag{16}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{m\times n}$, and $D \in \mathbb{R}^{m\times p}$.

The following assumptions are made:

- **A1** (A, B) is stabilizable and (C, A) is detectable;
- **A2** A has no eigenvalues in open right half plane;
- **A3** The system matrix associated to $G(s)$, defined by

$$\Gamma(s) := \begin{bmatrix} -sI + A & B \\ C & D \end{bmatrix} \tag{17}$$

has full normal column rank $n + p$.

From assumption **A3**, $m \geq p$, i.e., $G(s)$ is a tall or square system but D is not necessarily full column rank and $G(s)$ may have invariant zeros on the $j\omega$-axis.

The main objective of [12] is to solve the spectral factorization for

$$\Phi(s) = G^\sim(s)G(s), \tag{18}$$

i.e., to find a *spectral factor* $\Pi(s)$ such that

$$\Phi(s) = \Pi^\sim(s)\Pi(s) \tag{19}$$

and $\Pi(s)$ has neither poles nor zeros in $\mathbb{C}_- \cup \Omega$. Define

$$E_a := \begin{bmatrix} I_n & 0 \\ 0 & 0_{p\times p} \end{bmatrix}, \quad A_a := \begin{bmatrix} A & B \\ 0 & I_p \end{bmatrix}, \tag{20}$$

$$B_a := \begin{bmatrix} 0_{n\times p} \\ -I_p \end{bmatrix}, \quad Q_a := \begin{bmatrix} C^TC & C^TD \\ D^TC & D^TD - I_p \end{bmatrix}. \tag{21}$$

We recall the following result from [12]:

Lemma 1: [12] Consider the proper system with its stabilizable and detectable realization given in (17) which satisfies the assumptions **A1–3**. Then there exists a spectral factorization of the spectral density matrix $\Phi(s)$ satisfying (19) such that $\Pi(s)$ has neither poles nor zeros in $\mathbb{C}_- \cup \Omega$. Moreover, GARE

$$\begin{cases} X^TE_a = E_a^TX, \\ X^TA_a + A_a^TX - X^TB_aB_a^TX + Q_a = 0 \end{cases} \tag{22}$$

always has a strong solution X given by

$$X = \begin{bmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{bmatrix}, \; X_{11} = X_{11}^T \in \mathbb{R}^{n\times n}, \; X_{22} \in \mathbb{R}^{p\times p}, \tag{23}$$

and the spectral factor is given by

$$\Pi(s) = \left[\begin{array}{c|c} -sI + A & B \\ \hline -X_{21} & I_p - X_{22} \end{array}\right]. \tag{24}$$

We present the following remark about GARE (22).

Remark 1: In comparison with the GARE arising from LQ optimal for descriptor systems, the following differences can be observed. If D is not of full column rank, Q_a in (22) is not positive semidefinite, also there exists no solution such that $-sE_a + A_a - B_aB_a^TX$ is impulsive free. Furthermore, if $G(s)$ has invariant zeros on the finite imaginary axis, $-sE_a + A_a - B_aB_a^TX$ has eigenvalues on the finite imaginary axis for any solution of GARE (22). Thus, GARE (22) may not have an admissible solution under the assumptions **A1–3**.

In [12], the existence of a strong solution of GARE (22) is shown by providing a solution via solving a generalized eigenvector problem. However, whether the strong solution X in Lemma 1 is unique or not is not discussed.

Here, we show that the strong solution X is not unique, but E_a^TX is unique. Moreover, we give a parameterization of all the solutions to GARE (22). By applying Theorems 2 and 3, we have the following results.

Theorem 4: With the quantities as defined in Lemma 1. Suppose that

$$X = \begin{bmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{bmatrix}$$

is a strong solution of GARE (22). Then

(i) E_a^TX is unique and $E_a^TX \geq 0$, i.e., X_{11} is unique and $X_{11} \geq 0$.

(ii) Every strong solution of the GARE is

$$\begin{bmatrix} X_{11} & 0 \\ VX_{21} & I_p - V(I_p - X_{22}) \end{bmatrix}, \tag{25}$$

where $V \in \mathbb{R}^{p\times p}$ is any unitary matrix, i.e., $V^T V = I_p$.

Proof: (i) From (20) and (21), we obtain

$$\begin{bmatrix} -sE_a + A_a & -B_a B_a^T \end{bmatrix} = \begin{bmatrix} -sI_n + A & B & 0 & 0 \\ 0 & I_p & 0 & -I_p \end{bmatrix}.$$

Hence, the stabilizability of (A, B) implies that of $(E_a, A_a, -B_a B_a^T)$. Since X is a strong solution of (22), by using Theorem 3, we know that

$$E_a^T X = \begin{bmatrix} X_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

is unique, i.e., X_{11} is unique. It is easy to see that

$$X_1 := \begin{bmatrix} 0_{n\times n} & 0 \\ 0 & I_p \end{bmatrix}$$

is a solution of

$$\begin{cases} E_a^T X_1 = X_1^T E_a, \\ A_a^T X_1 + X_1^T A_a - X_1^T B_a B_a^T X_1 + Q_{a1} = 0, \end{cases} \tag{26}$$

where

$$Q_{a1} := \begin{bmatrix} 0_{n\times n} & 0 \\ 0 & -I_p \end{bmatrix}.$$

Since only the constant matrices in GARE (26) and GARE (22) are different, and $Q_a - Q_{a1} \geq 0$, then condition (11) in Theorem 2 holds for the above two GAREs, thus

$$E_a^T X = \begin{bmatrix} X_{11} & 0 \\ 0 & 0 \end{bmatrix} \geq E_a^T \begin{bmatrix} 0 & 0 \\ 0 & I_p \end{bmatrix} = 0_{(n+p)\times(n+p)}.$$

(ii) Since X_{11} is unique, the freedom of X (if it has) is contained only in X_{12} and X_{22}. It follows from (22) that

$$\begin{cases} X_{11}A + A^T X_{11} + C^T C = X_{21}^T X_{21}, \\ B^T X_{11} + D^T C = -(I_p - X_{22})^T X_{21}, \\ D^T D = (I_p - X_{22})^T (I_p - X_{22}), \end{cases} \tag{27}$$

which follows that

$$L := \begin{bmatrix} X_{11}A + A^T X_{11} + C^T C & X_{11}B + C^T D \\ B^T X_{11} + D^T C & D^T D \end{bmatrix} \tag{28}$$

is unique and

$$L = U^T U \geq 0, \tag{29}$$

where

$$U := \begin{bmatrix} -X_{21} & I_p - X_{22} \end{bmatrix} \in \mathbb{R}^{p\times(n+p)}. \tag{30}$$

From the assumption **A3**, the system matrix associated to $\Pi(s)$, defined by

$$\begin{bmatrix} -sI + A & B \\ -X_{21} & I_p - X_{22} \end{bmatrix}$$

has full normal rank $n + p$. Hence, together with (29) we obtain

$$\operatorname{rank} L = \operatorname{rank} U = \operatorname{rank} \begin{bmatrix} -X_{21} & I_p - X_{22} \end{bmatrix} = p. \tag{31}$$

Let

$$\widehat{X} = \begin{bmatrix} X_{11} & 0 \\ \widehat{X}_{21} & \widehat{X}_{22} \end{bmatrix}$$

be any another strong solution and let

$$\widehat{U} := \begin{bmatrix} -\widehat{X}_{21} & I_p - \widehat{X}_{22} \end{bmatrix} \in \mathbb{R}^{p\times(n+p)}.$$

Then,

$$L = \widehat{U}^T \widehat{U} = U^T U \tag{32}$$

Thus, $\operatorname{Ker} U = \operatorname{Ker} \widehat{U}$ and $\operatorname{Im} U^T = \operatorname{Im} \widehat{U}^T$. Therefore, there exists a $V \in \mathbb{R}^{p\times p}$ such that $\widehat{U}^T = U^T V^T$. From (32), we have $U^T(I - V^T V)U = 0$. Since U has full row rank, it follows that $V^T V = I_p$. We obtain from $\widehat{U} = VU$ that

$$\widehat{X}_{21} = V X_{21}, \tag{33}$$

$$\widehat{X}_{22} = I_p - V(I_p - X_{22}). \tag{34}$$

On the other hand, it is easy to check that (25) is a strong solution of GARE (22) for any unitary matrix V. This completes the proof. ■

B. Numerical Example

Consider the spectral factorization of (18) with

$$G(s) = \begin{bmatrix} \dfrac{s(s-2)}{(s+1)^3} \\ \dfrac{s(s-2)}{(s+1)^3} \end{bmatrix} = \left[\begin{array}{ccc|c} -3 & -3 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1 & -2 & 0 & 0 \\ 1 & -2 & 0 & 0 \end{array}\right].$$

Solving GARE (22), we obtain a solution X expressed in (23) with

$$X_{11} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$X_{21} = \begin{bmatrix} -\sqrt{2}, & -2\sqrt{2}, & 0 \end{bmatrix}, \quad X_{22} = 1.$$

We obtain a spectral factor

$$\Pi(s) = \left[\begin{array}{ccc|c} -3 & -3 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline \sqrt{2} & 2\sqrt{2} & 0 & 0 \end{array}\right] = \frac{\sqrt{2}(s+2)}{(s+1)^3}.$$

V. CONCLUSIONS

A comparison theorem for the solutions of two different GAREs has been proposed in this paper. It has been shown that the strong solutions are maximal. The results obtained in this paper generalize of the existing monotonicity results of algebraic Riccati equations developed in [19]. As an application of the results, a parameterization of all strong solutions of the GARE related to the singular spectral factorization has been provided. The results obtained give a new insight into the solutions of GAREs.

Appendix A Proof of Theorem 1

For a given real X, there exist nonsingular matrices M and N such that

$$M^T X N = \begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix}, \quad |W| \neq 0.$$

Accordingly, define

$$\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} := M^{-1}EN,$$
$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} := M^{-1}AN,$$
$$\begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix} := M^{-1}RM^{-T},$$
$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} = N^T Q N.$$

Pre-multiplying and post-multiplying two equations in (5) by N^T and N, respectively, we have

$$\begin{bmatrix} E_{11}^T W & 0 \\ E_{12}^T W & 0 \end{bmatrix} = \begin{bmatrix} W^T E_{11} & W^T E_{12} \\ 0 & 0 \end{bmatrix}, \tag{A1}$$

$$\begin{bmatrix} A_{11}^T W + W^T A_{11} + W^T R_{11} W + Q_{11} & W^T A_{12} + Q_{12} \\ A_{12}^T W + Q_{12}^T & Q_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{A2}$$

From (A1) and $|W| \neq 0$, we have $E_{12} = 0$ and $E_{11}^T W = W^T E_{11}$, i.e.,

$$W^{-T} E_{11}^T = E_{11} W^{-1}. \tag{A3}$$

From (A2), we obtain $Q_{22} = 0$. Hence, it follows from $Q \geq 0$ and $N^T Q N \geq 0$ that $Q_{12} = 0$. Consider the block $(1,2)$ of (A2). Due to $|W| \neq 0$, $A_{12} = 0$. Pre-multiplying and post-multiplying block $(1,1)$ of (A2) by W^{-T} and W^{-1}, respectively, we obtain

$$W^{-T} A_{11}^T + A_{11} W^{-1} = -\widehat{R}_{11}, \tag{A4}$$

where

$$\widehat{R}_{11} := R_{11} + W^{-T} Q_{11} W^{-1} \geq 0 \tag{A5}$$

due to $R_{11} \geq 0$ and $Q_{11} \geq 0$. From

$$\begin{aligned} E^T X &= N^{-T} N^T E^T M^{-T} M^T X N N^{-1} \\ &= N^{-T} \begin{bmatrix} E_{11}^T W & 0 \\ 0 & 0 \end{bmatrix} N^{-1}, \end{aligned}$$

we know that to show $E^T X \geq 0$ it is equivalently to show $E_{11}^T W \geq 0$. To this end, we need the following results via two steps:

Step 1 Show that $-sE_{11} + A_{11}$ is regular and $\sigma_f(-sE_{11} + A_{11}) \in \mathbb{C}_-$ holds.

Step 2 Show that $(E_{11}, A_{11}, \widehat{R}_{11})$ is stabilizable and $-sE_{11} + A_{11}$ is impulsive free.

As to **Step 1**, since $E_{12} = 0$ and $A_{12} = 0$, we have

$$M^{-1}(-sE + A)N = \begin{bmatrix} -sE_{11} + A_{11} & 0 \\ * & -sE_{22} + A_{22} \end{bmatrix}.$$

Thus, $-sE + A$ is regular, so is $-sE_{11} + A_{11}$; and $\sigma_f(-sE + A) \in \mathbb{C}_- \cup \Omega$ holds, so does $\sigma_f(-sE_{11} + A_{11}) \in \mathbb{C}_- \cup \Omega$. On contrary assume that$-sE_{11} + A_{11}$ has a finite imaginary eigenvalue $j\omega$, then there exists ξ such that

$$\xi^*(-j\omega E_{11} + A_{11}) = 0. \tag{A6}$$

Pre-multiplying and post-multiplying (A4) by ξ^* and ξ, respectively, we have

$$\xi^*(W^{-T} A_{11}^T + A_{11} W^{-1})\xi = -\xi^* R_{11}\xi - \xi^* W^{-T} Q_{11} W^{-1}\xi.$$

It follows from (A3) and (A6) that the left hand side of the above equation is 0. Since $R_{11} \geq 0$ and $Q_{11} \geq 0$, we have $\xi^* R_{11} = 0$. It implies that (E_{11}, A_{11}, R_{11}) is not stabilizable. However, this is not true which is shown as follows. Thus,

$$\sigma_f(-sE_{11} + A_{11}) \subset \mathbf{C}_-. \tag{A7}$$

To show (E_{11}, A_{11}, R_{11}) is stabilizable, since (E, A, R) is stabilizable, we have

$$\begin{aligned} &\operatorname{rank} \begin{bmatrix} -sE + A & R \end{bmatrix} \\ = \;& \operatorname{rank} \begin{bmatrix} M^{-1}(-sE + A)N & M^{-1}RM^{-T} \end{bmatrix} \\ = \;& \operatorname{rank} \begin{bmatrix} -sE_{11} + A_{11} & 0 & R_{11} & R_{12} \\ * & -sE_{22} + A_{22} & R_{12}^T & R_{22} \end{bmatrix} \\ = \;& \operatorname{rank} \begin{bmatrix} -sE_{22} + A_{22} & * & R_{12}^T & R_{22} \\ 0 & -sE_{11} + A_{11} & R_{11} & R_{12} \end{bmatrix} \\ = \;& n, \quad \forall s \in \mathbf{C}_+ \cup \mathbf{\Omega}_e. \end{aligned} \tag{A8}$$

Also since $R \geq 0$, then $\operatorname{rank} \begin{bmatrix} R_{11} & R_{12} \end{bmatrix} = \operatorname{rank} R_{11}$. Hence,

$$\operatorname{rank} \begin{bmatrix} -sE_{11} + A_{11} & R_{11} \end{bmatrix} = r, \quad \forall s \in \mathbf{C}_+ \cup \mathbf{\Omega}, \tag{A9}$$

where $r = \operatorname{rank} E$. For $s = \infty$, applying (4) to (A8) yields

$$\operatorname{Im} \begin{bmatrix} E_{11} & 0 \\ E_{21} & E_{22} \end{bmatrix} + \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \operatorname{Ker} \begin{bmatrix} E_{11} & 0 \\ E_{21} & E_{22} \end{bmatrix} + \operatorname{Im} \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^{\mathrm{T}} & R_{22} \end{bmatrix} = \mathbb{R}^n.$$

Therefore, we obtain

$$\operatorname{Im} E_{11} + A_{11} \operatorname{Ker} E_{11} + \operatorname{Im} R_{11} = \mathbb{R}^r. \tag{A10}$$

This shows that (E_{11}, A_{11}, R_{11}) is stabilizable.

As to **Step 2**, first, we show that $(E_{11}, A_{11}, \widehat{R}_{11})$ is also stabilizable. From (A7), we have

$$\operatorname{rank} \begin{bmatrix} -sE_{11} + A_{11} & \widehat{R}_{11} \end{bmatrix} = r, \ \forall s \in \mathbf{C}_+ \cup \mathbf{\Omega}. \tag{A11}$$

On the other hand, from (A5), we obtain $\operatorname{Ker} \widehat{R}_{11} \subset \operatorname{Ker} R_{11}$. This yields $\operatorname{Im} \widehat{R}_{11}^{\mathrm{T}} \supset \operatorname{Im} \mathbb{R}_{11}^{\mathrm{T}}$. Since $\widehat{R}_{11}$ and R_{11} are symmetric, $\operatorname{Im} \widehat{R}_{11} \supset \operatorname{Im} R_{11}$. Thus, it follows from (A10) that

$$\operatorname{Im} E_{11} + A_{11} \operatorname{Ker} E_{11} + \operatorname{Im} \widehat{R}_{11} = \mathbb{R}^r. \tag{A12}$$

This with (A11) shows that $(E_{11}, A_{11}, \widehat{R}_{11})$ is also stabilizable.

Next, we prove that $-sE_{11} + A_{11}$ is impulsive free. The proof is similar to that in [15]. Without loss of generality, we assume that E_{11} and A_{11} are given by the following form:

$$E_{11} =: \begin{bmatrix} I & 0 \\ 0 & \Lambda \end{bmatrix}, \quad A_{11} =: \begin{bmatrix} A_s & 0 \\ 0 & I_l \end{bmatrix}, \qquad \text{(A13)}$$

where A_s is strictly stable and Λ is a nilpotent matrix. It suffices to show $\Lambda = 0$. Now decompose W^{-1} and $\widehat{R}_{11}$ in accordance with (A13) as

$$W^{-1} =: \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}, \ \widehat{R}_{11} =: \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^T & Z_{22} \end{bmatrix}. \quad \text{(A14)}$$

From (A3) and (A4), we have

$$W_{11}^T A_s^T + A_s W_{11} = -Z_{11} \le 0, \quad W_{11} = W_{11}^T, \qquad \text{(A15)}$$

$$W_{12} = W_{21}^T \Lambda^T, \qquad \text{(A16)}$$

$$W_{22}^T + W_{22} = -Z_{22} \le 0, \quad W_{22}^T \Lambda^T = \Lambda W_{22}. \qquad \text{(A17)}$$

Now on contrary, assume $\Lambda \neq 0$. Since Λ is nilpotent, let $\alpha := \min\{k \ |\Lambda^k = 0, \ k \ge 2\}$. Using the stabilizability of $(E_{11}, A_{11}, \widehat{R}_{11})$ and (A12)-(A14) gives

$$\operatorname{Im}\Lambda + \operatorname{Ker}\Lambda + \operatorname{Im} Z_{22} = \mathbb{R}^l. \qquad \text{(A18)}$$

This yields

$$\operatorname{Im}\Lambda^{\alpha-1} = \operatorname{Im}\Lambda^{\alpha-1} Z_{22}. \qquad \text{(A19)}$$

Then pre-multiplying and post-multiplying the first equation of (A17) by $\Lambda^{\alpha-1}$ and $(\Lambda^T)^{\alpha-1}$, respectively, we obtain

$$\Lambda^{\alpha-1}(W_{22}^T + W_{22})(\Lambda^T)^{\alpha-1} = -\Lambda^{\alpha-1} Z_{22} (\Lambda^T)^{\alpha-1}.$$

Using the second equation of (A17), it follows that the left hand side of the above equation satisfies

$$\Lambda^{\alpha} W_{22}^T (\Lambda^T)^{\alpha-2} + \Lambda^{\alpha-2} W_{22} (\Lambda^T)^{\alpha} = 0.$$

Thus, together with $Z_{22} \ge 0$, we have $\Lambda^{\alpha-1} Z_{22} = 0$. Moreover, from (A19), $\Lambda^{\alpha-1} = 0$. This contradicts the minimality of α. Thus, $\Lambda = 0$.

Finally, we are ready to show $E_{11}^T W \ge 0$. From (A16), $W_{12} = 0$, then

$$W^{-1} = \begin{bmatrix} W_{11} & 0 \\ W_{21} & W_{22} \end{bmatrix}$$

implies that W_{11} is nonsingular. According to Lyapunov equation (A15), the stability of A_s leads to Thus $W_{11} > 0$. Then, by using (A13) with $\Lambda = 0$, we have

$$E_{11}^T W = \begin{bmatrix} W_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \ge 0.$$

This completes the proof of Theorem 1. ■

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