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Set Stability and Coordination of Nonlinear Multi-agent Systems with Switching Structure

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***Abstract*— In this paper, we discuss the aggregation of a group of autonomous agents related to a convex target set. The convergence of the whole agent group, consisting of leaders (informed agents) and followers, to a desired region is investigated with switching interconnection topologies described by the connectivity on $[t,\infty)$ for any time t. Set stability analysis is given with help of graph theory and non-smooth analysis, and moreover, sufficient conditions to achieve the coordination are proposed for some important cases. The results show that simple local rules can make the networked agents with first-order nonlinear individual dynamics converge to the target set.**

***Index Terms*— Multi-agent systems, joint connection, set stability, convex analysis, nonlinear dynamics.**

I. INTRODUCTION

Multi-agent coordination has become a hot topic in recent years (referring to [6], [8], [13], [16]). Target aggregation is one of the important coordination problems in the studies of multiple mobile agents, which shows how large groups of agents move to a target region. Sometimes, it can be viewed as a leader-follower aggregation problem with multiple (virtual) leaders. In fact, a "leader" in the multi-agent systems may be a special (informed) agent or a moving target, or a reference point to guide the whole group. Although, there is usually a single leader for the leader-following formulation in many existing results, but in some practical situations, the multiple (virtual) leaders can be found or may be needed in the flocking to a target region. In [10], Lin *et al.* discussed an interesting model for a group of agents with straight-line formation containing two "edge leaders", where all the agents converge to the line segment specified by the two edge leaders. A model for the fish school was given to simulate foraging and demonstrate that, the larger the group is, the smaller the proportion of "leaders" is needed to guide the group in [3], while moving targets can be also viewed as multiple "leaders" in pursuit evasion operations [14].

Many efforts have been made to deal with variable intra-agent communication structures. For example, [9] proved the consensus of a simplified Vicsek model with joint-connection assumption. Moreover, [8] investigated the jointly connected coordination for second-order agent dynamics. However, the problem becomes much more difficult if the agent dynamics are nonlinear. Moreau studied the stability and state agreement problems for nonlinear discrete-time agents with time-varying interconnection in [13]. However, for nonlinear continuous-time agent dynamics with jointly-connected interaction graphs, the results seem harder to be obtained. Lin *et al.* provided sufficient conditions to ensure state agreement for directed graph under uniform connectivity in [11].

In this paper, we consider a group of continuous-time agents, consisting of informed agents (leaders) and follower agents, with variable interconnection and nonlinear agent dynamics. Also with joint-connection assumptions given on $[t,\infty)$, the set stability of the networked agents is investigated. By neighborhood rules, we show that a group of agents can flock to a convex target set probably known by the leaders in some important cases.

II. PRELIMINARIES

In this section, we introduce some preliminary knowledge for the following discussion.

A directed graph (or digraph) is usually denoted as $\mathcal{G}=(\mathcal{N},\mathcal{E})$, where $\mathcal{N}=\{1,2,\cdots,n\}$ is the set of nodes and $\mathcal{E}$ is the set of arcs, each element of which is an ordered pair of distinct nodes in $\mathcal{N}$ (see [7]). (i,j) denotes an arc leaving from node v_i (or simply i) and entering node v_j (or j). A walk in digraph $\mathcal{G}$ is an alternating sequence $i_1e_1i_2e_2\cdots e_{k-1}i_k$ of nodes i_m and arcs $e_m=(i_m,i_{m+1})\in\mathcal{E}$ for $m=1,2,\cdots,k$. If there exists a walk from node i to node j then node j is said to be reachable from node i. In particular, each node is thought to be reachable by itself. A node v which is reachable from any node of $\mathcal{G}$ is called a globally reachable node of $\mathcal{G}$. $\mathcal{G}$ is said to be *weakly connected* if every two nodes are joined by a semiwalk (ignoring the orientation of each arc); $\mathcal{G}$ is said to be *fully weakly connected* if for every two nodes i and j there is an arc (i,j) or (j,i); $\mathcal{G}$ is said to be *quasi-strongly connected* if for every two nodes i and j there is a node k from which i and j are reachable. Given a digraph $\mathcal{G}$, its opposite graph $\mathcal{G}^*$ is the digraph formed by changing the orientation of each arc in $\mathcal{G}$. It is known that $\mathcal{G}$ is quasi-strongly connected if and only if $\mathcal{G}^*$ has a globally reachable node ([2]). If $\mathcal{G}_1=(\mathcal{V},\mathcal{E}_1)$ and $\mathcal{G}_2=(\mathcal{V},\mathcal{E}_2)$ have the same node set, the union of the two digraphs is defined as $\mathcal{G}_1\cup\mathcal{G}_2=(\mathcal{V},\mathcal{E}_1\cup\mathcal{E}_2)$. A time-varying digraph is defined as $\mathcal{G}_{\sigma(t)}=(\mathcal{V},\mathcal{E}_{\sigma(t)})$ with $\sigma:t\to\mathcal{Q}$ as a piecewise constant function, where $\mathcal{Q}$ is a finite set with all the possible digraphs with node set $\mathcal{V}$. Additionally, $\mathcal{G}([t_1,t_2])$ denotes the joint digraph $\mathcal{G}([t_1,t_2])=(\mathcal{V},\cup_{t\in[t_1,t_2]}\mathcal{E}(t))$.

A set $K\subset R^m$ is said to be convex if $(1-\gamma)x+\gamma y\in K$ whenever $x\in K, y\in K$ and $0<\gamma<1$. For any set

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$S \subset R^m$, the intersection of all convex sets containing S is called the *convex hull* of S, denoted by $co(S)$. Particularly, the convex hull of a finite set of points $x_1, \cdots, x_n \in R^m$ is a polygon, denoted by $co\{x_1, \cdots, x_n\}$ (see [17] for details). The following two lemmas can be found in [1].

Lemma 1 (Best-Approximation Theorem): Let K be a closed convex subset of a Hilbert space X. We can associate to any $x \in X$ a unique element $\pi_k(x) \in K$ satisfying $\|x-\pi_k(x)\| = \min_{y\in K} \|x-y\|$, where the map π_k is called the projector onto K. Moreover,

$$\langle \pi_k(x) - x, \pi_k(x) - y\rangle \le 0, \quad \forall y \in K.$$

Lemma 2: Let K be a closed convex subset of a Hilbert space X and d_K the function defined on X by $d_K(x) \triangleq \inf\{\|x-y\| \,|\, y \in K\}$. Then $d_K^2(x) = \inf\{\|x-y\|^2 | y \in K\}$ is continuously differentiable and

$$\nabla d_K^2(x) = 2(x - \pi_k(x)),$$

where $\nabla d_K^2(x)$ denotes the gradient of d_K^2 at x.

Then, we introduce a lemma related to Dini derivative.

Lemma 3 ([4], [11]): Let $V_i(t,x) : R \times R^n \to R$ be C^1 for $i = 1, 2, \cdots, n$ and $V(t,x) = \max_{i=1,2,\cdots,n} V_i(t,x)$. If $\mathcal{I}(t) = \{i \in \{1, 2, \cdots, n\} : V(t, x(t)) = V_i(t, x(t))\}$ is the set of indices where the maximum is reached at t, then

$$D^+V(t, x(t)) = \max_{i\in\mathcal{I}(t)} \dot{V}_i(t, x(t)).$$

where D^+ denotes the upper Dini derivative.

Consider a system

$$\dot{x} = f(t, x), \tag{1}$$

where $f : R \times R^n \to R^n$ is piecewise continuous in t and continuous in x. $\Omega_0 \subset R^n$ is called a positively invariant set of (1) if, for any $t_0 \in R$ and any $x^0 \in \Omega_0$, $x(t, t_0, x^0) \in \Omega_0$ when $t > t_0$. Then system (1) is said to be (set) stable with respect to Ω_0 if, for any t_0 and any $\varepsilon > 0$, there is $\delta > 0$ such that

$$d_{\Omega_0}(x^0) < \delta \Longrightarrow d_{\Omega_0}(x(t)) < \varepsilon, \ \forall t \ge t_0. \tag{2}$$

Moreover, if δ in (2) does not depend on t_0, then system (1) is said to be uniformly stable with respect to Ω_0. System (1) is said to be (globally) attractive with respect to Ω_0 if $x^0 \in R^n \Longrightarrow \lim_{t\to+\infty} d_{\Omega_0}(x(t)) = 0$.

III. Problem Formulation

A convex region, denoted by $\Omega \in R^m$, is considered as a target set (maybe viewed as a food source or next site) for the group of agents. There may be many challenges in the achievement of a multi-agent aggregation with respect to target Ω for the following reasons (referring to [3]):

(i) In practice, only some of these n agents are potential leaders or "informed" agents that know the location of Ω, while the others (called "followers") cannot.

(ii) Although the leaders can "see" the target, they may lose their sights from time to time (due to uncertainties in the environment or tradeoffs with other agents, for instance), which makes the "connection" between Ω and the "leaders" keep changing; the interconnection topology is also because the neighbors of the agents are time-varying due to the complex agent dynamics.

(iii) The followers cannot recognize the leaders, and therefore, the local rules are applied to all the agents without any difference, and the informed agents are also affected by the followers if there are connections.

The state of the agent i (that is, node v_i), is denoted as $x_i \in R^m$ $(i = 1, \cdots, n)$; and set Ω is regarded as a generalized (agent) node, denoted as v_0 (see Fig. 1). Define two sets $\mathcal{N} = \{v_1, v_2, \cdots, v_n\}$ and $\bar{\mathcal{N}} = \{v_0, v_1, v_2, \cdots, v_n\}$.

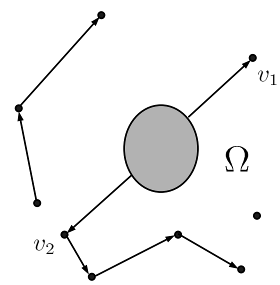

Fig. 1. v_1 and v_2 are the "leaders" that can see Ω at time t

At time t, if node v_i can "see" node v_j, then there is an arc (v_j, v_i) (marking the information flow) from v_j to v_i; and in this way, v_j is said to be a neighbor of v_i. Likewise, if "informed" agent v_i "sees" Ω at time t, then there is an arc (v_0, v_i) leaving from v_0 and entering v_i; and v_0 (that is, Ω) is said to be a (generalized) neighbor of v_i. In what follows, when there is no confusion, we will identify the index i with node v_i for convenience.

Denote $\bar{\mathcal{P}}$ as the set of all possible interconnection topologies, and $\sigma : [0, +\infty) \to \bar{\mathcal{P}}$ as a piecewise constant switching signal function to describe the switchings between the topologies. Thus, the interaction topology of the considered multi-agent network is described by $\bar{\mathcal{G}}_{\sigma(t)} = (\bar{\mathcal{N}}, \bar{\mathcal{E}}_{\sigma(t)})$. Moreover, as done in some existing works (e.g., [9], [8]), we assume that there is a dwell time, denoted by a constant τ_D for $\sigma(t)$, as a lower bound between two switching times.

Let $N_i(\sigma(t))$ represent the neighbor set of node i at time t and $a_{ij}(x) > 0$ denote the weight of arc (j, i), $i, j = 1, ..., n$ if there is an arc. Then, the dynamics of each agent v_i with state x_i $(i = 1, ..., n)$ is described as

$$\dot{x}_i = \sum_{j\in N_i(\sigma(t))} a_{ij}(x)(x_j - x_i) + \lambda(x_i)\chi_i(\sigma(t))f^i(x_i, \Omega), \tag{3}$$

where $x_i(t) \in R^m$ denotes the position vector of agent i at time t, and $\lambda(x_i)$ and $\chi_i(\sigma(t))$ are Boolean variables, defined respectively, as follows:

$$\lambda(x_i) = \begin{cases} 1, & \text{if } x_i \text{ is a (potential) informed agent} \\ 0, & \text{otherwise} \end{cases}$$

to mark the informed agents, and

$$\chi_i(\sigma(t)) = \begin{cases} 1, & \text{if } x_i \text{ is connected with } \Omega \text{ at } t \\ 0, & \text{otherwise} \end{cases}$$

to mark when the informed agents can see the target.

For $a_{ij}(x), i,j \in \mathcal{N}$, we assume

A1). $a_{ij}(x)$ is locally Lipschitz, $\forall x \in R^{mn}$;

A2). There are $a^* > 0$ and $a_* > 0$ such that

$$a_* \le a_{ij}(x) \le a^*, \forall x \in R^{mn}. \tag{4}$$

Moreover, for $f^i(x_i, \Omega), i = 1, ..., n$, we assume

F1). $f^i(x_i, \Omega)$ is locally Lipschitz, which secures the uniqueness of the solution to system (3);

F2). There is a $\mathcal{K}$-class function κ such that

$$\langle x_i - \pi_\Omega(x_i), f^i(x_i, \Omega)\rangle \le -\kappa(d^2_\Omega(x_i)), \tag{5}$$

with d_Ω defined as in Lemma 2.

Note that Assumption F2 can be easily satisfied. If $x_0 \in \Omega$ is a fixed point, we choose $f^i(x_i, \Omega) = x_0 - x_i$, and then, from Lemma 1,

$$\langle x_i - \pi_\Omega(x_i), f^i\rangle \le \langle x_i - \pi_\Omega(x_i), \pi_\Omega(x_i) - x_i\rangle = -d^2_\Omega(x_i).$$

Denote

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \; f_{\sigma(t)}(x) = \begin{pmatrix} \lambda_1\chi_1 f^1 \\ \vdots \\ \lambda_n \chi_n f^n \end{pmatrix} \in R^{mn}.$$

Let $A_{\sigma(t)}(x) = (\tilde{a}_{ij}(\sigma(t), x))$ be a matrix in $R^{n\times n}$ with its (i,j) entry $\tilde{a}_{ij}(\sigma(t), x) = a_{ij}(x) \cdot \chi_{ij}(\sigma(t))$, in which

$$\chi_{ij}(\sigma(t)) = \begin{cases} 1, & \text{if there is an arc from } x_i \text{ to } x_j \text{ at } t \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

Define a diagonal matrix

$$D_{\sigma(t)}(x) = diag\{\tilde{d}_1(\sigma(t), x), \cdots, \tilde{d}_n(\sigma(t), x)\} \in R^{n\times n}$$

with $\tilde{d}_i(\sigma(t), x) = \sum_{j\in N_i(\sigma(t))} a_{ij}(x)$, $i = 1, \cdots, n$. Taking the (nonlinear) Laplacian $L_{\sigma(t)}(x) = D_{\sigma(t)}(x) - A_{\sigma(t)}(x)$, system (3) can be rewritten as:

$$\dot{x} = -(L_{\sigma(t)}(x) \otimes I_m)x + f_{\sigma(t)}(x) \tag{7}$$

where $\otimes$ denotes the Kronecker product and I_m denotes the identity matrix in $R^{m\times m}$. Since $A_{\sigma(t)}(x)$ and $f_{\sigma(t)}(x)$ are piecewise constant with respect to t and locally Lipschitz with respect to x, the solution to system (7) is unique for any initial condition.

In this paper, we will consider the (global) target aggregation with respect to Ω, that is, $\lim_{t\to+\infty} d_\Omega(x_i(t)) = 0$, $i = 1, ..., n$ for any initial condition $x^0 \in R^{mn}$.

IV. Stability analysis

In this section, we will consider the target aggregation coordination of system (7) with a proposed method to analyze its limit set. Since we consider the system in mn-dimensional space, the considered convex set $\Omega \in R^m$ also be transformed to $\Omega^n = \Omega \times ... \times \Omega \in R^{mn}$. Define

$$\hbar(x(t)) \triangleq \max_{i=1,\ldots,n} \{\hbar^i(x_i(t))\}, \quad \hbar^i(x_i(t)) \triangleq \frac{d^2_\Omega(x_i(t))}{2}.$$

Lemma 4: $\hbar(x(t))$ is non-increasing for system (7).

Proof: According to Lemmas 1, 2 and 3, we have

$$\begin{aligned} D^+\hbar(x(t)) &= \max_{i\in\Upsilon(t)} \frac{d}{dt}(\hbar^i(x_i(t))) \\ &= \max_{i\in\Upsilon(t)} \langle x_i - \pi_\Omega(x_i), \sum_{j\in N_i(\sigma(t))} a_{ij}(x_j - x_i) \\ &\quad + \lambda(x_i)\chi_i(\sigma(t)) f^i(x_i, \Omega)\rangle. \end{aligned} \tag{8}$$

where $\Upsilon(t)$ denotes the set containing all the agents that reach the maximal distance away from Ω at time t.

Define a convex set $\bar{\Omega}(t) \triangleq \{p \mid \frac{1}{2}d^2_\Omega(p) \le \hbar(x(t))\}$. If $x_c(t) \in \Upsilon(t)$, then $\hbar^c(x_c(t)) = \hbar(x(t))$. For any $x_0 \in \bar{\Omega}(t)$,

$$\begin{aligned} \|2x_c - \pi_\Omega(x_c) - x_0\|^2 &= \|x_c - \pi_\Omega(x_c)\|^2 + \|x_c - x_0\|^2 \\ &\quad + 2\langle x_c - \pi_\Omega(x_c), x_c - x_0\rangle. \end{aligned} \tag{9}$$

By Lemma 2, $\langle x_c - \pi_\Omega(x_c), x_c - x_0\rangle \ge 0$. Therefore,

$$\|2x_c - \pi_\Omega(x_c) - x_0\|^2 \ge \|x_c - \pi_\Omega(x_c)\|^2 + \|x_c - x_0\|^2 \tag{10}$$

Since (10) reaches the minimum when $x_0 = x_c$, $x_c = \pi_{\bar{\Omega}}(2x_c - \pi_\Omega(x_c))$. For any $x_j \in \bar{\Omega}$, again by Lemma 1,

$$\begin{aligned} &\langle x_c - \pi_\Omega(x_c), x_j - x_c\rangle \\ = &\langle (2x_c - \pi_\Omega(x_c)) - \pi_{\bar{\Omega}}(2x_c - \pi_\Omega(x_c)), \\ &x_j - \pi_{\bar{\Omega}}(2x_c - \pi_\Omega(x_c))\rangle \le 0. \end{aligned} \tag{11}$$

On the other hand, when x_c is an informed agent,

$$\langle x_c - \pi_\Omega(x_c), f^c(x_c, \Omega)\rangle \le -\kappa(d^2_\Omega(x_c)). \tag{12}$$

Based on (11) and (12) as well as $a_{ij}(x) > 0$,

$$\begin{aligned} &\langle x_c - \pi_\Omega(x_c), \sum_{j\in N_c(\sigma(t))} a_{cj}(x)(x_j - x_c) \\ &\quad + \lambda(x_c)\chi_c(\sigma(t)) f^c(x_c, \Omega)\rangle \le 0. \end{aligned} \tag{13}$$

Since x_c is chosen arbitrarily in $\Upsilon(t)$, by (8) and (13), we have $D^+\hbar(x(t)) \le 0$, which implies the conclusion. □

Lemma 4 shows that, for any initial condition $x(t_0) = x^0$, the trajectory of system (7) cannot go to infinity in finite time, which implies the existence of the solution during $t \in (t_0, +\infty)$. By Lemma 4, Ω^n is positively invariant for (7), and $\hbar(x(t)) \le \hbar(x(t_0))$, $\forall t \ge t_0$. Therefore, we have

Theorem 1: (7) is uniformly stable with respect to Ω^n.

Remark 1: By Lemma 4, set $\Gamma \triangleq \{x \mid \hbar(x(t)) \le \hbar(x^0)\}$ is positively invariant for system (7) for any initial condition x^0. Therefore, if Ω is compact, so is Γ. Moreover, since $a_{ij}(x) > 0$, $i,j = 1, 2, \cdots, n$, we can find $a_* > 0$ and $a^* > 0$ such that

$$a_{ij}(x(t)) \in [a_*, a^*], \quad \forall t \in (t_0, +\infty), \forall i, j \in \mathcal{N},$$

which means A1 implies A2 (i.e., (4)) if Ω is compact.

Moreover, according to Lemma 4,

$$\lim_{t\to+\infty} \hbar(x(t)) = \hbar^*,$$

where $\hbar^* \ge 0$. Note that system (7) is globally attractive with respect to Ω^n if and only if $\hbar^* = 0$ given any initial condition. Moreover, $\hbar^* = 0$ if and only if the positive limit set of system (7) is within Ω. Therefore, the set attractivity of system (7) highly depends on its limit set. Define

$$\ell(x(t)) = \min_{i\in\mathcal{N}}\{\hbar^i(x_i(t))\},\ \ell^* \triangleq \liminf_{t\to+\infty} \ell(x(t)).$$

Obviously, $0 \le \ell^* \le \hbar^*$.

Here we propose a method to analyze the limit set in order to study the set stability and attractivity, which will be used throughout the whole paper.

Lemma 5: Suppose $\Omega \in R^m$ is convex and $x_a, x_b \in R^m$. Then

$$\langle x_a - \pi_\Omega(x_a), x_b - x_a\rangle \le d_\Omega(x_a)|d_\Omega(x_a) - d_\Omega(x_b)|. \quad (14)$$

Moreover, if $d_\Omega(x_a) > d_\Omega(x_b)$, then

$$\langle x_a - \pi_\Omega(x_a), x_b - x_a\rangle \le -d_\Omega(x_a)[d_\Omega(x_a) - d_\Omega(x_b)]. \quad (15)$$

Proof: (14) and (15) become obvious when $d_\Omega(x_a) = 0$. So we only consider $d_\Omega(x_a) > 0$ in the following.

Define

$$\Omega_a \triangleq \{v|d_\Omega(v) \le d_\Omega(x_a)\},\ \Omega_b \triangleq \{v|d_\Omega(v) \le d_\Omega(x_b)\}$$
$$H_1 \triangleq \{v|\langle x_a - \pi_\Omega(x_a), v - x_a\rangle > 0\}.$$

By Lemma 2, for any $v \in H_1$, $d_\Omega(x_a) < d_\Omega(v)$. Thus, $H_1 \cap \Omega_a = \emptyset$.

Define

$$\tilde{x}_a \triangleq \pi_\Omega(x_a) + \frac{d_\Omega(x_b)}{d_\Omega(x_a)} \cdot (x_a - \pi_\Omega(x_a)),$$
$$H_2 \triangleq \{v|\langle x_a - \pi_\Omega(x_a), v - \tilde{x}_a\rangle > 0\}.$$

Clearly, $\pi_\Omega(\tilde{x}_a) = \pi_\Omega(x_a)$ and we can get $H_2 \cap \Omega_b = \emptyset$ through similar analysis.

If $x_b \in \Omega_b \backslash H_1$, we have $\langle x_a - \pi_\Omega(x_a), x_b - x_a\rangle \le 0$ and (14) follows.

On the other hand, if $x_b \in \Omega_b \cap H_1$, then $x_b \notin H_2$ because $H_2 \cap \Omega_b = \emptyset$. Hence, $\langle x_a - \pi_\Omega(x_a), x_b - \tilde{x}_a\rangle \le 0$. Therefore, by the Cauchy-Schwarz Inequality, we have

$$\begin{aligned}\langle x_a - \pi_\Omega(x_a), x_b - x_a\rangle &\le \langle x_a - \pi_\Omega(x_a), \tilde{x}_a - x_a\rangle \\ &= d_\Omega(x_a)\cdot \mid d_\Omega(x_b) - d_\Omega(x_a) \mid .\end{aligned}$$

Furthermore, if $d_\Omega(x_a) > d_\Omega(x_b)$, it is not hard to see $\tilde{x}_a = \pi_{\Omega_b}(x_a)$, and

$$x_a - \tilde{x}_a = \frac{d_\Omega(x_a) - d_\Omega(x_b)}{d_\Omega(x_a)}(x_a - \pi_\Omega(x_a)).$$

Thus, by Lemma 1,

$$\begin{aligned}\langle x_a - \pi_\Omega(x_a), x_b - x_a\rangle &= \frac{d_\Omega(x_a)}{d_\Omega(x_a) - d_\Omega(x_b)}\langle x_a - \pi_{\Omega_b}(x_a), \\ &\quad x_b - \pi_{\Omega_b}(x_a)\rangle \\ &\quad + \langle x_a - \pi_\Omega(x_a), \tilde{x}_a - x_a\rangle \\ &\le \langle x_a - \pi_\Omega(x_a), \tilde{x}_a - x_a\rangle \\ &= -d_\Omega(x_a)[d_\Omega(x_a) - d_\Omega(x_b)],\end{aligned}$$

which completes the proof. □

Denote $\bar{\mathcal{G}}([t, +\infty)) = (\bar{\mathcal{N}}, \cup_{s\in[t,+\infty)}\bar{\mathcal{E}}_{\sigma(s)})$ as the joint topology from t to $+\infty$. Then we have the following result.

Theorem 2: If the joint topology $\bar{\mathcal{G}}([t, +\infty))$ is quasi-strongly connected for any t and $\hbar^* > 0$, then $\ell^* < \hbar^*$.

Proof: We will prove $\ell^* < \hbar^*$ by contradiction. Suppose $\ell^* = \hbar^*$. Then $\lim_{t\to+\infty} \hbar^i(x_i(t)) = \hbar^*$, $i = 1, ..., n$. Therefore, for any $\varepsilon > 0$, there is $T(\varepsilon) > 0$ such that, when $t > T(\varepsilon)$,

$$\hbar^i(x_i(t)) \in (\hbar^* - \varepsilon, \hbar^* + \varepsilon), \quad i = 1, ..., n. \quad (16)$$

Since $\bar{\mathcal{G}}([t, +\infty))$ is quasi-strongly connected for any t, there is a sequence

$$T < t_1 < t_2 < \cdots < t_p < \cdots, \quad t_{p+1} > t_p + \tau_D, \quad (17)$$

such that, at each time t_p, there is an arc from Ω to an agent node. Since the total number of the agents is finite, we can find a subsequence of (17) as follows

$$T < t_{p_1} < \cdots < t_{p_k} < \cdots, \quad (18)$$

such that there is an arc from Ω pointing to a fixed agent (supposed to be v_i) at each time t_{p_k}. With the dwell time assumption, we can also assume that the system topology does not change at $(t_{p_k}, t_{p_k} + \tau_D)$ for any p. Therefore, $\chi_i(\sigma(t)) = 1$ in each time interval $(t_{p_k}, t_{p_k} + \tau_D)$. Based on Lemma 5 and (5),

$$\begin{aligned}\langle x_i - \pi_\Omega(x_i), x_j - x_i\rangle &\le (\sqrt{2(\hbar^* + \varepsilon)} - \sqrt{2(\hbar^* - \varepsilon)}) \\ &\quad \times \sqrt{2(\hbar^* + \varepsilon)} \le 4\varepsilon. \quad (19)\end{aligned}$$

$$\langle x_i - \pi_\Omega(x_i), f^i(x_i, \Omega)\rangle \le -\kappa(2(\hbar^* - \varepsilon)). \quad (20)$$

Thus, when $t \in (t_{p_k}, t_{p_k} + \tau_D)$, $\forall k$, by taking ε sufficiently small to render $\kappa(2(\hbar^* - \varepsilon)) + 4(n-1)a^*\varepsilon < -\kappa(\hbar^*)$, we have

$$\frac{d}{dt}\hbar^i(x_i) \le -\kappa(\hbar^*), \quad \forall t \in (t_{p_k}, t_{p_k} + \tau_D),\ \forall k$$

which implies that, for any t_{p_k},

$$\hbar^i(x_i(t_{p_k} + \tau_D)) \le \hbar^i(x_i(t_{p_k})) - \kappa(\hbar^*)\tau_D. \quad (21)$$

Choose ε even smaller, if necessary, to make $\kappa(\hbar^*)\tau_D > 2\varepsilon$. Then we can find that (21) contradicts (16). □

V. Three Convergence Cases

In this section, we will consider several cases to guarantee the convergence to set Ω^n

Here is the result for the first case.

Theorem 3: System (7) is globally attractive with respect to Ω^n if Ω is a neighbor of every agent in the joint topology $\bar{\mathcal{G}}([t, +\infty))$ for any t with $\mathcal{N}_1 \triangleq \{i \in \mathcal{N}| \lim_{t\to+\infty} \hbar^i(x_i(t)) = \hbar^*\} \neq \emptyset$.

Proof: As it was shown before, the attractivity of system (7) is equivalent to $\hbar^* = 0$.

Suppose $\hbar^* > 0$. Then, for any $\varepsilon_1 > 0$, there is $T_1(\varepsilon_1) > 0$ such that, if $t > T_1(\varepsilon_1)$,

$$\hbar^i(x_i(t)) \in (\hbar^* - \varepsilon_1, \hbar^* + \varepsilon_1), \quad \forall i \in \mathcal{N}_1, \quad (22)$$

$$\hbar^i(x_i(t)) \in [0, \hbar^* + \varepsilon_1), \quad \forall i \in \mathcal{N} \setminus \mathcal{N}_1. \quad (23)$$

Moreover, if Ω is a neighbor of every agent in the joint topology $\bar{\mathcal{G}}([t, +\infty))$ for any t, there is a time sequence $T_1 <$

$\tilde{t}_1 < \cdots < \tilde{t}_p < \cdots$ with $\tilde{t}_{p+1} > \tilde{t}_p + \tau_D$ such that there is an arc from Ω to some agent in $\mathcal{N}_1$ at $t = \tilde{t}_p$ for any p. Since the number of agents in $\mathcal{N}_1$ is finite, we can select a subsequence

$$T_1 < \tilde{t}_{p_1} < \cdots < \tilde{t}_{p_k} < \cdots$$

of $\{\tilde{t}_p\}$ such that there is an arc (v_0, v_c) leaving from Ω to a fixed node $v_c \in \mathcal{N}_1$ at each moment $\tilde{t}_{p_k}$. Without loss of generality, we assume the system topology does not change in $[\tilde{t}_{p_k}, \tilde{t}_{p_k} + \tau_D)$. For $t \in (\tilde{t}_{p_k}, \tilde{t}_{p_k} + \tau_D)$, by Lemma 5,

$$\begin{aligned} & \langle x_c - \pi_\Omega(x_c), \sum_{j \in N_c(\sigma(t))} a_{cj}(x)(x_j - x_c)\rangle \\ \leq & \sum_{j \in N_c(\sigma(t))} 2a_{cj}\sqrt{\hbar^* + \varepsilon_1}(\sqrt{\hbar^* + \varepsilon_1} - \sqrt{\hbar^* - \varepsilon_1}) \\ \leq & \ 4(n-1)a^*\varepsilon_1, \end{aligned} \tag{24}$$

$$\begin{aligned} & \langle x_c(t) - \pi_\Omega(x_c(t)), f^c(x_c(t), \Omega)\rangle \\ \leq & -\kappa(d^2_\Omega(x_c(t))) \leq -\kappa(2(\hbar^* - \varepsilon_1)). \end{aligned} \tag{25}$$

As ε_1 is sufficiently small, $-\kappa(2(\hbar^* - \varepsilon_1)) + 4(n-1)a^*\varepsilon_1 \leq -\kappa(\hbar^*)$, which yields $\frac{d\hbar^c(x_c(t))}{dt} \leq -\kappa(\hbar^*)$ for any $t \in (\tilde{t}_{p_k}, \tilde{t}_{p_k} + \tau_D)$ and any k. Thus,

$$\hbar^c(x_c(\tilde{t}_{p_k} + \tau_D)) \leq \hbar^c(x_c(\tilde{t}_{p_k})) - \kappa(\hbar^*)\tau_D, \tag{26}$$

which contradicts (22) when $\kappa(\hbar^*)\tau_D > 2\varepsilon_1$. As a result, $\hbar^* > 0$ is not true, and the conclusion follows. □

The following simple example shows that the connectivity we gave for the global convergence condition in in Theorem 3 is comparatively tight.

Example 1. Consider a multi-agent system (3) with two agents $x_1, x_2 \in R$ and $\Omega = \{x_0\}$ with $x_0 \equiv 0 \in R$:

$$\begin{cases} \dot{x}_1(t) = -\chi_{12}(\sigma(t))(x_2 - x_1) - \chi_{10}(\sigma(t))x_1 \\ \dot{x}_2(t) = -\chi_{21}(\sigma(t))(x_2 - x_1) - \chi_{20}(\sigma(t))x_2 \end{cases} \tag{27}$$

$\bar{\mathcal{G}}(\sigma(t))$ is the interconnection topology and $\chi_{ij}(\sigma)$ is in (6).

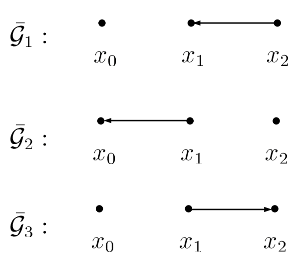

Fig. 2. Three possible topologies

Define $\bar{\mathcal{P}} = \{\bar{\mathcal{G}}_1, \bar{\mathcal{G}}_2, \bar{\mathcal{G}}_3\}$ as the set of all the possible interaction graphs, where $\bar{\mathcal{G}}_1, \bar{\mathcal{G}}_2, \bar{\mathcal{G}}_3$ are shown in Fig. 2.

Set $x_1(t_0) = \frac{1}{2}$ and $x_2(t_0) = 1$ and let $\bar{\mathcal{G}}(\sigma(t_0)) = \bar{\mathcal{G}}_1$. Then we will define the signal $\sigma(t)$ by induction as follows:

Set $m = 0$ in the beginning (that is $t = t_0$), and then we repeat the following steps:

Step 1. Once $x_2(t) = 1 - \sum_{k=0}^{k=m} (\frac{1}{2})^{k+2}$ at some time denoted as t_{3m+1}, we change the interconnection by setting $\bar{\mathcal{G}}(\sigma(t)) = \bar{\mathcal{G}}_2$ at $t = t_{3m+1}$. Then go to Step 2.

Step 2. Once $x_1(t) = \frac{1}{4}$ at some moment denoted as t_{3m+2}, we change the topology again by re-setting $\bar{\mathcal{G}}(\sigma(t)) = \bar{\mathcal{G}}_3$ at $t = t_{3m+2}$. Then go to Step 3.

Step 3. Once $x_1(t) = \frac{1}{2}$ at some moment denoted as t_{3m+3}, we change the interaction structure by setting $\bar{\mathcal{G}}(\sigma(t)) = \bar{\mathcal{G}}_1$ at $t = t_{3m+3}$. Let $m = m + 1$ and return to Step 1.

With the above procedure, we get a time sequence $t_0 < t_1 < \cdots < t_k \cdots$. Note that this switching signal has a dwell time no shorter than $\frac{1}{2}$. Moreover, for any $t > 0$, x_0 is not the neighbor of x_2 in $\bar{\mathcal{G}}([t, +\infty))$. However, $\lim_{t\to+\infty} x_2(t) = \frac{1}{2}$. Hence, system (27) is not attractive with respect to $\Omega = x_0$ for $x_1(0) = \frac{1}{2}$ and $x_2(0) = 1$. □

Next, we consider the case, "uniformly quasi-strongly connected case", which has been widely studied in multi-agent networks (for example, see [8], [9], [11]). In fact, if we can find a constant $T > 0$ such that $\mathcal{G}(t, t+T)$ is quasi-strongly connected for any t, then $\mathcal{G}(t)$ is called to be uniformly quasi-strongly connected.

Theorem 4: System (7) is globally attractive with respect to Ω^n if $\bar{\mathcal{G}}(\sigma(t))$ is uniformly quasi-strongly connected.

Proof: For any $\varepsilon_2 > 0$ and any $\mathcal{K}$-class function κ_0, there exists $T_2(\varepsilon_2) > 0$ such that, when $t > T_2(\varepsilon_2)$,

$$\hbar^i(x_i(t)) \in [0, \hbar^* + \kappa_0(\varepsilon_2)), \quad \forall i \in \mathcal{N}. \tag{28}$$

Note that, if, for any $\varepsilon_2 > 0$, there is $T^* > 0$ such that

$$\hbar^i(x_i(t)) \in (\hbar^* - \kappa_0(\varepsilon_2), \hbar^* + \kappa_0(\varepsilon_2)), \quad \forall t > T^*, \tag{29}$$

then $\lim_{t\to+\infty} \hbar^i(x_i(t)) = \hbar^*$, $i = 1, ..., n$. Thus, $\hbar^* = 0$ holds from Theorem 2.

On the other hand, there is a $\mathcal{K}$-class function κ_0^* such that, for

$$\mathcal{N}_1^{\varepsilon_2}(t) \triangleq \{i \in \mathcal{N} \mid \hbar^i(x_i(t)) \in (\hbar^* - \kappa_0^*(\varepsilon_2), \hbar^* + \kappa_0^*(\varepsilon_2))\},$$

we have

$$\begin{aligned} \forall \varepsilon_2 > 0, \forall \tilde{T}_2 > 0, \exists t_1 > \tilde{T}_2 \quad s.t. \quad & \mathcal{N}_1^{\varepsilon_2}(t_1) \neq \mathcal{N}, \\ and \ \ & \hbar^i(x_i(t_1)) \leq \tilde{h}, \forall i \in \mathcal{N} \setminus \mathcal{N}_1^{\varepsilon_2}(t_1) \end{aligned} \tag{30}$$

for some $\tilde{h} < \hbar^*$. Due to (28), $\mathcal{N}_1^{\varepsilon_2}(t)$ will not be empty for any $t > T_2$. Then we claim, if $\hbar^* > 0$, we can find a finite time sequence such that $\mathcal{N}_1^{\varepsilon_2}(t)$ is strictly decreasing when ε_2 is sufficiently small (see Fig. 3).

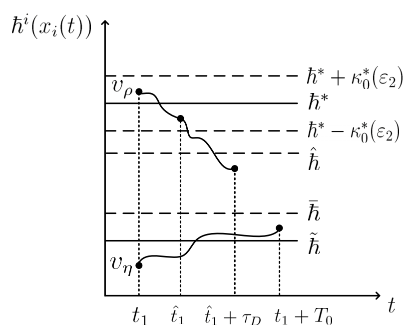

Fig. 3. $\mathcal{N}_1^{\varepsilon_2}(t_1) \supseteq \mathcal{N}_1^{\varepsilon_2}(\hat{t}_1 + \tau_D)$ and $v_\rho \in \mathcal{N}_1^{\varepsilon_2}(t_1) \setminus \mathcal{N}_1^{\varepsilon_2}(\hat{t}_1 + \tau_D)$ for sufficiently small ε_2

Without loss of generality, suppose $t_1 > T_2$. Then, based on Lemma 5, $\forall v_\eta \in \mathcal{N} \setminus \mathcal{N}_1^{\varepsilon_2}(t_1)$, when $t \in (t_1, t_1 + T_0)$,

$$\begin{aligned} \frac{d\hbar^\eta(x_\eta(t))}{dt} &\le -\lambda(x_\eta)\chi_\eta(\sigma(t))\kappa(2\hbar^\eta(x_\eta(t))) \\ &\quad +(n-1)a^*\sqrt{2\hbar^\eta(x_\eta(t))} \\ &\quad \cdot(\sqrt{2(\hbar^* + \kappa_0^*(\varepsilon_2))} - \sqrt{2\hbar^\eta(x_\eta(t))}) \\ &\le -2(n-1)a^*\hbar^\eta(x_\eta(t)) \\ &\quad +2(n-1)a^*\sqrt{\hbar^\eta(x_\eta(t))(\hbar^* + \kappa_0^*(\varepsilon_2))}, \end{aligned}$$

which is equivalent to

$$\frac{d\sqrt{\hbar^\eta(x_\eta(t))}}{dt} \le (n-1)a^*(\sqrt{\hbar^* + \kappa_0^*(\varepsilon_2)} - \sqrt{\hbar^\eta(x_\eta(t))}). \tag{31}$$

From (31), for sufficiently small ε_2, we obtain

$$\hbar^\eta(x_\eta(t_1 + T_0)) \le \bar{\hbar} \tag{32}$$

for some $\bar{\hbar} < \hbar^* - \kappa_0^*(\varepsilon_2)$. Since $\bar{\mathcal{G}}([t_1, t_1 + T_0])$ is quasi-strongly connected, in $\bar{\mathcal{G}}([t_1, t_1 + T_0])$, there has to be an arc from a node in $\mathcal{N} \setminus \mathcal{N}_1^{\varepsilon_2}(t_1)$ or Ω and entering a node in $\mathcal{N}_1^{\varepsilon_2}(t_1)$ at some time $\hat{t}_1 \in [t_1, t_1 + T_0]$. Then we have the two cases:

1) If $v_\rho \in \mathcal{N}_1^{\varepsilon_2}(t_1)$ has a neighbor in $\mathcal{N} \setminus \mathcal{N}_1^{\varepsilon_2}(t_1)$ at time $\hat{t}_1$, by $a_* \le a_{ij}(x(t)), \forall i, j \in \mathcal{N}$, during $t \in (\hat{t}_1, \hat{t}_1 + \tau_D)$, similar to the above analysis, we have that, for sufficiently small ε_2,

$$\hbar^\rho(x_\rho(\hat{t}_1 + T_0)) \le \hat{\hbar}, \quad \hat{\hbar} \in [\bar{\hbar}, \hbar^* - \kappa_0^*(\varepsilon_2)). \tag{33}$$

2) If Ω is a neighbor of $v_\rho \in \mathcal{N}_1^{\varepsilon_2}(t_1)$ at $\hat{t}_1$, when $t \in (\hat{t}_1, \hat{t}_1 + \tau_D)$, we can similarly obtain

$$\hbar^\rho(x_\rho(\hat{t}_1 + \tau_D)) \le \hat{\hbar}. \tag{34}$$

Denote $t_2 = \hat{t}_1 + \tau_D$. Based on (32), (33), and (34),

$$\mathcal{N}_1^{\varepsilon_2}(t_1) \supseteq \mathcal{N}_1^{\varepsilon_2}(t_2) \text{ and } v_\rho \in \mathcal{N}_1^{\varepsilon_2}(t_1) \setminus \mathcal{N}_1^{\varepsilon_2}(t_2). \tag{35}$$

Regarding t_2 as t_1 and through similar analysis, we can find $t_3 > t_2 + \tau_D$ such that

$$\mathcal{N}_1^{\varepsilon_2}(t_2) \supseteq \mathcal{N}_1^{\varepsilon_2}(t_3) \text{ and } \mathcal{N}_1^{\varepsilon_2}(t_2) \setminus \mathcal{N}_1^{\varepsilon_2}(t_3) \ne \emptyset. \tag{36}$$

Repeating the upper process yields a time sequence

$$\max\{T_2, \tilde{T}_2\} < t_1 < t_2 < \cdots < t_k$$

such that

$$\mathcal{N}_1^{\varepsilon_2}(t_i) \supseteq \mathcal{N}_1^{\varepsilon_2}(t_{i+1}) \text{ and } \mathcal{N}_1^{\varepsilon_2}(t_i) \setminus \mathcal{N}_1^{\varepsilon_2}(t_{i+1}) \ne \emptyset, \tag{37}$$

until $\mathcal{N}_1^{\varepsilon_2}(t_k) = \emptyset$, which leads to a contradiction. Thus, $\hbar^* = 0$ and the proof is completed. □

The next theorem is regarded as a partially converse result of Theorems 3 and 4.

Theorem 5: If $\bar{\mathcal{G}}([t, +\infty))$ is not quasi-strongly connected for some t, (7) is not globally attractive with respect to Ω^n.

Proof: Suppose $\bar{\mathcal{G}}([T_0, +\infty))$ is not quasi-strongly connected for some T_0. Then there is one agent x_{i_0} that cannot reach Ω in $\bar{\mathcal{G}}([T_0, +\infty))$. Construct a subgraph $\mathcal{G}_{i_0} \ne \emptyset$ of $\bar{\mathcal{G}}([T_0, +\infty))$, composed of all the agents from which x_{i_0} is reachable. Note that Ω is not in $\mathcal{G}_{i_0}$. Therefore, the agents in $\mathcal{G}_{i_0}$ will not be influenced by Ω after T_0, which yields the conclusion. □

Finally, we will consider the bidirectional graph case, where x_i is a neighbor of x_j if and only if x_j is a neighbor of x_i, but the weight of arc (x_i, x_j) may not be equal to that of arc (x_j, x_i) [13]. Obviously, an undirected topology is a special case of bidirectional topologies.

Theorem 6: System (7) with switching bidirectional topologies is globally attractive with respect to Ω^n if and only if its joint topology $\bar{\mathcal{G}}([t, +\infty))$ is connected for any t.

The proof idea is almost the same as that given in Theorems 3, 4, and 5. The detailed proof is omitted for the space limitations.

In some sense, the obtained results (e.g. Theorems 3, 4, 5, and 6) extend and are consistent with some related existing results, for example, in [9], [11], [13].

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