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Internal Model Control of Infinite Dimensional Systems

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Abstract— **An internal model control (IMC) design method is proposed that is applicable to linear infinite-dimensional systems. An infinite-dimensional filter is coupled with the inverse of the irrational transfer function between the process input and output, to produce a physically realizable controller with a tuning parameter to trade off nominal performance with robustness. The proposed IMC design technique is applied to two boundary control problems for mass transport described by partial differential equations, in which the proposed method provides much better robust performance compared to regular IMC design applied to the finite-dimensional approximation of the transfer function.**

I. INTRODUCTION

The most commonly applied approach for the design of control systems for distributed parameter systems (DPS) is to approximate the partial differential equations (PDEs) by ordinary differential equations (ODEs) and apply finite-dimensional control design methods. The advantage of this approach is that a finite set of state-space equations or a finite-dimensional transfer function is produced for which many control design methods (e.g., linear quadratic gaussian control, H_∞-optimal control, model predictive control, differential geometric methods, robust optimal control) are directly applicable. A drawback of this approach is that, depending on the spatial dynamics of the particular process, the dimension of the ODEs can be high resulting in high computational cost or the ODEs may not represent the DPS as accurately as desired so that important model behaviors are missed. Also, such an approach can hide the underlying structure and dynamics of the optimal controller for the DPS, with a loss in understanding, elegance, and efficiency that would be obtained by direct solution of the DPS optimal control problem (e.g., see discussion by [1] and citations therein).

Internal model control (IMC) is a control design method that matured in the 1980s to 1990s in which an analytical solution for an optimal controller for a nominal process model is combined with a low-order filter with an adjustable tuning parameter for trading off control quality with robustness to model uncertainties [6]. This method can be interpreted as augmenting the inverse or an approximate inverse of the transfer function of the nominal process model with a low-order low-pass filter to make the IMC controller proper and provide the tuning parameter. Other features are that the IMC controller can be easily modified to provide antiwindup

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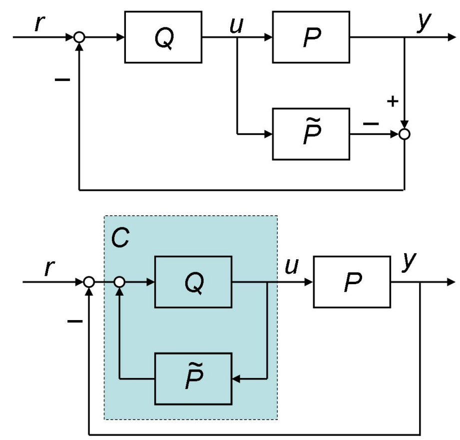

Fig. 1. IMC (a, top) and classical (b, bottom) feedback control structures.

compensation during control input saturation [7], and that reference prefilters, cascade controllers, and other multiple degree-of-freedom controllers can be designed independently [6]. Certainly it would be beneficial to have design methods for DPS that share these features.

The above considerations motivate this paper's development of an IMC design method that is directly applicable to linear infinite-dimensional systems without approximation of the PDEs, in which the IMC controller is designed by analytical solution of an optimal control problem or by inverting the irrational transfer function for the process. Two approaches are developed for designing a physically realizable IMC controller for infinite-dimensional systems: (1) augmenting the IMC controller with an infinite-dimensional filter, or (2) designing the IMC controller for the process model that has been augmented to be semiproper. Section II provides the definitions and robust analysis results used in Section III which describes the IMC design method. This is followed by illustrative examples, conclusions, and directions for future investigation.

II. PRELIMINARIES

The definitions and structure for infinite-dimensional systems described in this section mirror those for finite-dimensional systems [6].

Definition 2.1 (Nominal Performance): The closed-loop system in Fig. 1b attains *nominal performance* if it is stable and

$$\|W_1 S\|_\infty = \sup_{\omega\in\mathbb{R}} |W_1(j\omega)S(j\omega)| < 1, \tag{1}$$

where W_1 is a weighting function, $S = 1 - \tilde{P}Q =$

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$1/(1+\tilde{P}C)$ is the sensitivity function, and $\tilde{P}$, Q, and C are the nominal process model, IMC, and classical feedback controller transfer functions, respectively.

The weighting function W_1 is typically selected to be large at low frequencies and small at high frequencies, so as to emphasize performance at low frequencies. A pole at $s=0$ forces the controller to have integral action. The H_∞-norm is the induced system norm for input and output signals quantified in terms of the L_2-norm, regardless of whether the weight and sensitivity are finite or infinite dimensional [1].

Some more definitions and a more formal definition of stability for infinite-dimensional systems is needed in the subsequent statement of the generalized Nyquist stability criterion. For $\sigma \in \mathbb{R}$, define

$$L_{1,\sigma} \equiv \{f(\cdot)|f(\cdot):\mathbb{R}_+ \to \mathbb{C}, \int_0^\infty |f(t)e^{-\sigma t}|dt < \infty\}. \quad (2)$$

The convolution algebra $\mathfrak{A}(\sigma)$ consists of the elements of the form

$$f(t) = \begin{cases} 0, & t<0, \\ f_a(t) + \sum_{i=0}^{\infty} f_i \delta(t-t_i), & t \geq 0, \end{cases} \quad (3)$$

where

- $f_a(\cdot) \in L_{1,\sigma}$,
- $t_0 = 0$ and $t_i > 0,\ \forall i = 1, 2, \ldots,$
- $f_i \in \mathbb{C}$ and $\delta(t-t_i)$ is the Dirac delta distribution applied at t_i, and
- $\sum_{i=0}^{\infty} |f_i| e^{-\sigma t_i} < \infty$.

Definition 2.2 ($\mathfrak{A}(\sigma)$-stability): Let $\mathbb{C}_{\sigma+}$ denote the closed right-half complex plane $\{s \in \mathbb{C}|\ \text{Re}\ s \geq \sigma\}$. Then $f(\cdot)$ is said to belong to $\mathfrak{A}_-(\sigma)$ iff there exists $\sigma_1 \in \mathbb{R}$ with $\sigma_1 < \sigma$ such that $f(\cdot) \in \mathfrak{A}(\sigma_1)$. Further, define $\hat{\mathfrak{A}}(\sigma) = \{\hat{f}|f \in \mathfrak{A}(\sigma)\}$, where $\hat{f}$ is the Laplace transform of f. A system is said to be $\mathfrak{A}(\sigma)$-stable if its transfer function belongs to $\hat{\mathfrak{A}}(\sigma)$.

A closed-loop system that is $\mathfrak{A}(\sigma)$-stable with $\sigma < 0$ is exponentially stable. The generalized Nyquist stability criterion provides an analytical condition for $\mathfrak{A}(\sigma)$-stability of a closed-loop infinite-dimensional system in terms of a path in the complex s-plane.[1]

Theorem 2.3 (Generalized Nyquist Stability Criterion [3]): Suppose the following conditions are satisfied:

1) $\hat{G}(s) = \hat{n}(s)/\hat{d}(s)$, where $\hat{n}(s) \in \hat{\mathfrak{A}}_-(\sigma_0)$ and $\hat{d}(s) \in \hat{\mathfrak{A}}_-(\sigma_0)$,
2) There exist $\hat{u}(s) \in \hat{\mathfrak{A}}_-(\sigma_0)$ and $\hat{v}(s) \in \hat{\mathfrak{A}}_-(\sigma_0)$ such that $\hat{u}(s)\hat{n}(s) + \hat{v}(s)\hat{d}(s) = 1$,
3) $\hat{d}(s)$ is analytic and bounded away from zero at ∞ in $\mathbb{C}_{\sigma_0+}$,
4) $\hat{G}(s)$ approaches zero as $|s| \to \infty$ in $\mathbb{C}_{\sigma_0+}$.

Then, the closed-loop system with open-loop transfer function $\hat{G}(s)$ in unity feedback is $\mathfrak{A}(\sigma)$-stable iff, for some $\sigma < 0$,

- $1 + \hat{G}(s) \neq 0,\ \forall s \in \bar{N}_\infty$,
- $1+\hat{G}(s)$ encircles the origin k times in the counterclockwise sense, where k is the number of open right-half plane zeros of $\hat{d}(s)$, counting multiplicities,

where $\bar{N}_\infty$ is the Nyquist path, which follows the imaginary axis including the points $+j\infty$ and $-j\infty$ except with ϵ-indentations in the left-half plane around any poles of $\hat{G}(s)$ on the imaginary axis.

Model uncertainties in the form of unmodeled dynamics are represented in terms of a family of processes:

$$\begin{aligned}\Pi = \{P : P = (1+\Delta W_2)\tilde{P},\ \text{where}\ \|\Delta\|_\infty \leq 1; \\ P \text{ and } \tilde{P} \text{ have the same number of unstable poles}\}, \quad (4)\end{aligned}$$

where $\tilde{P}$ is strictly proper and the uncertainty weight W_2 and perturbation Δ are stable proper (possibly irrational) transfer functions. Below are robustness analysis conditions for infinite-dimensional systems controlled by proper controllers Q and C. The following is the definition of robust stability.

Definition 2.4 (Robust Stability): The closed-loop system in Fig. 1b is *robust stable* if it is $\mathfrak{A}(\sigma)$-stable for all $P \in \Pi$.

The next result applies the generalized Nyquist stability criterion to derive a necessary and sufficient analytical condition for the robust stability of infinite-dimensional systems. The proof, which is simpler than that in Ref. [2], follows a similar argument as used for finite-dimensional systems [4], [6], but with more care in distinguishing operators from complex numbers.

Theorem 2.5 (Robust Stability): The closed-loop system in Fig. 1b is robust stable iff the nominal system is $\mathfrak{A}(\sigma)$-stable and

$$\|W_2 T\|_\infty < 1, \quad (5)$$

where $T = \tilde{P}C/(1+\tilde{P}C)$ is the complementary sensitivity function.

Proof: ($\Leftarrow$) Suppose

$$\|W_2 T\|_\infty = \sup_{\omega \in \mathbb{R}} |W_2(j\omega)T(j\omega)| < 1, \quad (6)$$

then

$$\|\Delta W_2 T\|_\infty \leq \|\Delta\|_\infty \|W_2 T\|_\infty < 1. \quad (7)$$

Now consider the equality

$$\begin{aligned} 1 + PC &= 1 + (1+\Delta W_2)\tilde{P}C \\ &= (1+\tilde{P}C)(1+\Delta W_2 T). \end{aligned} \quad (8)$$

Nominal $\mathfrak{A}(\sigma)$-stability and the generalized Nyquist stability criterion imply that $1+\tilde{P}C$ along the Nyquist path is not equal to zero and encircles the origin the k times required for $\mathfrak{A}(\sigma)$-stability. The inequality (7) implies that $1+\Delta W_2 T$ is not equal to zero along the Nyquist path and hence does not change the number of encirclements of the origin. That is, $N(1+\tilde{P}C) = N(1+PC)$, where $N(\hat{G}(s))$ is the net number of clockwise encirclements of the origin by the image of the Nyquist path under $\hat{G}(s)$.

($\Rightarrow$) Proof by contrapositive. Suppose

$$\|W_2 T\|_\infty \geq 1, \quad (9)$$

[1]The result is the same as that of Ref. [3] except for the strengthening of Condition 4 as done in Ref. [2].

then $\hat{\Delta} = 1/\|W_2T\|_\infty$ satisfies $\|\hat{\Delta}\|_\infty \leq 1$ and $\|\hat{\Delta}W_2T\|_\infty = 1$. Let $\hat{\omega}$ be a frequency in which this equality holds, then $|\hat{\Delta}W_2T|_{\hat{\omega}}$ is located on a unit circle centered at origin. Select the parameter θ in $\bar{\Delta} = e^{-s\theta}\hat{\Delta}$ so that $|1+\bar{\Delta}W_2T|_{\hat{\omega}} = 0$. Then $\|\bar{\Delta}\|_\infty \leq 1$ and, by (8), $1+PC = 0$ which implies that the system is not robust stable (from the generalized Nyquist stability criterion). ∎

Definition 2.6 (Robust Performance): The closed-loop system in Fig. 1b attains *robust performance* if it is $\mathfrak{A}(\sigma)$-stable and the performance objective (1) is satisfied $\forall P \in \Pi$.

Mathematically, robust performance is equivalent to the satisfaction of the inequalities

$$\|W_2T\|_\infty < 1 \text{ and } \left\|\frac{W_1S}{1+\Delta W_2T}\right\|_\infty < 1,\ \forall \|\Delta\|_\infty \leq 1, \tag{10}$$

with the first condition being the test for robust stability (Thm. 2.5) and the second condition being the performance condition (Defn. 2.1) applied to all processes within the uncertainty description (4).

Theorem 2.7 (Robust Performance): The closed-loop system in Fig. 1b attains robust performance if and only if it is $\mathfrak{A}(\sigma)$-stable and

$$\||W_1S| + |W_2T|\|_\infty < 1. \tag{11}$$

Proof: The proof is very similar to that on pages 56-57 of [4] with substitution of Thm. 2.5 for the analysis of the robust stability for infinite-dimensional systems. ∎

III. IMC Design for DPS

The analysis conditions for infinite-dimensional systems in the previous section are used to develop an IMC design method for DPS. As in the design method for ODEs, the method can be interpreted as determining an exact or approximate inverse of the nominal process model which is augmented with a filter to derive a proper (physically realizable) controller. Extension of this approach to an infinite-dimensional process model can require the use of an infinite-dimensional filter. The required information for controller design are:

1) nominal process model $\tilde{P}$,
2) performance specification (weighting function W_1), and
3) uncertainty weight W_2.

Two methods are proposed for creation of a physically realizable IMC controller.

A. Method 1

The IMC controller consists of the optimal controller for the nominal process model followed by filtering to provide robustness.

1) Nominal performance: Determine the operator $\tilde{Q}$ for the nominal process $\tilde{P}$ that optimizes the nominal performance:

$$\min \|W_1\tilde{S}\|_\infty, \tag{12}$$

where $\tilde{S} = 1 - \tilde{P}\tilde{Q}$ is the nominal sensitivity function. Readers are referred to Ref. [2] for algorithms for solving this minimization for infinite-dimensional processes. If the nominal process is minimum-phase, then the solution is

$$\tilde{Q} = \tilde{P}^{-1}. \tag{13}$$

This nominal operator $\tilde{Q}$ is usually improper since the nominal process is usually strictly proper.

2) Robust stability and performance: Design the IMC controller

$$Q(s) = \tilde{Q}(s)F(s,\lambda) \tag{14}$$

which augments $\tilde{Q}$ with a filter F to detune the optimal controller and trade off speed of response with robustness to model uncertainty, insensitivity to measurement noise, and smoothness of the control action. The classical feedback controller C is given by comparison of the IMC and classical control structures in Fig. 1:

$$C = \frac{Q}{1-\tilde{P}Q}. \tag{15}$$

Usually an infinite-dimensional filter is needed so that the IMC controller Q is proper and the feedback controller C is physically realizable. The filter F should be selected so that the closed-loop system retains desired asymptotic properties as Q is detuned for robustness. In particular, for the error signal resulting from a step input to approach zero at steady-state, the filter should satisfy

$$\lim_{s\to 0} F(s,\lambda) = 1. \tag{16}$$

To provide a one-to-one correspondence to the IMC design method for finite-dimensional systems, the tuning parameter λ in the infinite-dimensional filter F should be defined so that the optimal nominal performance is achieved as $\lambda \to 0$:

$$\lim_{\lambda\to 0} F(s,\lambda) = 1. \tag{17}$$

The specific finite value for the tuning parameter λ can be selected in a number of ways, corresponding to the same criteria used to tune IMC controllers for finite-dimensional systems [6]. For example, λ can be selected as small as possible while satisfying the robust stability or robust performance conditions

$$\|W_2T\|_\infty < 1, \tag{18}$$

$$\||W_1S| + |W_2T|\|_\infty < 1, \tag{19}$$

or to minimize the robust performance condition

$$\min_{\lambda>0} \||W_1S| + |W_2T|\|_\infty \tag{20}$$

with the design being acceptable if the attained objective is less than one. For stable processes, increasing λ slows the closed-loop dynamics and increases robustness to model uncertainties.

B. Method 2

This approach first defines a super-set of the nominal process model. For a minimum-phase nominal process model, construct $\tilde{P}_s(s,\lambda) \supset \tilde{P}(s)$ for $\lambda > 0$ such that $\tilde{P}_s(s,\lambda)$ is minimum-phase and semiproper and satisfies

$$\lim_{\lambda \to 0} \tilde{P}_s(s,\lambda) = \tilde{P}(s). \tag{21}$$

Then the IMC controller is

$$Q(s,\lambda) = \tilde{P}_s^{-1}(s,\lambda), \tag{22}$$

where the IMC tuning parameter λ is selected as described in Method 1. If the nominal process model is non-minimum phase, then $\tilde{P}_s(s,\lambda)$ should be constructed so that the IMC controller Q optimizes the nominal performance (12) as $\lambda \to 0$.

C. Implementation

Method 1 is closest in character to the IMC method for finite-dimensional systems [6] whereas Method 2 is more convenient when inspecting Laplace transform tables to identify suitable forms for the IMC controller. As in the standard IMC method, the IMC controller is uniquely defined as $\lambda \to 0$ but the form of the filter is up to the designer. The transfer function of the classical controller is determined by (15) with the time-domain equations constructed by analytical or numerical solution of the inverse Laplace transform. When the processes in Π are stable, then the control system can be implemented using either the IMC or classical feedback structure in Fig. 1. Only the classical feedback structure can be implemented when the processes are unstable, and any unstable poles in the nominal process must be canceled by unstable zeros in Q, similarly as in the finite-dimensional case [6].

The allowable error in any finite-dimensional approximation of the infinite-dimensional controller can be quantified by using similar analysis methods as developed in Section II.

IV. Example 1

Consider the diffusion equation

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2},\ \forall x \in (0,a),\ \forall t > 0, \tag{23}$$

with Dirichlet and Neumann boundary conditions

$$C(0,t) = u(t), \tag{24}$$

$$\left.\frac{\partial C}{\partial x}\right|_{x=a} = 0, \tag{25}$$

nominal diffusion coefficient $\tilde{D} = 10^{-5}$ m^2/s and the distance across the domain $a = 10^{-2.5}$ m. The minimum-phase transfer function for the nominal process

$$\tilde{P}(s) = \frac{1}{\cosh\sqrt{s/\tilde{D}}} \tag{26}$$

for the control input $u(t)$ and output $C(a,t)$ is obtained by taking Laplace transforms of the PDE (23) and boundary conditions (24) and (25), and solving for the Laplace transform for the output.

The performance weight

$$W_1 = 0.5\frac{0.06s+1}{0.06s}, \tag{27}$$

is selected to specify zero steady-state error for a step input (that is, integral action), a peak sensitivity less than 2, and a closed-loop time constant of 0.06 s.

The model uncertainty is described by the frequency-dependent bound

$$\left|\frac{P(j\omega) - \tilde{P}(j\omega)}{\tilde{P}(j\omega)}\right| \le |W_2(j\omega)|,\ \forall \omega \in \mathbb{R}, \tag{28}$$

with

$$W_2 = \frac{\cosh\sqrt{s/\tilde{D}}}{\cosh\sqrt{s/1.2\tilde{D}}} - 0.8, \tag{29}$$

which requires that the closed-loop system is robust to variations in the diffusion coefficient, $0.72 \times 10^{-5} \le D \le 1.2 \times 10^{-5}$ or 20% uncertainty in the steady-state gain.

An invertible semiproper super-set of the nominal process model

$$\tilde{P}_s(s,\lambda) = \frac{\cosh\lambda\sqrt{s/\tilde{D}}}{\cosh\sqrt{s/\tilde{D}}}, \tag{30}$$

follows naturally from (26). Hence the DPS IMC controller is

$$Q(s,\lambda) = \frac{\cosh\sqrt{s/\tilde{D}}}{\cosh\lambda\sqrt{s/\tilde{D}}} \tag{31}$$

which is the optimal solution of (12) for any fixed $\lambda \ge 0$ with $\tilde{P}_s(s,\lambda)$ in place of the nominal process model.

The nominal sensitivity function and complementary sensitivity function for the above Q and $\tilde{P}$ (26) are

$$S(s) = 1 - \tilde{P}Q = 1 - \frac{1}{\cosh\lambda\sqrt{s/\tilde{D}}}, \tag{32}$$

$$T(s) = \tilde{P}Q = \frac{1}{\cosh\lambda\sqrt{s/\tilde{D}}}. \tag{33}$$

Fig. 2 shows that $\lambda = 0.3$ satisfies the robust stability (18) and robust performance conditions (19), and nearly minimizes (20).

The DPS IMC controller (31) was compared to the finite-dimensional (FD) IMC controller [6] designed from the second-order and tenth-order transfer functions obtained by applying a second-order finite-difference spatial discretization to the PDE (23) with grid size $dx = 0.5$ and $dx = 0.1$, respectively. This discretization provided a much more accurate fit to the frequency response of the nominal infinite-dimensional process and gave much better control system performance than using modal decomposition (details will

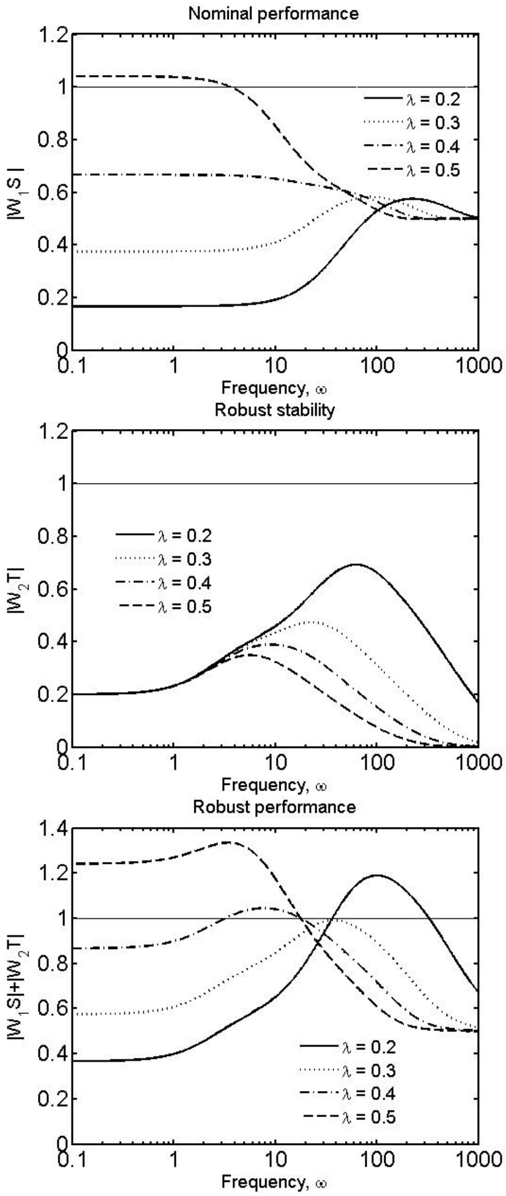

Fig. 2. Bode plots for the evaluation of nominal performance, robust stability, and robust performance for the DPS IMC controller (Example 1).

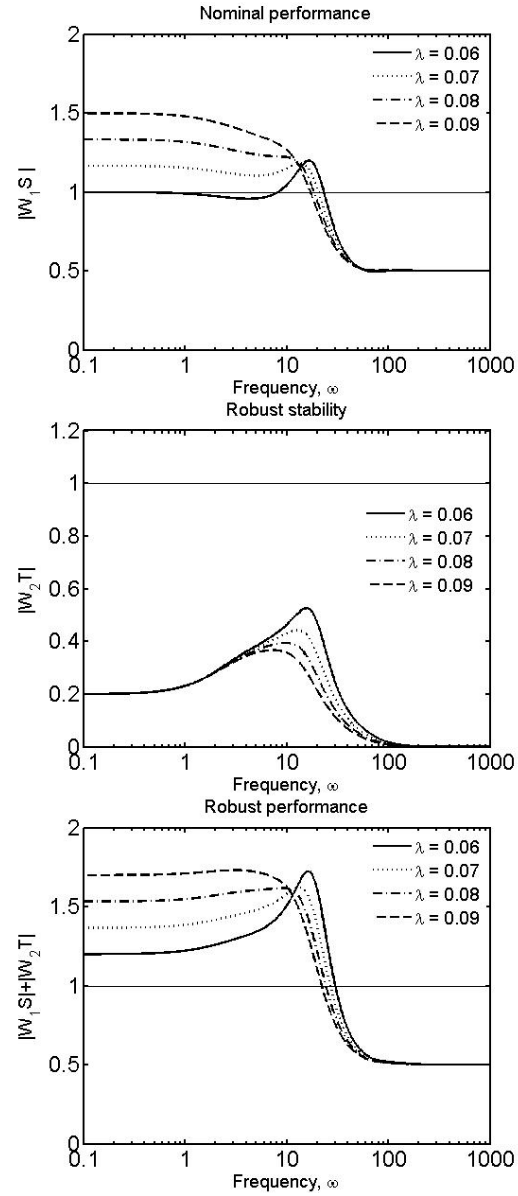

Fig. 3. Bode plots for the evaluation of nominal performance, robust stability, and robust performance for the FD IMC controller with $dx = 0.5$ (Example 1).

be shown in the journal version of this paper). The FD IMC controllers used the filter [6]

$$F(s, \lambda) = \frac{1}{(\lambda s + 1)^n} \tag{34}$$

with $n = 2$ or 10 to produce a proper Q with physically realizable controller C. In both cases, there did not exist *any* IMC tuning parameter λ that achieved robust performance (see Figs. 3 and 4). The values $\lambda = 0.08$ and $\lambda = 0.01$ were selected for the second- and tenth-order transfer functions, respectively, to minimize the violation of the robust performance condition (19), that is, to minimize (20).

To investigate the effects of model uncertainty, setpoint tracking responses were simulated assuming that the real process was (23) with the diffusion coefficient $D = 1.2 \times 10^{-5}$ which is covered by the model uncertainty description. The DPS IMC controller (31) provided much better closed-loop performance than the FD IMC controllers for both sinusoidal and step setpoints (see Fig. 5 and Table I). Further analysis (not shown here for brevity) indicated that the FD IMC controllers do not provide acceptable robust performance for *any* order for the finite-difference discretization of the PDE (23), no matter how high. As the order increases, the frequency response of the finite-difference process model approaches the frequency response of the PDE (23), however, the finite-order IMC filter (34) is poorly matched to the dynamics of the PDE and this mismatch becomes worse as the order is increased. While IMC filters other than (34) could be proposed to attempt to better match the infinite-

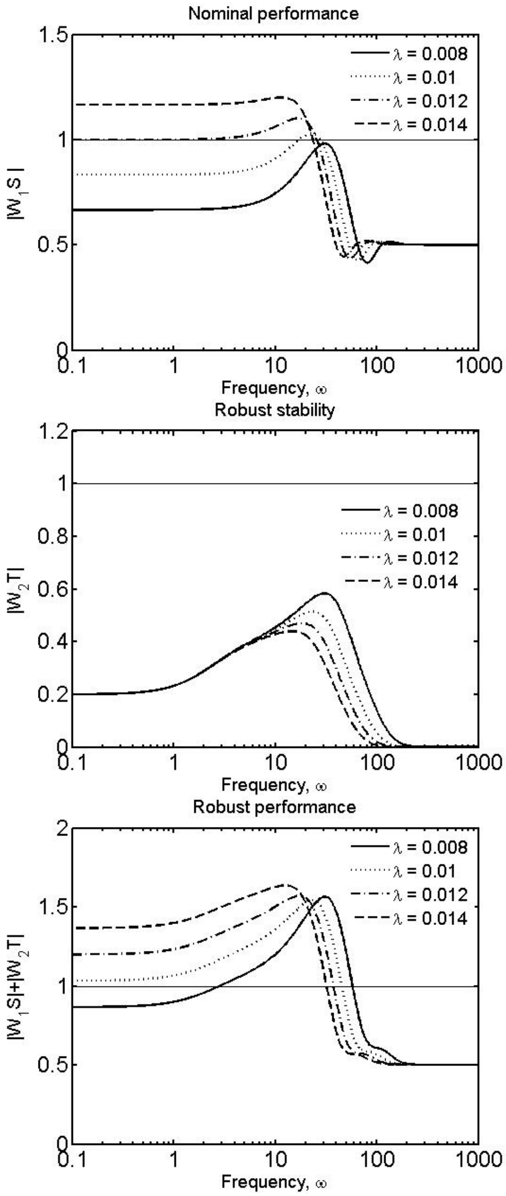

Fig. 4. Bode plots for the evaluation of nominal performance, robust stability, and robust performance for the FD IMC controller with $dx = 0.1$ (Example 1).

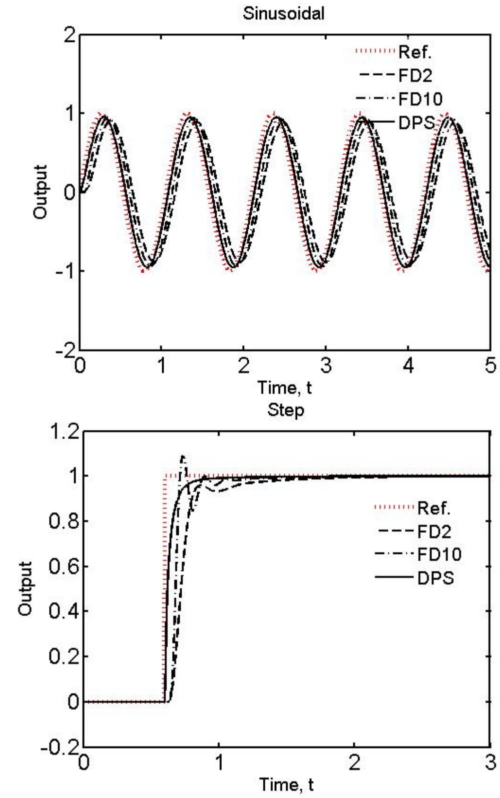

Fig. 5. Closed-loop responses for setpoint tracking of sinusoidal and step signals for Example 1. The “exact” process was modeled by the finite-difference method with grid size $dx = 0.02$ and $D = 1.2 \times 10^{-5}$. FD2 and FD10 are the second- and tenth-order IMC controllers designed based on a finite-dimensional process model. The output is written in terms of deviation variables.

TABLE I
ERRORS IN TIME-DOMAIN SIGNAL NORMS ON THE CONTROLLED VARIABLE (EXAMPLE 1).

setpoint	FD2 IMC	FD10 IMC	DPS IMC
Sinusoidal $\|\cdot\|_\infty$	0.6879	0.4766	0.2199
Sinusoidal $\|\cdot\|_2$	0.0338	0.0235	0.0108
Step $\|\cdot\|_2$	0.0100	0.0085	0.0043

dimensional dynamics of the process, such guesswork is not required during the DPS IMC design where an appropriate form for the infinite-dimensional filter (31) follows naturally from the transfer function of the nominal process model (26).

The IMC controllers designed for robust performance applied to finite-dimensional models of increasing order do not converge to the controller designed by the DPS IMC method (see Fig. 6), regardless of how accurate the finite-dimensional model approximates the infinite-dimensional model. Even if the ultimate goal is to design a finite-dimensional controller, this example indicates that it can be much more efficient and effective to design the infinite-dimensional controller based on the infinite-dimensional process model and then determine a finite-dimensional approximation of the controller. It is straightforward to show, from the continuity of the performance objective with respect to the controller transfer function evaluated at each frequency, that a sufficiently high order finite-dimensional approximation of the DPS IMC controller will satisfy the robust performance criterion (19) provided that the criterion is strictly satisfied for the DPS IMC controller and that the frequency response of the finite-dimensional controller converges to the frequency response of the DPS IMC controller as the order is increased.

V. EXAMPLE 2

This example illustrates the application of the DPS IMC method to an unstable process. Consider a series connection of two processes (A) and (B), where process (A) is governed

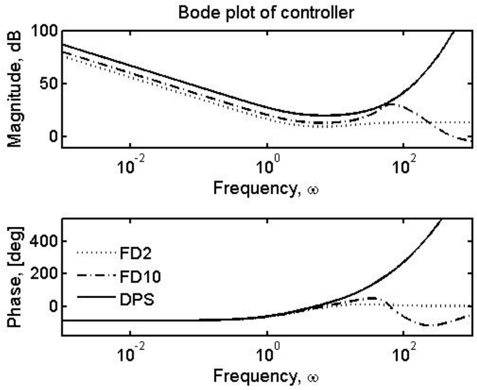

Fig. 6. Bode plots for the controllers $C(s) = \frac{Q}{1-Q\tilde{P}}$ designed by three ways for Example 1.

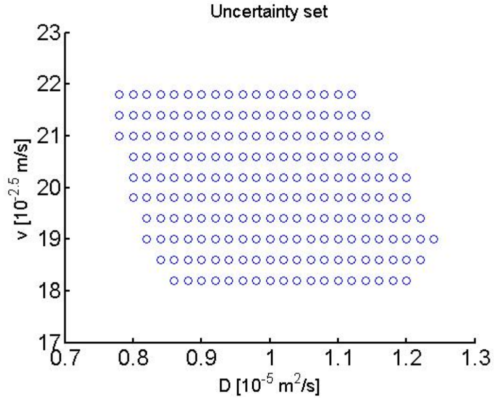

Fig. 7. Uncertainty set covered by W_2 for Example 2.

by the convection equation,

$$\frac{\partial C_A}{\partial t} + v\frac{\partial C_A}{\partial x}, \ \forall x \in (0, a), \ \forall t > 0, \tag{35}$$

and process (B) is governed by the diffusion equation,

$$\frac{\partial C_B}{\partial t} = D\frac{\partial^2 C_B}{\partial x^2}, \ \forall x \in (a, b), \ \forall t > 0, \tag{36}$$

with initial conditions

$$C_A(x, 0) = 0, \tag{37}$$

$$C_B(x, 0) = 0, \tag{38}$$

and boundary conditions

$$C_A(0, t) = u(t), \tag{39}$$

$$\left.\frac{\partial C_B}{\partial x}\right|_{x=b} = 0, \tag{40}$$

$$vC_A(a, t) = -D\left.\frac{\partial C_B}{\partial x}\right|_{x=a}. \tag{41}$$

For the control input $u(t)$ and output $C_B(b, t)$, the nominal process transfer function is

$$\tilde{P}(s) = \frac{\tilde{v}e^{-as/\tilde{v}}}{\sqrt{\tilde{D}s}\sinh\sqrt{\frac{s}{\tilde{D}}}(b-a)} = \tilde{P}_a(s)\tilde{P}_m(s), \tag{42}$$

where

$$\tilde{P}_a(s) = e^{-as/\tilde{v}}, \tag{43}$$

$$\tilde{P}_m(s) = \frac{\tilde{v}}{\sqrt{\tilde{D}s}\sinh\sqrt{\frac{s}{\tilde{D}}}(b-a)}. \tag{44}$$

This is a non-minimum-phase process involving a hyperbolic PDE coupled with a parabolic PDE, which describes the transport of molecules through adjacent gas and liquid films. For $a = 10^{-2.5}$ m, $b = 2{\times}10^{-2.5}$ m, and nominal parameters $\tilde{D} = 10^{-5}$ m/s and $\tilde{v} = 2 \times 10^{-1.5}$ m/s, these nominal transfer functions are

$$\tilde{P}(s) = \frac{20e^{-s/20}}{\sqrt{s}\sinh\sqrt{s}}, \tag{45}$$

$$\tilde{P}_a(s) = e^{-s/20}, \tag{46}$$

$$\tilde{P}_m(s) = \frac{20}{\sqrt{s}\sinh\sqrt{s}}. \tag{47}$$

The performance weight

$$W_1 = 0.3\frac{0.093s + 1}{0.093s} \tag{48}$$

specifies zero steady-state gain error for a step input, a closed-loop time constant of 0.093, and a maximum disturbance amplification of 10/3. The uncertainty weight

$$W_2 = 1.1\frac{\sinh\sqrt{s}}{\sqrt{1.2}\sinh\sqrt{\frac{s}{1.2}}} - 1 \tag{49}$$

covers the set of parameters in Fig. 7.

The invertible semiproper super-set of $\tilde{P}_m(s)$

$$\tilde{P}_{ms}(s, \lambda) = \frac{20\cosh\lambda\sqrt{s}}{\sqrt{s}\sinh\sqrt{s}} \tag{50}$$

follows naturally from (47), which results in the DPS IMC controller

$$Q(s, \lambda) = \frac{\sqrt{s}\sinh\sqrt{s}}{20\cosh\lambda\sqrt{s}}. \tag{51}$$

which is the optimal solution of (12) for any fixed $\lambda \geq 0$ with $\tilde{P}_{ms}(s, \lambda)$ in place of the nominal process model.

The nominal sensitivity function and complementary sensitivity function for the above Q and $\tilde{P}$ (45) are

$$S(s) = 1 - \tilde{P}Q = 1 - \frac{e^{-s/20}}{\cosh\lambda\sqrt{s}}, \tag{52}$$

$$T(s) = \tilde{P}Q = \frac{e^{-s/20}}{\cosh\lambda\sqrt{s}}. \tag{53}$$

As discussed in Section III-C, the control system is implemented using the classical feedback control structure. The zero at $s = 0$ in the IMC controller Q cancels the pole at $s = 0$ in the nominal process when determining the classical controller from (15), and S has a zero at $s = 0$, which is required for nominal stability.

It can be verified that $\lambda = 0.25$ satisfies the robust stability (18) and robust performance conditions (19), and

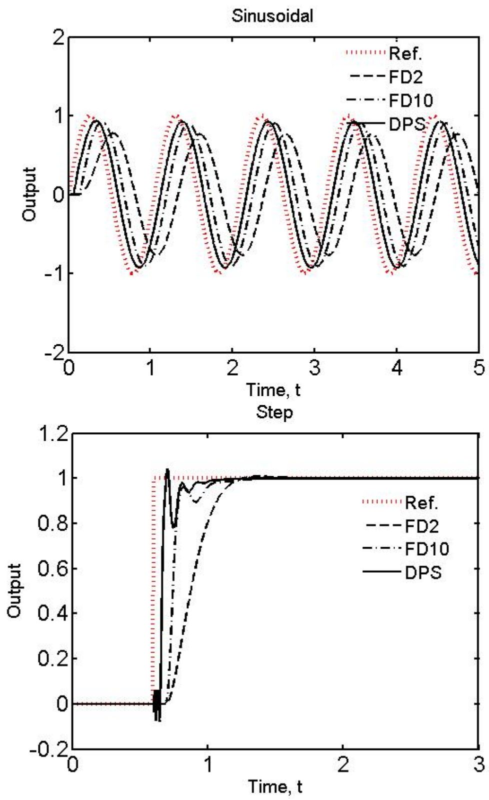

Fig. 8. Closed-loop responses for setpoint tracking of sinusoidal and step signals (Example 2). The "exact" process was modeled by the finite-difference method with grid size $\Delta x = (b-a)/50$, $D = 1.24 \times 10^{-5}$ m^2/s, and $v = 1.9 \times 10^{-1.5}$ m/s. FD2 and FD10 are the second- and tenth-order IMC controllers designed based on a finite-dimensional process model. The output is written in terms of deviation variables.

TABLE II

ERRORS IN TIME-DOMAIN SIGNAL NORMS ON THE CONTROLLED VARIABLE (EXAMPLE 2).

setpoint	FD2 IMC	FD10 IMC	DPS IMC
Sinusoidal $\|\cdot\|_\infty$	1.3563	0.8449	0.4497
Sinusoidal $\|\cdot\|_2$	0.0667	0.0418	0.0223
Step $\|\cdot\|_2$	0.0151	0.0115	0.0080

nearly minimizes (20). By following the same procedure as the Example 1, the values $\lambda = 0.08$ and $\lambda = 0.01$ were selected for the second- and tenth-order transfer functions, respectively. The DPS IMC controller (51) provides better setpoint tracking than the FD IMC controllers (see Fig. 8 and Table II).

VI. CONCLUSIONS AND FUTURE DIRECTIONS

The Internal Model Control method was generalized to the design of controllers for linear distributed parameter systems. The proposed controller provided improved setpoint tracking and robust performance compared to the IMC method designed from an approximate rational transfer function for two DPS. It can be shown that this new IMC design method can be used to extend all of the features of IMC for finite-dimensional systems listed in the Introduction to infinite-dimensional systems, which means that antiwindup compensation, reference prefilter design, cascade control design, and feedforward-feedback control design which is very well-developed for finite-dimensional systems can be applied to distributed parameter systems in a very similar manner. More theoretical work is needed to extend the DPS IMC approach to nonlinear systems, to mirror the developments for finite-dimensional systems [5]. To do this, systematic techniques to invert nonlinear infinite-dimensional operators need to be developed.

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