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Application of Reachability Analysis for Stochastic Hybrid Systems to Aircraft Conflict Prediction

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Abstract— **In this paper, the problem of aircraft conflict prediction is formulated as a reachability analysis problem for a stochastic hybrid system. A switching diffusion model is introduced to predict the future positions of an aircraft following a given flight plan. The weak approximation of the switching diffusion through a Markov chain allows us to develop a numerical algorithm for computing an estimate of the probability that the aircraft enters an unsafe region of the airspace or come too close to another aircraft. Simulation results are reported to show the efficacy of the approach.**

I. Introduction

The rapidly increasing demand for air travel in recent years has been a great challenge to the current Air Traffic Management (ATM) systems. The primary tasks of ATM systems are to maintain smooth air traffic flows and to ensure air safety by avoiding the occurrence of *aircraft conflicts*, namely, aircraft coming within a minimal allowed separation or aircraft entering a forbidden zone. It is thus of central importance to develop highly automated tools and methodologies for the ATM systems to predict future aircraft conflict, both for advance alerting and for conflict resolution.

The development of conflict prediction methods needs to consider several characteristics of aircraft dynamics. First, specified by the air traffic controller by a sequence of timed way-points, the nominal path of an aircraft is typically a piecewise linear one. Second, aircraft motions are subject to various random perturbations such as wind, air turbulence, etc., and thus may deviate from the nominal path. This cross-track deviation may be corrected by the onboard Flight Management System (FMS). In addition, aircraft dynamics may exhibit several distinct modes, for example, keeping a constant heading, turning, ascending, descending, and may switch modes at proper times when following the nominal paths. To accommodate these characteristics, we adopt the modeling framework of *stochastic hybrid systems*, [1], [2].

Stochastic hybrid systems are hybrid systems with continuous dynamics governed by stochastic differential equations and with random discrete mode transitions governed by Markov chains, and they are well-suited for modeling the aircraft dynamics (see, e.g., [3], [4]) due to the random perturbations and the mode-switching behavior exhibited in the aircraft motion when reaching way-points. In this paper, we introduce a simplified version of the simulation model proposed in [3] to predict the aircraft future positions. This prediction model belongs to the family of the *switching diffusion* systems. The aircraft conflict prediction problem is formulated as a *reachability analysis* problem for the switching diffusion system, namely, estimating the probability that the system state enters a certain subset of the state space called the *unsafe set*.

Various previous works exist in studying aspects of the reachability problem of stochastic hybrid systems. In [5], [6], theoretical issues regarding the measurability of the reachability events are addressed. In [6], [7], upper bounds on the probability of reachability events are derived based on the theory of Dirichlet forms associated with a right-Markov process and on certain functions of the state of the system known as barrier certificates, respectively. Another reachability quantity, the exit time, is studied in [8]. In [9], stochastic reachability is addressed in the discrete time case by dynamic programming.

In this paper, we develop a numerical algorithm to compute an asymptotically convergent estimate of the probability that an aircraft conflict occurs. This algorithm is derived based on the methodology for reachability computation of stochastic hybrid systems introduced by the same authors in [10]. The proposed algorithm is based on reachability computations on a Markov chain approximation of the solution to the switching diffusion modeling the aircraft motion.

II. A switching diffusion model to predict the aircraft position

Consider an aircraft flying at some constant altitude in some region of the airspace during the time horizon $T = [0, t_f]$. The aircraft position can be described through a two-dimensional state vector $\mathbf{x} \in \mathbb{R}^2$ of coordinates with respect to some global reference frame $(0, x_1, x_2)$ in the horizontal plane. The aircraft is assigned some flight plan to follow that consists of an ordered sequence of way-points $\{O_i, i = 1, 2, \dots, M+1\}$: $O_i = (x_{1i}, x_{2i}) \in \mathbb{R}^2$, $i = 1, 2, \dots, M+1$. Ideally, the aircraft should fly at some constant speed along the reference path composed of the concatenation of the ordered sequence $\{I_i, i = 1, 2, \dots, M\}$ of line segments I_i with starting point O_i and ending point O_{i+1}, $i = 1, 2, \dots, M$. Deviations from the reference path may be caused by the wind affecting the aircraft position and by limitations in the aircraft dynamics in performing sharp turns, resulting in cross-track error. The onboard 3D FMS tries to reduce the cross-track error by issuing corrective actions based on the aircraft's current geometric deviation

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from the nominal path (without taking into account timing specifications, however). Thus, the state of the aircraft at any time instant t is given by a continuous component $\mathbf{x}(t) \in \mathbb{R}^2$ representing its position, and a discrete component $\mathbf{q}(t) \in \mathcal{Q} := \{1, 2, \ldots, M\}$ depending on which line segment the aircraft is currently tracking.

The aircraft motion is affected by different sources of uncertainty, the main one being the wind. We assume that the wind disturbance acts additively on the aircraft velocity through some nominal contribution $f : \mathbb{R}^2 \to \mathbb{R}^2$ that depends on the aircraft position, and some stochastic contribution modeled by a 2-D standard Brownian motion $\mathbf{w}(t)$.

Under the assumption that the aircraft velocity is constant and equal to $v \in \mathbb{R}^+$, the aircraft position $\mathbf{x} \in \mathbb{R}^2$ during the time horizon T is governed by

$$d\mathbf{x}(t) = v[\cos(\theta(t)) \; \sin(\theta(t))]^T dt + f(\mathbf{x}(t))dt + \sigma d\mathbf{w}(t), \tag{1}$$

where $\theta(t)$ is the heading angle at time $t \in T$.

In our model, the corrective actions of the 3D FMS are modeled by setting the heading angle θ as an appropriate function of $\mathbf{x}$ at any given time $t \in T$. For each segment I_i of the reference path $\{I_i, i \in \mathcal{Q}\}$, we define as reference heading the angle $\Psi_i = \arg(x_{1i+1} - x_{1i} + j(x_{2i+1} - x_{2i}))$ that segment I_i makes with the positive x_1 axis of the reference coordinate frame (see Figure 1).

Suppose that the aircraft is tracking the line segment I_i, for some $i \in \mathcal{Q}$, and is currently at a position x not on I_i. For the aircraft to get on the reference segment I_i as quickly as possible, it should assume a heading, called correction heading, that is orthogonal to and points towards I_i:$\Psi_c(x, i) = \Psi_i - \mathrm{sgn}(d(x, i))\frac{\pi}{2}$. Here, $\mathrm{sgn} : \mathbb{R} \to \{-1, 0, +1\}$ denotes the sign function with $\mathrm{sgn}(0) = 0$, and $d : \mathbb{R}^2 \times \mathcal{Q} \to \mathbb{R}$ denotes the cross-track error function $d((x_1, x_2), i) = -\sin(\Psi_i)(x_1 - x_{1i}) + \cos(\Psi_i)(x_2 - x_{2i})$.

On the other hand, the aircraft should also head towards its next destination way-point O_{i+1}. To compromise between these two objectives of reducing the cross-track error and moving towards the next destination way-point, the heading θ as specified by the FMS is modeled by a convex combination of the reference heading Ψ_i and the correction heading Ψ_c:

$$\theta = u(x, i) = \gamma(x, i)\Psi_c(x, i) + (1 - \gamma(x, i))\Psi_i, \tag{2}$$

where the coefficient of the convex combination is a growing function of the absolute value of the cross-track error:

$$\gamma(x, i) = \min\left(1, \frac{|d(x, i)|}{d_{\mathrm{m}}}\right). \tag{3}$$

Here, $d_{\mathrm{m}} > 0$ is a threshold value for the cross-track error: the more closely it approaches d_{m}, the more the aircraft will follow the correction heading $\Psi_c(x, i)$ rather than its reference heading Ψ_i. Note that the resulting function $u(\cdot, i)$ is continuous because $\gamma(\cdot, i)$ and $d(\cdot, i)$ are continuous.

Let $\mathbf{q}(t) \in \mathcal{Q}$ be the index of the reference line segment at time $t \in T$. Then the dynamics of the aircraft during T can be obtained by plugging (2) into (1):

$$d\mathbf{x}(t) = a(\mathbf{x}(t), \mathbf{q}(t))dt + \sigma d\mathbf{w}(t). \tag{4}$$

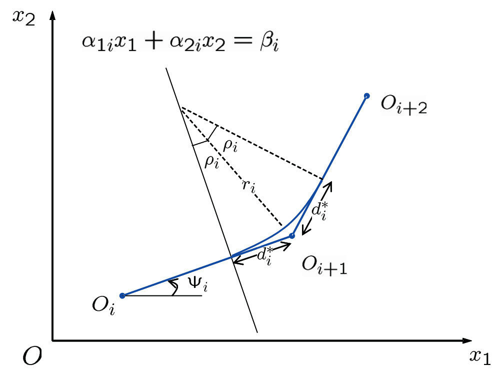

Fig. 1. Reference frame for the "fly-by" turning method.

where we set $a(x, q) = v[\cos(u(x, q)) \; \sin(u(x, q))]^T + f(x)$. The switching law from line segment I_i to the next one I_{i+1} is determined according to the commonly used "fly-by" method of performing turns, where the aircraft turns from I_i to I_{i+1} without passing over the way-point O_{i+1} but by "cutting the corner." In the higher-order aircraft model proposed in [3], the turn starts when the aircraft enters the half-plane $\{(x_1, x_2) \in \mathbb{R}^2 : \alpha_{1i}x_1 + \alpha_{2i}x_2 \geq \beta_i\}$, whose boundary line $\alpha_{1i}x_1 + \alpha_{2i}x_2 = \beta_i$ is chosen so that an aircraft tracking the reference line segment I_i can fly with constant velocity v along an arc of circle joining I_i with I_{i+1} (see Figure 1). If we denote by d_i^* the distance from the way-point O_{i+1} at which an aircraft flying exactly on line segment I_i should start turning, then, $\alpha_{1i} = \frac{x_{1i+1} - x_{1i}}{\|x_{i+1} - x_i\|}$, $\alpha_{2i} = \frac{x_{2i+1} - x_{2i}}{\|x_{i+1} - x_i\|}$, $\beta_i = \frac{x_{1i+1}(x_{1i+1} - x_{1i}) + x_{2i+1}(x_{2i+1} - x_{2i})}{\|x_{i+1} - x_i\|} - d_i^*$. The following expression for d_i^*, $d_i^* = \frac{v^2}{g \tan(\bar{\phi})} \tan\left(\frac{|\Psi_{i+1} - \Psi_i|}{2}\right)$, is derived in [3] from $d_i^* = r_i \tan(\rho_i)$, where $\rho_i = \frac{|\Psi_{i+1} - \Psi_i|}{2}$ and r_i is computed as the velocity v divided by the (constant) angular velocity $\frac{g}{v}\tan(\bar{\phi})$, which is obtained from a higher order aircraft model by assuming that the bank angle is kept constant and equal to $\bar{\phi}$.

Ideally, crossing the switching boundary $\alpha_{1i}x_1 + \alpha_{2i}x_2 = \beta_i$ while tracking I_i should cause a jump in the state component $\mathbf{q}$ of equation (4) from i to $i + 1$. In practice, however, the switching time instant can be uncertain. For this reason, we assume that $\mathbf{q}$ is a Markov chain with switching rates $\lambda_{ij} : \mathbb{R}^2 \to \mathbb{R}$, $i, j \in \mathcal{Q}$, $i \neq j$, that depend on the aircraft position $\mathbf{x}$. More specifically, for any $x = (x_1, x_2) \in \mathbb{R}^2$

$$\lambda_{ij}(x) = \begin{cases} \bar{\lambda}\, g(\alpha_{1i}x_1 + \alpha_{2i}x_2 - \beta_i), & j = i + 1, i < M \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

where $\bar{\lambda}$ is some positive real constant and $g : \mathbb{R} \to [0, 1]$ is a continuous function increasing monotonically from 0 to 1. Thus, the switching rate from I_i to I_{i+1} grows from 0 to $\bar{\lambda}$ while crossing the switching boundary.

As detailed next, the described stochastic hybrid system modeling the aircraft motion generates a switching diffusion process $(\mathbf{x}(t), \mathbf{q}(t))$, $t \in T$, for any initial condition (x_0, q_0).

A. Switching diffusions

A switching diffusion is a stochastic hybrid system with state $\mathbf{s}$ characterized by a continuous component $\mathbf{x}$ and a discrete component $\mathbf{q}$ that take values, respectively, in the Euclidean space $\mathbb{R}^n$ and in the finite set $\mathcal{Q} = \{1, 2, \dots M\}$. Thus, the hybrid state space is given by $\mathcal{S} := \mathbb{R}^n \times \mathcal{Q}$.

The evolution of the discrete state component $\mathbf{q}$ is piecewise constant and right continuous, i.e., for each trajectory of $\mathbf{q}$ there exists a sequence of consecutive left closed, right open time intervals $\{T_i,\, i = 0, 1, \dots\}$, such that $\mathbf{q}(t) = q_i$, $\forall t \in T_i$, with $q_i \in \mathcal{Q}$, and $q_i \neq q_{i\pm1}$.

During each time interval T_i when $\mathbf{q}(t)$ is constant and equal to $q_i \in \mathcal{Q}$, the continuous state component $\mathbf{x}$ evolves according to the stochastic differential equation (SDE)

$$d\mathbf{x}(t) = a(\mathbf{x}(t),\, q_i)dt + b(\mathbf{x}(t),\, q_i)\,\Sigma\, d\mathbf{w}(t), \qquad (6)$$

initialized with $\mathbf{x}(t_i^-) = \lim_{h\to0^+} \mathbf{x}(t_i - h)$ at time $t_i := \inf\{t :\ t \in T_i\}$. Functions $a(\cdot, q_i) : \mathbb{R}^n \to \mathbb{R}^n$ and $b(\cdot, q_i) : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ are the drift and diffusion terms, and matrix $\Sigma \in \mathbb{R}^{n\times n}$ is diagonal with positive entries modulating the variance of the standard n-dimensional Brownian motion $\mathbf{w}(t)$. During the time interval T_i between consecutive jumps in $\mathbf{q}$, then $\mathbf{x}(t)$ behaves as a diffusion process with local properties determined by $a(\cdot, q_i)$ and $b(\cdot, q_i)$.

A jump in the discrete state may occur during the continuous state evolution with an intensity and according to a probabilistic reset map that both depend on the current value taken by $\mathbf{s}$. Specifically, $\mathbf{q}$ is a continuous time process, whose evolution at time t is conditionally independent on the past given $\mathbf{s}(t^-) = (x, q) \in \mathcal{S}$, and is governed by the transition probabilities

$$P\{\mathbf{q}(t + \Delta) = q'|\mathbf{s}(t^-) = (x, q)\} = \lambda_{qq'}(x)\Delta + o(\Delta),$$

for $q' \neq q \in \mathcal{Q}$, where $\lambda_{qq'} : \mathbb{R}^n \to \mathbb{R}$ is the transition rate function.

The transition rate functions determine switching intensity and reset map of the discrete state $\mathbf{q}$. More precisely, during the infinitesimal time interval $[t, t+\Delta]$, $\mathbf{q}(t)$ will jump once with probability $\lambda(s)\Delta + o(\Delta)$, and two or more times with probability $o(\Delta)$, starting from $\mathbf{s}(t^-) = s$, where $\lambda : \mathcal{S} \to [0, +\infty)$ is the jump intensity function given by

$$\lambda(s) = \sum_{q'\in\mathcal{Q}, q'\neq q} \lambda_{qq'}(x), \quad s = (x, q) \in \mathcal{S}. \qquad (7)$$

If $s \in \mathcal{S}$ is such that $\lambda(s) = 0$, then no instantaneous jump can occur from s. Let $s \in \mathcal{S}$ be such that $\lambda(s) \neq 0$. Then, the distribution of $\mathbf{q}(t)$ over $\mathcal{Q}$, after a jump indeed occurs at time t from $\mathbf{s}(t^-) = (x, q)$, is given by the reset function $R : \mathcal{S} \times \mathcal{Q} \to [0, 1]$:

$$R((x, q), q') = \begin{cases} \frac{\lambda_{qq'}(x)}{\lambda(s)}, & q' \neq q \\ 0, & q' = q. \end{cases} \qquad (8)$$

Assumption 1: $\lambda_{qq'}(\cdot)$ is a non-negative function, which is bounded and Lipschitz continuous for each $q, q' \in \mathcal{Q}$, $q \neq q'$. $a(\cdot, q)$, $b(\cdot, q)$ are bounded and Lipschitz continuous for each $q \in \mathcal{Q}$. □

Under Assumption 1, the stochastic hybrid system described above initialized with $s_0 = (x_0, q_0) \in \mathcal{S}$ admits a unique strong solution $\mathbf{s}(t) = (\mathbf{x}(t), \mathbf{q}(t))$, $t \geq 0$, representing a switching diffusion process. Moreover, $\mathbf{s}$ is a Markov process and the trajectories of the continuous component $\mathbf{x}$ are continuous.

III. AIRCRAFT CONFLICT PREDICTION BY REACHABILITY COMPUTATIONS

Our objective is to evaluate the possibility that the aircraft will enter some forbidden area of the airspace $D \subset \mathbb{R}^2$, characterized, for example, by Special Use Airspace (SUA) areas, bad weather or congested zones that could make the flight uncomfortable or even unsafe, during the look-ahead time horizon $T = [0, t_f]$.

With the aircraft dynamics modeled by a switching diffusion process with state $\mathbf{s} = (\mathbf{x}, \mathbf{q})$, the aircraft conflict prediction problem can be reformulated as the following stochastic reachability problem: Given the unsafe set $D \subset \mathbb{R}^n$, determine the probability that the continuous component $\mathbf{x}(t)$ solving (6) reaches D during the look-ahead time horizon $T = [0, t_f]$ when the switching diffusion is initialized with $s_0 = (x_0, q_0) \in \mathcal{S}$:

$$P_{s_0}\big\{\mathbf{x}(t) \in D \text{ for some } t \in T\big\}, \qquad (9)$$

where P_{s_0} is the probability measure induced by the switching diffusion $\mathbf{s}$ with initial condition s_0. If D is measurable and closed, the problem is well-posed since the reachability event "$\mathbf{x}(t) \in D$ for some $t \in T$" is measurable given that the process $\mathbf{x}$ has continuous trajectories, [6].

To evaluate the probability (9) numerically, we introduce a bounded open set $U \subset \mathbb{R}^n$ containing D that is chosen large enough so that the situation can be declared safe once $\mathbf{x}$ wonders outside U. Let U^c denote the complement of U in $\mathbb{R}^n$. Then, with reference to the domain U, the probability of entering D can be approximated by

$$\mathrm{P}_{s_0} := P_{s_0}\big\{\mathbf{x} \text{ hits } D \text{ before hitting } U^c \text{ within } T\big\}. \qquad (10)$$

Hence, for the purpose of computing (10), we can assume that $\mathbf{x}$ in (6) is defined on the open domain $U \setminus D$ with initial condition $x_0 \in U \setminus D$, and that $\mathbf{x}$ is stopped as soon as it hits the boundary $\partial U^c \cup \partial D$ of $U \setminus D$.

We now describe a method to estimate P_{s_0} by weakly approximating the switching diffusion process $\mathbf{s}$ using the piecewise constant interpolation of a suitably defined discrete time Markov chain.

A. Markov chain approximation

The discrete time Markov chain $\{\mathbf{v}_k, k \geq 0\}$ is characterized by a two-component state: $\mathbf{v} = (\mathbf{z}, \mathbf{m})$, where $\mathbf{z}$ takes on values in a finite set $\mathcal{Z}_\delta$ obtained by gridding $U \setminus D$, whereas $\mathbf{m}$ takes on values in the finite set $\mathcal{Q}$. Note that the two components of the Markov chain state $\mathbf{v} = (\mathbf{z}, \mathbf{m})$ are introduced to approximate the two components of the switching diffusion $\mathbf{s} = (\mathbf{x}, \mathbf{q})$, respectively. The interpolation time interval Δ_δ is a positive function of the gridding scale parameter δ and tends to zero faster than δ: $\Delta_\delta = o(\delta)$.

In order to take into account the properties of the pure jump process $\mathbf{q}$ when defining the transition probabilities of the approximating Markov chain $\{\mathbf{v}_k, k \geq 0\}$, it is convenient to introduce an enlarged Markov chain process $\{(\mathbf{v}_k, \mathbf{j}_k), k \geq 0\}$. The discrete time process $\{\mathbf{j}_k, k \geq 0\}$ is an i.i.d. Bernoulli process that represents the jump occurrences: if $\mathbf{j}_k = 1$, then a jump, possibly of zero entity, occurs at time k; whereas if $\mathbf{j}_k = 0$, then no jump occurs at time k. Under the assumption that $\mathbf{j}_k$ is independent of the past variables $\mathbf{v}_i$, $i = 0, 1, \dots, k$, $\forall k \geq 0$, then, it is easily shown that $\{\mathbf{v}_k, k \geq 0\}$ is a Markov chain. Also, the transition probabilities of the Markov chain $\{\mathbf{v}_k, k \geq 0\}$ under the grid scale δ are given by $P_\delta\{\mathbf{v}_{k+1} = v' \mid \mathbf{v}_k = v\} = \sum_{j \in \{0,1\}} P_\delta\{\mathbf{v}_{k+1} = v' \mid \mathbf{v}_k = v, \mathbf{j}_k = j\} P_\delta\{\mathbf{j}_k = j\}$, which are specified by the jump probability $P_\delta\{\mathbf{j}_k = 1\}$, the inter macro-states transition probability $P_\delta\{\mathbf{v}_{k+1} = v' \mid \mathbf{v}_k = v, \mathbf{j}_k = 1\}$, and the intra macro-states transition probability $P_\delta\{\mathbf{v}_{k+1} = v' \mid \mathbf{v}_k = v, \mathbf{j}_k = 0\}$.

Jump probability: We set, for each $k = 0, 1, \dots$,

$$P_\delta\{\mathbf{j}_k = 1\} = 1 - e^{-\lambda_{\max}\Delta_\delta} = \lambda_{\max}\Delta_\delta + o(\Delta_\delta), \qquad (11)$$

where $\lambda_{\max} := \max_{x \in \mathbb{R}^n} \sum_{q, q' \in \mathcal{Q}, q \neq q'} \lambda_{qq'}(x)$.

Inter macro-states transition probability: If $\mathbf{j}_k = 1$ (a jump occurs at time k), then, $\mathbf{z}_{k+1} = \mathbf{z}_k$ since the continuous state component $\mathbf{x}$ of the diffusion process $\mathbf{s}$ is reinitialized with the same value prior to the jump occurrence; whereas the value of $\mathbf{m}_{k+1}$ is determined based on that of $\mathbf{v}_k$ through the (conditional) transition probabilities $p_\delta(q \to q'|z) := P_\delta\{\mathbf{m}_{k+1} = q' \mid \mathbf{v}_k = (z, q), \mathbf{j}_k = 1\}$. In other words,

$$P_\delta\{(\mathbf{z}_{k+1}, \mathbf{m}_{k+1}) = (z', q') \mid \mathbf{v}_k = (z, q), \mathbf{j}_k = 1\} = \begin{cases} 0, & z' \neq z \\ p_\delta(q \to q'|z), & z' = z. \end{cases}$$

We set

$$p_\delta(q \to q'|z) = \begin{cases} \frac{\lambda_{q\,q'}(z)}{\lambda_{\max}}, & q' \neq q \\ 1 - \frac{1}{\lambda_{\max}} \sum_{q^* \in \mathcal{Q}, q^* \neq q} \lambda_{q\,q^*}(z), & q' = q. \end{cases}$$

This way, the probability distribution of $\mathbf{m}_{k+1}$ when a jump of non-zero entity occurs at time k from (z, q) is $P_\delta\{\mathbf{m}_{k+1} = q' \mid \mathbf{v}_k = (z, q), \mathbf{j}_k = 1, \mathbf{m}_{k+1} \neq \mathbf{m}_k\} = R((z, q), q')$, where $R(\cdot, \cdot)$ is the reset function defined in (8). Also, the probability that a jump of non-zero entity occurs at time k from (z, q) is given by $P_\delta\{\mathbf{j}_k = 1, \mathbf{m}_{k+1} \neq q \mid \mathbf{v}_k = (z, q)\} = \lambda((z, q))\Delta_\delta + o(\Delta_\delta)$, where $\lambda(\cdot)$ is the jump intensity function defined in (7).

Intra macro-state transition probability: If $\mathbf{j}_k = 0$ (no jump occurs at time k), then $\mathbf{m}_{k+1} = \mathbf{m}_k$; whereas the value of $\mathbf{z}_{k+1}$ is determined from that of $\mathbf{v}_k$ through the (conditional) transition probabilities $p_\delta(z \to z'|q) := P_\delta\{\mathbf{z}_{k+1} = z' \mid \mathbf{v}_k = (z, q), \mathbf{j}_k = 0\}$ describing the evolution of $\mathbf{z}$ within the "macro-state" $q \in \mathcal{Q}$. In other words,

$$P_\delta\{(\mathbf{z}_{k+1}, \mathbf{m}_{k+1}) = (z', q') \mid \mathbf{v}_k = (z, q), \mathbf{j}_k = 0\} = \begin{cases} 0, & q' \neq q \\ p_\delta(z \to z'|q), & q' = q. \end{cases}$$

For the weak convergence result to hold, the probabilities $p_\delta(z \to z'|q)$ should be suitably selected so as to approximate locally the evolution of the $\mathbf{x}$ component of the switching diffusion $\mathbf{s} = (\mathbf{x}, \mathbf{q})$ with absorption on the boundary $\partial U^c \cup \partial D$ when no jump occurs in $\mathbf{q}$.

To clarify this "local consistency" notion, we need first to introduce some notations. Let $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \dots, \sigma_n)$ with $\sigma_i > 0$, $i = 1, \dots, n$. Fix $\delta > 0$. Denote by $\mathbb{Z}^n_\delta$ the integer grids of $\mathbb{R}^n$ scaled according to δ and the positive diagonal entries of matrix Σ as follows $\mathbb{Z}^n_\delta = \{(m_1\eta_1\delta, m_2\eta_2\delta, \dots, m_n\eta_n\delta) | \; m_i \in \mathbb{Z}, i = 1, \dots, n\}$, where $\eta_i := \frac{\sigma_i}{\sigma_{\max}}$, $i = 1, \dots, n$, with $\sigma_{\max} = \max_i \sigma_i$. For each grid point $z \in \mathbb{Z}^n_\delta$, define the immediate neighbors set as a subset of $\mathbb{Z}^n_\delta$ whose distance from z along the coordinate axis x_i is at most $\eta_i\delta$, $i = 1, \dots, n$, i.e., $\mathcal{N}_\delta(z) = \{z + (i_1\eta_1\delta, \dots, i_n\eta_n\delta) \in \mathbb{Z}^n_\delta | \; (i_1, \dots, i_n) \in \mathcal{I}\}$, where $\mathcal{I} \subseteq \{0, 1, -1\}^n \setminus \{(0, 0, \dots, 0)\}$. $\mathcal{N}_\delta(z)$ represents the set of states to which $\mathbf{z}$ can evolve in one time step within a macro-state, starting from z.

The finite set $\mathcal{Z}_\delta$ where $\mathbf{z}$ takes on values is defined as the set of all those grid points in $\mathbb{Z}^n_\delta$ that lie inside U but outside D: $\mathcal{Z}_\delta = (U \setminus D) \cap \mathbb{Z}^n_\delta$. The interior $\mathcal{Z}^\circ_\delta$ of $\mathcal{Z}_\delta$ consists of all those points in $\mathcal{Z}_\delta$ which have all their neighbors in $\mathcal{Z}_\delta$. The boundary $\partial\mathcal{Z}_\delta = \mathcal{Z}_\delta \setminus \mathcal{Z}^\circ_\delta$ of $\mathcal{Z}_\delta$ is the union of the sets $\partial\mathcal{Z}_{\delta U^c}$ and $\partial\mathcal{Z}_{\delta D}$, where $\partial\mathcal{Z}_{\delta U^c}$ is the set of points with at least one neighbor inside U^c and $\partial\mathcal{Z}_{\delta D}$ is the set of points with at least one neighbor inside D. The points that satisfy both these conditions, if any, are assigned to either $\partial\mathcal{Z}_{\delta D}$ or $\partial\mathcal{Z}_{\delta U^c}$, so as to make these two sets disjoint. This eventually introduces an error in the estimate of the probability of interest, which however becomes negligible if U is chosen sufficiently large.

For each $q \in \mathcal{Q}$, we define $p_\delta(z \to z'|q)$ so that:

- each state z in $\partial\mathcal{Z}_\delta$ is an absorbing state;
- from any state z in $\mathcal{Z}^\circ_\delta$, $\mathbf{z}$ moves to one of its neighbors in $\mathcal{N}_\delta(z)$ or remains at z according to probabilities determined by its current location:

$$p_\delta(z \to z'|q) = \begin{cases} \pi_\delta(z'|(z, q)), & z' \in \mathcal{N}_\delta(z) \cup \{z\} \\ 0, & \text{otherwise}, \end{cases} \quad z \in \mathcal{Z}^\circ_\delta, \qquad (12)$$

where the probability distributions $\pi_\delta(\cdot|(z, q)) : \mathcal{N}_\delta(z) \cup \{z\} \to [0, 1]$, $z \in \mathcal{Z}^\circ_\delta$, are appropriate functions of the drift and diffusion terms in (6) evaluated at (z, q).

Fix some time step k and consider the conditional mean and variance of the finite difference $\mathbf{z}_{k+1} - \mathbf{z}_k$ given that $\mathbf{v}_k = v$ and $\mathbf{j}_k = 0$ (intra macro-state evolution):

$$m_\delta(v) = E_\delta\big[\mathbf{z}_{k+1} - \mathbf{z}_k \mid \mathbf{v}_k = v, \mathbf{j}_k = 0\big]$$

$$V_\delta(v) = E_\delta\big[(\mathbf{z}_{k+1} - \mathbf{z}_k)(\mathbf{z}_{k+1} - \mathbf{z}_k)^T \mid \mathbf{v}_k = v, \mathbf{j}_k = 0\big]$$

For the local consistency property to hold, $\mathcal{N}_\delta(z)$ and $\pi_\delta(\cdot|(z, q)) : \mathcal{N}_\delta(z) \cup \{z\} \to [0, 1]$, $z \in \mathcal{Z}^\circ_\delta$, should be chosen so that $\frac{1}{\Delta_\delta} m_\delta(z,q) \to a(x,q)$, $\frac{1}{\Delta_\delta} V_\delta(z,q) \to b(x,q)\Sigma^2 b(x,q)^T$, as $\delta \to 0$, for all $x \in U \setminus D$, where, for any δ, z is a point in $\mathcal{Z}_\delta^\circ$ closest to x. Different choices are possible that satisfy the local consistency property (see [11]).

Discrete time Markov chain interpolation: Let $\{\boldsymbol{\Delta\tau}_k, k \geq 0\}$ be an i.i.d. sequence of random variables exponentially distributed with mean value Δ_δ, independent of $\{\mathbf{v}_k, k \geq 0\}$ and $\{\mathbf{j}_k, k \geq 0\}$. Denote by $\{\mathbf{v}(t), t \geq 0\}$ the continuous time stochastic process that is equal to $\mathbf{v}_k$ on each interval $[\boldsymbol{\tau}_k, \boldsymbol{\tau}_{k+1})$, with $\boldsymbol{\tau}_0 = 0$ and $\boldsymbol{\tau}_{k+1} = \boldsymbol{\tau}_k + \boldsymbol{\Delta\tau}_k$, $k \geq 0$.

Theorem 1 ([10]): Suppose that the approximating Markov chain $\{\mathbf{v}_k, k \geq 0\}$ is initialized at a point $v_0 \in \mathcal{Z}_\delta^\circ \times \mathcal{Q}$ closest to $s_0 \in (U \setminus D) \times \mathcal{Q}$ and satisfies the local consistency properties. Then, under Assumption 1, the process $\{\mathbf{v}(t), t \geq 0\}$ obtained by interpolation of $\{\mathbf{v}_k, k \geq 0\}$ converges weakly as $\delta \to 0$ to the switching diffusion process $\{\mathbf{s}(t) = (\mathbf{x}(t), \mathbf{q}(t)), t \geq 0\}$ associated with the initial condition s_0, with $\mathbf{x}(t)$ defined on $U \setminus D$ and absorption on the boundary $\partial U^c \cup \partial D$. □

Estimation of the probability of reaching the unsafe set: Consider the look-ahead time horizon $T = [0, t_f]$. Fix $\delta > 0$ so that $k_f := \frac{t_f}{\Delta_\delta}$ is an integer, and construct the approximating Markov chain $\{\mathbf{v}_k = (\mathbf{z}_k, \mathbf{m}_k), k \geq 0\}$ satisfying Theorem 1. Then, the estimate $\hat{\mathrm{P}}_{s_0} := P_\delta\{\mathbf{z}_k \text{ hits } \partial\mathcal{Z}_{\delta D} \text{ before } \partial\mathcal{Z}_{\delta U^c} \text{ within } [0, k_f]\}$ converges with probability one to the probability of interest P_{s_0} in (10). Since both the boundaries $\partial\mathcal{Z}_{\delta U^c}$ and $\partial\mathcal{Z}_{\delta D}$ are absorbing, then $\hat{\mathrm{P}}_{s_0}$ reduces to

$$\hat{\mathrm{P}}_{s_0} = P_\delta\{\mathbf{z}_{k_f} \in \partial\mathcal{Z}_{\delta D}\}. \tag{13}$$

B. Application to the aircraft conflict prediction

In order to complete the definition of the Markov chain approximating the diffusion process modeling the aircraft motion in Section II, we only need to specify the immediate neighbors set $\mathcal{N}_\delta(z)$, the family of distribution functions $\{\pi_\delta(\cdot|v) : \mathcal{N}_\delta(z) \cup \{z\} \to [0,1], z \in \mathcal{Z}_\delta^\circ\}$, and the interpolation time interval Δ_δ, so that the local consistency property holds.

Note that the diffusion term $b(x,q)$ in equation (4) governing the aircraft position $\mathbf{x}$ is given by $b(x,q) = \sigma I$, where I is the identity matrix of size 2. Then, the immediate neighbors set $\mathcal{N}_\delta(z)$, $z \in \mathcal{Z}_\delta$, can be confined to the set of points along each one of the x_i axis whose distance from q is δ, $i = 1, 2$: $z_{1_+} = z + (+\delta, 0)$, $z_{1_-} = z + (-\delta, 0)$, $z_{2_+} = z + (0, +\delta)$, and $z_{2_-} = z + (0, -\delta)$. The transition probability function $\pi_\delta(\cdot|v)$ over $\mathcal{N}_\delta(z) \cup \{z\}$ from $v = (z,q) \in \mathcal{Z}_\delta^\circ \times \mathcal{Q}$ can be chosen as follows:

$$\pi_\delta(z'|v) = \begin{cases} c(v)\,\xi_0(v), & z' = z \\ c(v)\,e^{+\delta\xi_i(v)}, & z' = z_{i_+},\, i = 1,2 \\ c(v)\,e^{-\delta\xi_i(v)}, & z' = z_{i_-},\, i = 1,2, \end{cases} \tag{14}$$

with $\xi_0(v) = \frac{2}{\rho\sigma^2} - 4$, $\xi_i(v) = \frac{[a(v)]_i}{\sigma^2}$, $i = 1,2$, $c(v) = \frac{1}{2\sum_{i=1}^2 \mathrm{csh}(\delta\xi_i(v)) + \xi_0(v)}$, where for any $y \in \mathbb{R}^n$, $[y]_i$ denotes the component of y along the x_i direction, $i = 1,2$. ρ is a positive constant that has to be chosen small enough such that $\xi_0(v)$ defined above is positive for all $v \in \mathcal{Z}_\delta^\circ \times \mathcal{Q}$. In particular, this is guaranteed if $0 < \rho \leq \frac{1}{2\sigma^2}$. As for the interpolation time interval Δ_δ, it can be set equal to $\Delta_\delta = \rho\delta^2$.

With the above choices, it is then easily verified that the local consistency property holds, which implies the weak convergence result in Theorem 1. The estimate $\hat{\mathrm{P}}_{s_0}$ in (13) of the probability of conflict can be computed by the iterative algorithm described hereafter.

Define a set of probability maps $\hat{\mathrm{p}}^{(k)} : \mathcal{Z}_\delta \times \mathcal{Q} \to [0,1]$, $k = 0, 1, \ldots, k_f$, where

$$\hat{\mathrm{p}}^{(k)}(v) := P_\delta\{\mathbf{z}_{k_f} \in \partial\mathcal{Z}_{\delta D} \mid \mathbf{v}_{k_f - k} = v\} \tag{15}$$

is the probability of $\mathbf{z}_k$ hitting $\partial\mathcal{Z}_{\delta D}$ before $\partial\mathcal{Z}_{\delta U^c}$ within the discrete time interval $[k_f - k, k_f]$ starting from v at time $k_f - k$. Then, $\hat{\mathrm{P}}_{s_0}$ can be computed as $\hat{\mathrm{P}}_{s_0} = \hat{\mathrm{p}}^{(k_f)}(v_0)$. Moreover, it is easily seen that $\hat{\mathrm{p}}_\delta^{(k)} : \mathcal{Z}_\delta \times \mathcal{Q} \to [0,1]$, $0 \leq k < k_f$, satisfies the recursion

$$\hat{\mathrm{p}}^{(k+1)}(v) = \sum_{v' \in \mathcal{Z}_\delta \times \mathcal{Q}} p_\delta(v \to v')\hat{\mathrm{p}}^{(k)}(v'), \quad v \in \mathcal{Z}_\delta \times \mathcal{Q}.$$

Hence $\hat{\mathrm{p}}^{(k_f)}$ can be computed by iterating this equation k_f times starting from $k = 0$, initialized with the indicator function of the set $\partial\mathcal{Z}_{\delta D} \times \mathcal{Q}$ by the definition (15) of $\hat{\mathrm{p}}^{(k)}$.

Recalling that any $v \in \partial\mathcal{Z}_\delta \times \mathcal{Q}$ is an absorbing state and that, for each $k = 0, \ldots, k_f$, $\hat{\mathrm{p}}^{(k)}(v) = 1$ if $v \in \partial\mathcal{Z}_{\delta D} \times \mathcal{Q}$, and $\hat{\mathrm{p}}^{(k)}(v) = 0$ if $v \in \partial\mathcal{Z}_{\delta U^c} \times \mathcal{Q}$, we get

$$\hat{\mathrm{p}}^{(k+1)}(v) = \begin{cases} \displaystyle\sum_{v' \in \mathcal{Z}_\delta \times \mathcal{Q}} p_\delta(v \to v')\hat{\mathrm{p}}^{(k)}(v'),\; v \in \mathcal{Z}_\delta^\circ \times \mathcal{Q} \\ 1,\; v \in \partial\mathcal{Z}_{\delta D} \times \mathcal{Q} \\ 0,\; v \in \partial\mathcal{Z}_{\delta U^c} \times \mathcal{Q}. \end{cases}$$

Remark 1 (Computational Complexity): The proposed iterative algorithm to compute $\hat{\mathrm{P}}_{s_0}$ determines all the $k_f + 1$ maps $\hat{\mathrm{p}}^{(k)}$, $k = 0, 1, \ldots, k_f$. Consider the general case where the continuous state space has dimension n and there is a total of M discrete modes. Then for a grid size δ, since $\Delta_\delta = \rho\delta^2$, the computational complexity of the above reachability computation as measured by the total number of recursive iterations is of the order $O(\frac{M}{\delta^{n+2}})$, which grows exponentially fast with the continuous state dimension. This unfavorable feature, however, is also shared by other reachability computation algorithms of *general* deterministic and stochastic hybrid systems. For practical purpose, the grid size δ should be chosen to balance the two conflicting considerations that large δ's may not allow for the simulation of fast moving processes and may lead to larger estimation errors, but for small δ's the running time may be too long.

Despite the computation intensity, our algorithm has the advantage over simulation-based methods, [8], that, after its completion, an estimate of the probability of conflict over the residual time horizon $[t_f - t, t_f]$ of length t is readily available for any $t \in (0, t_f)$, and is given by the map $\hat{\mathrm{p}}^{(\lfloor (t_f - t)/\Delta_\delta \rfloor)}$ evaluated at the state value at time $t_f - t$. This fact may enable one to design a resolution maneuver to avoid the unsafe region during $[0, t_f]$ by adaptively adjusting the aircraft heading based on the probability-to-go map $\hat{\mathrm{p}}^{(\lfloor (t_f - t)/\Delta_\delta \rfloor)}$ pre-computed at the very beginning of the time interval. For instance, the heading of the aircraft could be chosen as the negative gradient direction of $\hat{\mathrm{p}}^{(\lfloor (t_f - t)/\Delta_\delta \rfloor)}$, i.e., the direction along which the probability of conflict decreases the fastest. □

IV. Numerical Example

Consider the sequence of way-points $O_1 = (60, -40)$, $O_2 = (40, -20)$, $O_3 = (40, 0)$, and $O_4 = (60, 20)$ (all coordinates have the unit of km), and a disk D of radius 5 km centered at the point $(60, 15)$. Our goal is to estimate the probability P_{s_0} that an aircraft with flight plan $\{O_i,\, i = 1, 2, 3, 4\}$, velocity v equal to Mach 0.8, and located at an arbitrary initial position x_0 will ever enter the forbidden zone D within the time horizon $T = [0, 200]$ s. Note that we have chosen a case where the nominal flight path crosses the forbidden zone to allow a more prominent visualization of the influence of the FMS correction action on P_{s_0}.

We perform two experiments with the same set of parameters ($\bar{\phi} = 0.2°$, $g = 9.81$ m s^{-2}, $d_{\mathrm{m}} = 200$ km, $\sigma = 0.3$ km$^{1/2}$s^{-1}, $f(\cdot) = 0$, $\bar{\lambda} = 0.03$ s^{-1}, η= 0.5 km^{-1}s^2), except that in the second experiment we set $\gamma(x, i) = 1$ in (2) so that there is no cross-track error correction effort from the FMS in the aircraft dynamics. In both experiments, we choose the gridding scale parameter $\delta = 1$ km and the region U as the rectangle $U = (-10, 110) \times (-90, 30)$. In addition, we assume that the function $g(\cdot)$ in equation (5) is given by $g(y) = 1/(1 + 0.1e^{-500y})$, $y \in \mathbb{R}$. The estimated probability of conflict $\hat{\mathrm{P}}_{s_0}$ is plotted in Figure 2 (first experiment on the top and second on the bottom) as a function of the initial position x_0 within U. Note that the region with higher probability of conflict shrinks considerably in the case of no cross-track error correction. This is because, regardless of the aircraft initial location, the cross-track error correction term tends to cause the aircraft to converge along the reference path, which in itself will pass through the forbidden zone. Without the cross-track error correction, the aircraft will deviate from the reference path with increased probability, thus reducing the likelihood of a conflict.

V. Conclusions

We studied the problem of aircraft conflict prediction as a reachability analysis problem for a switching diffusion. A stochastic approximation scheme to estimate the probability that a single aircraft will enter a forbidden area of the airspace within a finite time horizon was presented. Extension of the approach to more complex aircraft conflict prediction problems is straightforward, although the increased problem dimension causes an exponential growth in the computational effort. The ongoing development of efficient model checkers for Markov chains is much relevant in this respect.

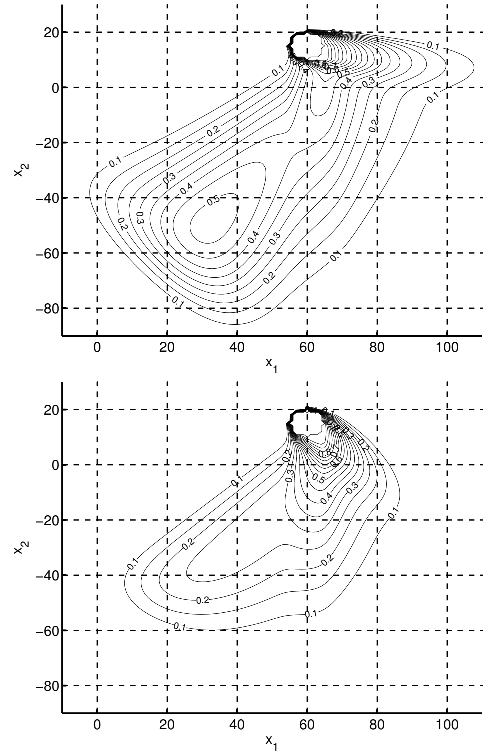

Fig. 2. 2-D contour plot of the estimated P_{s_0}. Top: with cross-track error correction. Bottom: with no cross-track error correction.

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