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Multiple Model Adaptive Control With Mixing: A Pedagogical Example

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***Abstract*— We apply the *adaptive mixing control* approach to a simple example, presenting concepts of the design and analysis of these schemes in a pedagogical manner. Unique to deterministic multiple model adaptive control schemes, adaptive mixing control does not switch discontinuously among candidate controllers. Continuous "mixing" is possible because the multicontroller is constructed by controller interpolation methods and the supervisor generates the mixing signal by monitoring an online estimate of the unknown parameters. This paper also presents stability and robustness results that can be extended to the general case with out difficulty.**

I. INTRODUCTION

Critical to the success of designing practical controllers for application with stringent requirements is an approach that specifies such requirements explicitly in the problem formulation. Modern linear time invariant (LTI) control theories, e.g. $\mathcal{H}_\infty$ and μ-synthesis [1]–[3], provide powerful tools when model uncertainties are sufficiently small. In the presense of "large"uncertainty, a single fixed LTI controller that achieves satisfactory closed-loop behavior may not exist. Adaptive control, on the other hand, is capable of coping with large parametric uncertainty by tuning controller gains in response to estimated changes in the model. Since in conventional adaptive control [4], [5] the controller gains are calculated in real time based on an estimated plant model, the complicated manner in which the plant parameters influence $\mathcal{H}_\infty$ and μ-synthesis controller gains has precluded the use of these modern robust synthesis techniques in a conventional adaptive control setting.

By using controllers designed off-line, multiple model adaptive control (MMAC) schemes avoid real-time controller synthesis issues and, therefore, provide an attractive framework for combining adaptive and modern robust tools. The general MMAC architecture, shown in Fig. 1, comprises two levels of control: (1) a low-level system $\mathbb{C}(\beta)$ called the multicontroller that is capable of generating finely-tuned candidate controls and (2) a high-level system Σ_S called the supervisor that influences the control u by adjusting the multicontroller based on processed plant input/output data. Existing MMAC approaches in literature include switching-based schemes: supervisory control [6]–[8], adaptive control with multiple models [9]–[11], unfalsified control [12], [13]; and robust MMAC [14]–[16], which is a stochastic approach. The switching-based schemes have the advantage

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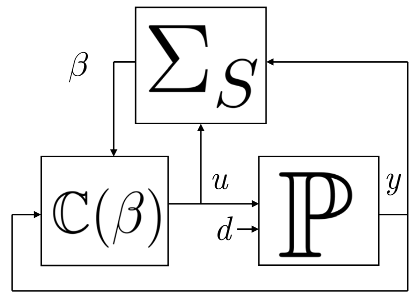

Fig. 1. Multiple model adaptive control architecture: Based on observed data, the supervisor Σ_S selects/blends/mixes candidate controls to apply to the unknown plant $\mathbb{P}$.

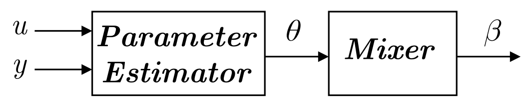

Fig. 2. Adaptive mixing control's supervisor Σ_S comprises an online parameter estimator and a mixer.

over conventional adaptive control of being capable of overcoming the *loss of stabilization problem* and responding rapidly to abrupt parameter changes. However, switching-based schemes may also exhibit behaviors, such as intermittent switching among "similar" models and persistent selection of a poorly performing controller despite data that suggest to switch, that lead to poor performance.

The focus of this paper is the application of the MMAC approach *adaptive mixing control* to a simple example in order to introduce its architecture, design, and analysis. With the aim of eliminating the undesirable behaviors of existing MMAC approaches, candidate controllers are "mixed" into the loop in a continuous manner, driven by a robust adaptive law. Not only is the multicontroller of the adaptive mixing control scheme capable of generating any of the candidate control laws, but also, by controller interpolation, a mix of candidate controllers. This allows the multicontroller to evolve from one controller to another in a continuous manner. The supervisor, shown in Fig. 2, generates the mixing signal $\beta(t)$ by processing the online estimate $\theta(t)$ of the unknown system parameter θ^* through a system called the mixer M that determines the level of participation of each candidate controller. This determination is a manifestation of certainty equivalence: at every fixed $t \geq 0$, the candidate controllers that were designed for $\theta^* = \theta(t)$ are mixed such that closed-loop objectives are met. A more in-depth exposition of adaptive mixing control is the subject of the follow-up paper [17].

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II. Notation and Preliminaries

For $A \in \mathbb{R}^{m\times n}$, the transpose of A is denoted by A^T. For the n-vector x, $|x|$ is the Euclidean norm $(x^Tx)^{1/2}$ and the corresponding induced matrix norm of A is denoted as $\|A\|$. If $y : \mathbb{R}^+ \to \mathbb{R}^n$, then the $\mathcal{L}_p$ norm of y is denoted as $\|y\|_p$ and the truncated $\mathcal{L}_{2\delta}$ norm is defined as

$$\|y_t\|_{2\delta} \triangleq \left(\int_0^t e^{-\delta(t-\tau)}y^T(\tau)y(\tau)d\tau\right)^{\frac{1}{2}} \qquad (1)$$

where $\delta \geq 0$ is a constant, provided that the integral in (1) exists. By $\|y_t\|_2$ we mean that $\|y_t\|_{2\delta}$ with $\delta = 0$, and we say that $y \in \mathcal{L}_{2e}$ if $\|y_t\|_2$ exists. Let $y \in \mathcal{L}_{2e}$, and consider the set

$$\mathcal{S}(\mu) = \left\{y : \int_t^{t+T} y^T(\tau)y(\tau)d\tau \leq c_0\mu T + c_1,\ \forall t, T \geq 0\right\}$$

for a given constant μ, where $c_0, c_1 \geq 0$ are some finite constants independent of μ. We say that y is μ-small in the mean square sense (m.s.s.) if $y \in \mathcal{S}(\mu)$. Furthermore, consider the signal $w : [0,\infty) \to \mathbb{R}^+$ and the set

$$\mathcal{S}(w) = \left\{y : \int_t^{t+T} y^T(\tau)y(\tau)d\tau \leq c_0\int_t^{t+T} w(\tau)d\tau + c_1, \quad \forall t, T \geq 0\right\}$$

where $c_0, c_1 \geq 0$ are some finite constants. We say that y is w-small in the m.s.s. if $y \in \mathcal{S}(w)$.

Let $H(s)$ and $h(t)$ be the transfer function and impulse response, respectively, of some LTI system. If $H(s)$ is a proper transfer function and analytic in $\mathcal{R}[s] \geq -\delta/2$ for some $\delta \geq 0$, where $\mathcal{R}[s]$ denotes the real part of s, then the $\mathcal{H}_\infty$ system norm is given by $\|H\|_\infty \triangleq \sup_{j\omega}|H(j\omega)|$. The $\|\cdot\|_{2\delta}$ system norm of $H(s)$ is defined as $\|H\|_{2\delta} \triangleq \frac{1}{\sqrt{2\pi}}\left\{\int_{-\infty}^{\infty}\left|H\left(j\omega - \frac{\delta}{2}\right)\right|^2 d\omega\right\}^{\frac{1}{2}}$. The induced $\mathcal{L}_\infty$ system norm of H is given by $\|H\|_{\infty-gn} = \|h\|_1$. If $y = H(s)u$ and $\|u\|_\infty = u_0$ then $\|y\|_\infty \leq \|H\|_{\infty-gn}u_0$. We say that $A : [0,\infty) \to \mathbb{R}^{n\times n}$ is *exponentially stable* (e.s.) if its transition matrix $\Phi(t,\tau)$ satisfies $\|\Phi(t,\tau)\| \leq \lambda_0 e^{-\alpha_0(t-\tau)}$for some $\lambda_0, \alpha_0 > 0$ for all $t \geq \tau \geq 0$. The following key results are used in the stability and robustness analysis of the adaptive mixing control scheme. The results are well known, and, unless stated otherwise, their proofs can be found in [5] and the references within.

Theorem 1: Let $\Omega \subset \mathbb{R}^{2n}$ be compact and p be any constant in Ω. Let the parameterized detectable pair $(C(p), A(p))$ be continuously differentiable with respect to $p \in \Omega$, where $A(p) \in \mathbb{R}^{n\times n}$ and $C(p) \in \mathbb{R}^{l\times n}$.

1) Then there exists an analytic matrix function $L : \Omega \to \mathbb{R}^{n\times l}$, such that $A_I(p) \triangleq A(p) - L(p)C(p)$ is a stability matrix uniformly in $p \in \Omega$, i.e., $A_I(p)$ satisfies
$$\max_i \mathcal{R}\{\lambda_i[A_I(p)]\} < -\sigma \qquad (2)$$
for some $\sigma > 0$ independent of p, where $\lambda_i(A_I(p))$ is the i^{th} eigenvalue of the matrix $A_I(p)$.
2) If $\theta(t) \in \Omega$ for all $t \geq 0$ and $\dot{\theta} \in \mathcal{L}_2$ is satisfied in addition to the conditions in 1), then the equilibrium $x_e = 0$ of $\dot{x} = A_I(\theta(t))x$ is e.s.
3) If $\theta(t) \in \Omega$ for all $t \geq 0$ and $\dot{\theta} \in \mathcal{S}(\mu^2)$ is satisfied in addition to the conditions in 1), then there exists a $\mu^* > 0$ such that if $\mu \in [0, \mu^*)$ the equilibrium $x_e = 0$ of $\dot{x} = A_I(\theta(t))x$ is e.s. □

The proof of Theorem 1 is a combination of the well-known results of [18] and the linear time varying (LTV) stability results found in [5].

The following results concern the LTV system given by

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u, \qquad x(0) = x_0 &(3)\\ y &= C(t)x + D(t)u &(4)\end{aligned}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^l$, $u(t) \in \mathbb{R}^m$, and the elements of the matrices $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$, $C(t) \in \mathbb{R}^{l\times n}$, and $D(t) \in \mathbb{R}^{l\times m}$ are bounded continuous functions of time.

Lemma 2: If the LTV system (3),(4) is e.s. and $u \in \mathcal{L}_{2e}$ then

1) for any $\delta \in [0, \delta_1)$ where $0 < \delta_1 < 2\alpha_0$ is arbitrary, we have
$$\|x_t\|_{2\delta} \leq \frac{c\lambda_0}{\sqrt{(\delta_1 - \delta)(2\alpha_0 - \delta)}}\|u_t\|_{2\delta} + \epsilon_t$$
where $c = \sup_t\|B\|$ and ϵ_t is an exponentially decaying to zero term accounting for the possibility that $x_0 \neq 0$.
2) $u \in \mathcal{L}_2 \Rightarrow x \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $\dot{x} \in \mathcal{L}_2$, and $\lim_{t\to\infty}|x(t)| = 0$
3) $u \in \mathcal{S}(\mu) \Rightarrow x \in \mathcal{S}(\mu) \cap \mathcal{L}_\infty$ □

Lemma 3: Consider the LTI system given by $y = H(s)u$ where $H(s)$ is a strictly proper rational function of s. If $H(s)$ is analytic in $\mathcal{R}[s] \geq -\delta/2$ for some $\delta \geq 0$ and $u \in \mathcal{L}_{2e}$ then we have $|y(t)| \leq \|H(s)\|_{2\delta}\|u_t\|_{2\delta}$. □

The following Bellman-Gronwall (B-G) lemma is useful for establishing boundedness.

Lemma 4 (B-G Lemma): Let c_1, c_2 be positive constants and $g(t)$ be a piece-wise continuous function of t. If for all $t \geq t_0 \geq 0$, the function $y(t)$ satisfies the inequality

$$y(t) \leq c_1 + c_2\int_{t_0}^t e^{-\delta(t-\tau)}g^2(\tau)y(\tau)d\tau$$

then for all $t \geq t_0 \geq 0$

$$y(t) \leq (c_0 + c_1)e^{-\alpha(t-t_0)}e^{c_2\int_{t_0}^t g^2(s)ds} + c_1\alpha\int_{t_0}^t e^{-\alpha(t-\tau)}e^{c_2\int_\tau^t g^2(s)ds}d\tau.$$

□

III. Example

A. Problem Statement

Consider the uncertain plant

$$y = \frac{1}{s-\theta^*}(1 + \Delta_m(s))(u + d), \quad \theta^* \in \Omega = [-2.5,\ 2.5] \qquad (5)$$

where θ^* is an unknown constant that belongs to the known interval Ω; d is a bounded disturbance, i.e., $|d(t)| \leq d_0\ \forall t \geq$

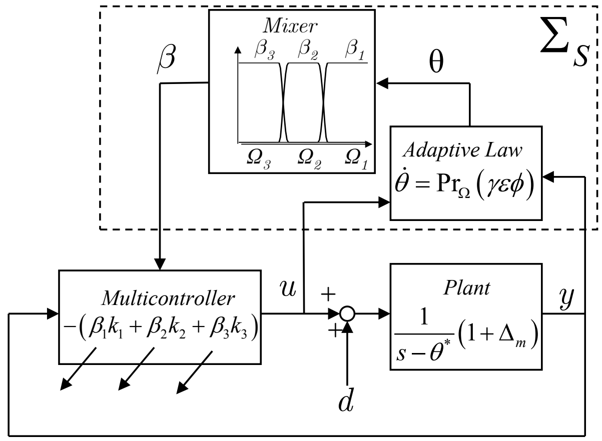

Fig. 3. Block diagram of the adaptive mixing control scheme.

0; and $\Delta_m(s)$ is a multiplicative plant uncertainty. $\Delta_m(s)$ is assumed to be a proper rational transfer function that is analytic in $\mathcal{R}[s] \geq -\delta_0/2$ for some known $\delta_0 > 0$. We refer to $G_0(s) = 1/(s - \theta^*)$ as the nominal model. The control objective is to place the pole of the closed-loop nominal plant in the interval $[-5, -3]$; guarantee that y and u are bounded; and when $d_0 = 0$, y and u converge to zero as $t \to \infty$. Given are three candidate-controller/parameter-subset pairs

$$(k_1, \Omega_1) = (5.5,\ \ [0.5\ \ , 2.5]) \tag{6}$$

$$(k_2, \Omega_2) = (4,\ \ [-1,\ \ 1]) \tag{7}$$

$$(k_3, \Omega_3) = (2.5,\ \ [-2.5,\ \ -0.5]). \tag{8}$$

The candidate index set is defined as $\mathcal{I} \triangleq \{1, 2, 3\}$. The candidate controllers were designed such that if $\theta^* \in \Omega_{i^*}$ for some $i^* \in \mathcal{I} = \{1, 2, 3\}$ then $u = -k_{i^*} y$ would meet the control objective. Let us now describe the design of the adaptive mixing control scheme shown in Fig. 3.

B. Adaptive Mixing Control Design

No candidate controller meets the control objective, but if the supervisor Σ_S could determine that $\theta^* \in \Omega_{i^*}$ for some $i^* \in \mathcal{I}$, then the control objective would be met by applying the candidate control $u_{i^*} = -k_{i^*} y$. Moreover, if θ^* belongs to the parameter overlap $\Omega_i \cap \Omega_{i+1}$ for some $i \in \{1, 2\}$, then any convex combination of controllers k_i and k_{i+1} would also meet the control objective. Thus, we construct the multicontroller $\mathbb{C}(\beta)$ as

$$u = -\mathbb{C}(\beta)y = -(\beta_1 k_1 + \beta_2 k_2 + \beta_3 k_3)y \tag{9}$$

where the mixing signal $\beta \triangleq [\beta_1\ \ \beta_2\ \ \beta_3]^T$ is tuned by the supervisor . Observe that $\mathbb{C}(\beta)$ is continuously differentiable with respect to β.

The supervisor Σ_S generates the mixing signal $\beta(t)$ by feeding an estimate $\theta(t)$ of θ^* to a system called the mixer M, which implements the mapping $\beta : \Omega \to [0, 1]^3$. For the purpose of exposition, we will first examine the mixer design for the known parameter case. When the adaptive design is pursued, we will combine the multicontroller $\mathbb{C}(\beta)$ and the mixer M with an online parameter estimator by replacing θ^* with $\theta(t)$.

If θ^* is known then one possible construction of $\theta^* \mapsto \beta(\theta^*)$ is

$$\beta(\theta^*) \triangleq \frac{[I_{\Omega_1}(\theta^*), I_{\Omega_2}(\theta^*), I_{\Omega_3}(\theta^*)]^T}{\sum_{i \in \mathcal{I}} I_{\Omega_i}(\theta^*)} \tag{10}$$

where, for any $i \in \mathcal{I}$, I_{Ω_i} is the indicator function: $I_{\Omega_i}(\theta^*) = 1$ if $\theta^* \in \Omega_i$; $I_{\Omega_i}(\theta^*) = 0$ if $\theta^* \notin \Omega_i$. Note that β is discontinuous in θ^* and, since θ^* will ultimately be replaced with its estimate $\theta(t)$ in the adaptive case, will lead to practical issues. We therefore define the mixing signal as

$$\beta(\theta^*) \triangleq \frac{[\psi(\frac{\theta^* - 1.75}{1.25}), \psi(\theta^*), \psi(\frac{\theta^* + 1.75}{1.25})]^T}{\psi(\frac{\theta^* - 1.75}{1.25}) + \psi(\theta^*) + \psi(\frac{\theta^* + 1.5}{1.25})} \tag{11}$$

where ψ is the smooth bump function: $\psi(\theta^*) = e^{-\frac{1}{1-\theta^{*2}}}$ if $|\theta^*| \leq 1$; $\psi(\theta^*) = 0$ if $|\theta^*| > 1$. Note that by construction $\beta : \Omega \to [0, 1]^3$ is smooth; $\sum_{i \in \mathcal{I}} \beta_i = 1$; $\beta_i \geq 0$ for all $i \in \mathcal{I}$; and if $\theta^* \notin \Omega_i$ for some $i \in \mathcal{I}$, then $\beta_i = 0$. When $u = -\mathbb{C}(\beta(\theta^*))y$, the nominal closed-loop pole is $s_{CL} = \theta^* - \mathbb{C}(\beta(\theta^*))$. From the above properties of β and (9), we have that

- if $\theta^* \in [-2.5,\ -1]$ then $\beta(\theta^*) = [0\ \ 0\ \ 1]^T$ and $-5 \leq s_{CL} \leq -3.5$
- if $\theta^* \in [-1,\ -0.5]$ then $\beta(\theta^*) = [0\ \ b_2^*\ \ b_3^*]^T$, where $b_2^*, b_3^* \geq 0$ and $b_2^* + b_3^* = 1$, and $-5 \leq s_{CL} \leq -3$
- if $\theta^* \in [-0.5,\ 0.5]$ then $\beta = [0\ \ 1\ \ 0]^T$ and $-4.5 \leq s_{CL} \leq -3.5$
- if $\theta^* \in [0.5, 1]$ then $\beta(\theta^*) = [b_1^* b_2^* 0]^T$, where $b_1^*, b_2^* \geq 0$ and $b_1^* + b_2^* = 1$, and $-5 \leq s_{CL} \leq -3$
- if $\theta^* \in [1,\ 2.5]$ then $\beta(\theta^*) = [1\ \ 0\ \ 0]^T$ and $-4.5 \leq s_{CL} \leq -3$.

Therefore $\mathbb{C}(\beta(\theta^*))$ meets the control objective if θ^* is known.

Now we consider θ^* to be unknown and use the certainty equivalence approach by combining the multicontroller $\mathbb{C}(\beta)$ and mixer M with a robust online parameter estimator. The linear parametric model (LPM) of the plant (5), constructed using the procedure of [5, Sec. 2.4.1], is given by

$$z = \theta^* \phi + \eta,\ \ z = sF(s)y - F(s)u,\ \ \phi = F(s)y \tag{12}$$

$$\eta = \Delta_m F(s)u + (1 + \Delta_m)F(s)d \tag{13}$$

where z, ϕ are measurable signals; η is an unknown modeling error term; $F(s) = \frac{\lambda}{s+\lambda} F_\eta(s)$; $\lambda > 0$; and $F_\eta(s)$ is a stable minimum phase filter. This LPM can be used to generate a wide-class of adaptive laws for generating the estimate θ of θ^* [5] that guarantee that $\theta(t) \in \Omega$ for all $t \geq 0$ (by the use of the projection operator) and when $\Delta_m, d_0 = 0$,

$$\epsilon, \epsilon m, \dot{\theta} \in \mathcal{L}_2 \cap \mathcal{L}_\infty, \quad \text{if } \Delta_m, d_0 = 0 \tag{14}$$

$$\epsilon, \epsilon m, \dot{\theta} \in \mathcal{S}(\eta^2/m^2) \cap \mathcal{L}_\infty, \quad \text{otherwise.} \tag{15}$$

For this example, let us choose $F_\eta(s) = 1$ and the adaptive law as the gradient algorithm with projection modification

$$\dot{\theta} = \Pr_{\Omega}(\gamma\epsilon\phi) = \begin{cases} \gamma\epsilon\phi, & |\theta| < 2.5 \text{ or } \gamma\epsilon\phi \operatorname{sgn}\theta \le 0 \\ 0, & \text{otherwise} \end{cases} \tag{16}$$

$$\epsilon = \frac{\epsilon_1}{m^2} = \frac{z - \theta\phi}{m^2} \tag{17}$$

$$m^2 = 1 + n_d, \quad \dot{n}_d = -\delta_0 n_d + u^2 + y^2 \tag{18}$$

where $\gamma > 0$ is the adaptive gain.

To complete the adaptive mixing control design, we combine the adaptive law (16) with the mixer M given in (11) so that $\beta(t)$ takes on the value

$$\beta(\theta(t)) = \frac{[\psi(\frac{\theta(t)-1.75}{1.25}),\ \psi(\theta(t)),\ \psi(\frac{\theta(t)+1.75}{1.25})]^T}{\psi(\frac{\theta(t)-1.75}{1.25}) + \psi(\theta(t)) + \psi(\frac{\theta(t)+1.5}{1.25})}. \tag{19}$$

C. Analysis

Let the plant be realized as by $(A_{\mathbb{P}}, B_{\mathbb{P}}, C_{\mathbb{P}}, D_{\mathbb{P}})$, where the pairs $(C_{\mathbb{P}}, A_{\mathbb{P}})$ and $(A_{\mathbb{P}}, B_{\mathbb{P}})$ are detectable and stabilizable, respectively. Let us define a system called the *estimation model* $\mathbb{E}(\theta)$ as the dynamical system $\epsilon_1 = z - \theta\phi$ with the realization $(A_{\mathbb{E}}, B_{\mathbb{E}}, C_{\mathbb{E}}, D_{\mathbb{E}})$, where $A_{\mathbb{E}}$ is a stability matrix. For the purpose of stability and robustness analysis and by following the tunability analysis approach of [19], we study the behavior of the system comprised of the plant, multicontroller, mixer, and estimation model, which we call the *parameterized system* $\Sigma(\theta)$. The output of Σ is defined as the unnormalized estimation error ϵ_1. The state-space realization of Σ is written compactly as

$$\dot{x} = A(\theta)x + Bd, \quad \epsilon_1 = C(\theta)x \tag{20}$$

where $x = [x_{\mathbb{P}}^T, x_{\mathbb{E}}^T]^T$ where $x_{\mathbb{P}}, x_{\mathbb{E}}$ are the states of the plant and $\mathbb{E}(\theta)$, respectively. We also define the *parameterized controller* as $k(\theta) \stackrel{\triangle}{=} \mathbb{C}(\beta(\theta))$, i.e., the mixer/multicontroller interconnection.

The stability analysis of adaptive mixing control is carried out in four steps. In the first three steps we consider the adaptive mixing scheme applied to the nominal plant $(\Delta_m, d_0 = 0)$ with the objective of establishing that $x \to 0$ as $t \to \infty$. Then in the fourth step we consider the application of the adaptive mixing scheme to the true system and analyze its robustness properties. For all steps, assume that θ^* is any constant in Ω.

Step 1: Establish that $\forall p \in \Omega$, $(C(p), A(p))$ *is a detectable pair.*

Consider the adaptive law initialization $\theta(0) = p$, where p is any constant in Ω. If we let $\epsilon_1 \equiv 0$ then from (16) there is no adaptation, i.e., $\theta \equiv p$; therefore the closed-loop system is an LTI system. Since $\epsilon_1 \equiv 0$, it follows from (12) that $z = p\phi$ and y,u satisfy

$$\frac{s\lambda}{s+\lambda}y - \frac{\lambda}{s+\lambda}u = p\frac{\lambda}{s+\lambda}y. \tag{21}$$

Likewise, because there is no adaptation, the parameterized controller $k(p)$ is constant and, therefore, u satisfies

$$u = -k(p)y. \tag{22}$$

By substituting (22) into (21), we can rewrite (21) as

$$(s - p + k(p))y = 0 \tag{23}$$

and since $k(p)$ was constructed to ensure that $(s-p+k(p))$ is Hurwitz, we have that $y \in \mathcal{L}_\infty$ and $y \to 0$ as $t \to \infty$. Therefore, it follows from (22) that $u \in \mathcal{L}_\infty$ and $u \to 0$ as $t \to \infty$, and in turn, because $A_{\mathbb{E}}$ is a stability matrix, we have that $x_{\mathbb{E}} \in \mathcal{L}_\infty$ and $x_{\mathbb{E}} \to 0$ as $t \to \infty$. Since $\epsilon_1 \equiv 0$ implies $x \to 0$ as $t \to \infty$, the parameterized closed-loop system $\Sigma(p)$ is detectable on Ω.

Step 2: Establish that along the solutions of (20),(16)-(18) there exists a vector-valued function $L : \Omega \to \mathbb{R}^{3\times 1}$ *such that* $A_I(t) \stackrel{\triangle}{=} A(\theta(t)) - L(\theta(t))C(\theta(t))$ *is exponentially stable.*

Since the adaptive law guarantees that $\theta(t) \in \Omega$, and the pair $(C(\theta), A(\theta))$ is affine, and consequently continuously differentiable, with respect to θ, it follows from result 1) of Theorem 1 that there exists a continuously differentiable function $L : \Omega \to \mathbb{R}^{3\times 1}$ such that for each fixed $t \ge 0$ that $A_I(t) \stackrel{\triangle}{=} A(\theta(t)) - L(\theta(t))C(\theta(t))$ is a stability matrix uniformly in $\theta(t) \in \Omega$, i.e., $\max_i \mathcal{R}\{\lambda_i[A_I(t)]\} < -\sigma$ for some $\sigma > 0$ and $\forall t \ge 0$, where $\lambda_i(A)$ is the i^{th} eigenvalue of the matrix $A \in \mathbb{R}^{n\times n}$. Observe that since L is analytic and Ω is compact, $\|L\| \in \mathcal{L}_\infty$. Because the adaptive law additionally guarantees that $\dot{\theta} \in \mathcal{L}_2$, it follows from result 2) of Theorem 1 that the transition matrix $\Phi(t,\tau)$ for the differential equation

$$z(t) = A_I(t)z(t) \tag{24}$$

satisfies $\|\Phi(t,\tau)\| \le \lambda_0 e^{-\alpha_0(t-\tau)}$ for some positive constants λ_0, α_0 and $t \ge \tau \ge 0$.

Step 3: Establish boundedness and convergence of x.

Let $\delta \in [0, \delta_1)$ where $\delta_1 < \min\{2\alpha_0, \delta_0\}$ and $c > 0$ denotes any finite constant.

By applying output injection, we rewrite (20) as

$$\dot{x} = A_I(t)x + L(\theta(t))\epsilon_1 \tag{25}$$

where in Step 2 we established e.s. of the homogeneous part of (25). For the purpose of establishing that $\epsilon_1 \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, we will show that $m \in \mathcal{L}_\infty$, which together with $\|L\| \in \mathcal{L}_\infty$, $\epsilon_1 = \epsilon m^2$, and the properties (14) guaranteed by the adaptive law will establish boundedness and convergence of x.

By result 1) of Lemma 2 and the e.s. property of A_I, we have that

$$\|x_t\|_{2\delta} \le c\|(\epsilon_1)_t\|_{2\delta} + c. \tag{26}$$

Since y is a subvector of x we have that

$$\|y_t\|_{2\delta} \le c\|(\epsilon_1)_t\|_{2\delta} + c \tag{27}$$

and therefore it follows from $u = -k(\theta(t))y$ and $k \in \mathcal{L}_\infty$ that

$$\|u_t\|_{2\delta} \le c\|(\epsilon_1)_t\|_{2\delta} + c. \tag{28}$$

Consider the fictitious normalization signal

$$m_f^2 \stackrel{\triangle}{=} 1 + \|u_t\|_{2\delta}^2 + \|y_t\|_{2\delta}^2. \tag{29}$$

Note that because $\delta < \delta_0$, it follows from the definitions of m and m_f that $m \leq m_f$. Substituting (27), (28), and $\epsilon_1 = \epsilon m^2$ into (29) yields

$$m_f^2 \leq c\|(\epsilon m^2)_t\|_{2\delta}^2 + c \leq c\|(\epsilon m m_f)_t\|_{2\delta}^2 + c \tag{30}$$

where the second inequality is obtained by using $m \leq m_f$. From the definition of $\|(\cdot)_t\|_{2\delta}$ it follows that

$$m_f^2 \leq c\int_0^t e^{-\delta(t-\tau)}(\epsilon(\tau)m(\tau))^2 m_f^2(\tau)d\tau + c. \tag{31}$$

Applying the B-G Lemma to (31) with $g(\tau) = \epsilon(\tau)m(\tau)$ yields

$$m_f^2 \leq ce^{-\delta t}e^{c\int_0^t g^2(\tau)d\tau} + c\delta\int_0^t e^{-\delta(t-s)}e^{c\int_s^t g^2(\tau)d\tau}ds. \tag{32}$$

Boundedness of m_f, and in turn m, follows from $g = \epsilon m \in \mathcal{L}_2$.

We now turn our attention to the injected system (25). The term $L(t)\epsilon(t)m^2(t)$ can be viewed as the input $\bar{u}$ into the exponentially stable linear system $\dot{x} = A_I(t)x + \bar{u}$. Because $\|L\|, m \in \mathcal{L}_\infty$ and $\epsilon m \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, the input $\bar{u} = L\epsilon m^2$ belongs to $\mathcal{L}_2 \cap \mathcal{L}_\infty$. Since the equilibrium solution $x_e = 0$ of $A_I(t)$ is e.s. and $L\epsilon m^2 \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, it follows from result 2) of Lemma 2 and (25) that $x \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $\dot{x} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, and $x \to 0$ as $t \to \infty$. From the convergence property of x, and consequently $L\epsilon_1$, it follows from (25) that $\dot{x} \to 0$ as $t \to 0$.

Step 4: Establish robustness claims

Now suppose that $\Delta_m, d_0 \neq 0$. Consequently, the robust adaptive law no longer guarantees that $\epsilon, \epsilon m, \dot{\theta} \in \mathcal{L}_2$, but rather, it guarantees that $\epsilon, \epsilon m, \dot{\theta} \in \mathcal{S}(\eta^2/m^2)$. Nonetheless, the analysis approach for the nominal case can be applied to the robustness analysis with only minor modifications.

It follows from Lemma 3, $d \in \mathcal{L}_\infty$, and (13) we have

$$|\eta(t)| \leq \Delta_1\|u_t\|_{2\delta_0} + \Delta_2 \tag{33}$$

$$\Delta_1 \stackrel{\triangle}{=} \|\Delta_m F\|_{2\delta_0}, \;\; \Delta_2 \stackrel{\triangle}{=} \|(1+\Delta_m)F\|_{\infty-gn}\, d_0 \tag{34}$$

and because $m^2 = 1 + \|u_t\|_{2\delta_0}^2 + \|y_t\|_{2\delta_0}^2 \geq 1$, it follows that $|\eta(t)|/m \leq \Delta_1 + \Delta_2$. Therefore, ϵ, ϵm, $\dot{\theta} \in \mathcal{S}(\mu^2)$, where $\mu^2 \stackrel{\triangle}{=} c(\Delta_1^2 + \Delta_2^2)$.

The detectability result established in Step 1 is still valid, a consequence of the certainty equivalence stabilization theorem (cf. [20]). Following Step 2, there exists a smooth function $L : \Omega \to \mathbb{R}^{\bar{n}\times 1}$, where $\bar{n}$ is the dimension of x, such that the system matrix $A(p) - L(p)C(p)$ is e.s. for any constant $p \in \Omega$. We apply output injection to the parameterized system to rewrite the dynamics of x as

$$\dot{x} = A_I(t)x + L(\theta(t))\epsilon_1 + Bd \tag{35}$$

In contrast to the ideal case where $\dot{\theta} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, here the robust adaptive law only guarantees that $\dot{\theta} \in \mathcal{S}(\mu^2) \cap \mathcal{L}_\infty$. It then follows from result 3) of Theorem 1 that $A_I(t)$ is e.s. provided that

$$c\left(\Delta_1^2 + \Delta_2^2\right) < \mu^* \tag{36}$$

for some μ^*. Condition (36) may not be satisfied, even for small Δ_1, unless d_0 is sufficiently small. One way to deal with the disturbance term is to design the component $F_\eta(s)$ of $F(s)$ such that Δ_2 is sufficiently small, say $c\Delta_2^2 < \mu^*/2$ so that for $c\Delta_1^2 < \mu^*/2$, condition (36) is always satisfied. The constant Δ_2 can be made arbitrarily small by choosing $F_\eta(s) = c$ where $c > 0$ is a small design constant. Because this would slow adaptation across all frequencies, a more practical approach would be to shape F_η such that $|F_\eta(j\omega)|$ is small in the frequency range of the disturbance. We continue with the assumption that condition (36) is satisfied. Therefore the homogeneous part of (35) is e.s., i.e., the transition matrix $\Phi(t,\tau)$ of $A_I(t)$ satisfies $\|\Phi(t,\tau)\| \leq \lambda_0 e^{-\alpha_0(t-\tau)}$ for some positive constants λ_0, α_0 and $t \geq \tau \geq 0$.

Now we prove the claim that $\epsilon_1 \in \mathcal{S}(\mu^2) \cap \mathcal{L}_\infty$. The bound (32) on m_f^2 still holds. From $m_f \in \mathcal{L}_\infty$ and $m \leq m_f$, we have that m is bounded. It follows from $m, \|L\| \in \mathcal{L}_\infty$ and $\epsilon m \in \mathcal{S}(\mu^2) \cap \mathcal{L}_\infty$ that $L\epsilon m^2 \in \mathcal{S}(\mu^2) \cap \mathcal{L}_\infty$. Therefore, it follows from (35) and the e.s. of $A_I(t)$ that $x \in \mathcal{L}_\infty$. Furthermore, if $\lim_{t\to\infty} d(t) = 0$ then it follows from result 3) of Lemma 2 that $x \in \mathcal{S}(\mu^2)$, i.e., the mean value of x is of the order of the modeling error characterized by μ^2. Also, from $x, L\epsilon m^2 \in \mathcal{S}(\mu^2) \cap \mathcal{L}_\infty$, it follows from (35) that $\dot{x} \in \mathcal{L}_\infty \cap \mathcal{S}(\mu^2)$.

The condition for stability is, therefore,

$$\mu^2 < \delta^* \stackrel{\triangle}{=} \min\{\mu^*, \delta/c\}, \quad 0 < \delta < \min\{\delta_0, 2\alpha_0\} \tag{37}$$

for some constant $c > 0$ and $\mu^* > 0$ is the bound for μ^2 for $A_I(t)$ to be e.s.

In summary, we have shown that in the absence of multiplicative uncertainty Δ_m and exogenous input d that the states $x_{\mathbb{P}}, x_{\mathbb{E}}, \theta$ remain bounded and $x_{\mathbb{P}}, x_{\mathbb{E}} \to 0$ as $t \to \infty$. When Δ_m and d satisfy condition $c\mu^2 < \delta^*$, then all closed-loop states are bounded and, if $d \to 0$ as $t \to \infty$, x and $\dot{x}$ are μ^2-small in the m.s.s.

D. Simulation

We now simulate the adaptive mixing control scheme (9),(19),(16)-(18) applied to the plant given by (5). For simulation purposes, we use the plant parameters $\theta^* = 2.5 \in \Omega_1$, $\Delta_m(s) = \frac{-2\mu s}{1+\mu s}$, $\mu = 0.1$, $d_0 = 0$, and $x_{\mathbb{P}}(0) = [1,\ 1]^T$. We use the control parameters $\gamma = 100$, $\lambda = 5$, $\delta_0 = 4$, and $\theta(0) = 0$. For comparison, we also simulate an adaptive pole-placement control (APPC) scheme with the control $u = -(\theta(t)+3)y$ and the same adaptive law and initialization as the adaptive mixing control scheme. The plant output is shown in Fig. 4. In this example, the adaptive mixing control exhibits faster regulation to zero and less oscillatory behavior compared to the APPC scheme. The improved performance is from the fact that oscillations in θ, shown in Fig. 5, caused by exciting Δ_m, are not seen in the control when $\theta(t) \geq 0.5$. Said another way, outside the model overlaps, the mixer output is constant and therefore the controller-supervisor loop is "open" in the sense that small deviations in the parameter estimate do not affect the control. This is not the case for the APPC scheme, where the oscillations

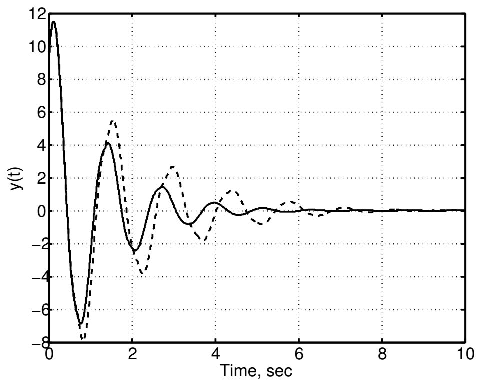

Fig. 4. Plant output: Adaptive mixing control (solid); Adaptive pole-placement control (dashed).

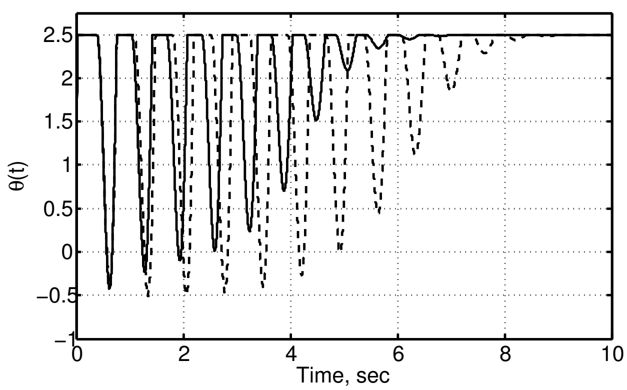

Fig. 5. Estimate $\theta(t)$ of $\theta^* = 2.5$: Adaptive mixing control (solid); Adaptive pole-placement control (dashed).

in u caused by θ further excites Δ_m. Simulations show that the APPC scheme remains stable for $\mu \leq 0.113$; adaptive mixing control remains stable for $\mu \leq 0.12$; and perfect identification $\theta(t) \equiv \theta^*$ remains stable for $\mu \leq 0.124$.

IV. Concluding Remarks

In this paper we applied the adaptive mixing control approach to a pedagogical example to illustrate its design and analysis. We established that the closed-loop states remain bounded when the scheme is applied to the true plant with multiplicative uncertainty and bounded disturbance. When the true system matches the nominal model and in the absence of an external disturbance, the plant output y and input u converge to zero. Simulation results demonstrated its robustness with respect to model uncertainty. Furthermore, for this simple example, the adaptive mixing control scheme exhibited improved robustness when compared to an analogous (conventional) adaptive pole-placement scheme.

This paper represents our first step towards the general exposition of adaptive mixing control [17]. While the example in this paper illustrates the concepts of adaptive mixing control approach, it does leave several issues unaddressed. First, because of its pole-placement control objective, this simple example does little to justify its use over conventional adaptive approaches. An application with demanding performance objectives that requires the use of modern controller synthesis methods would demonstrate the performance advantage of adaptive mixing control over conventional adaptive control. Second, the *ad hoc* output blending strategy used for constructing the multicontroller works for this specific example, but it is not guaranteed to work in general. These are among the topics addressed in [17] and [21]. Further work is focused in combining mixing and switching approaches with the aim of developing schemes that possess the beneficial properties of each, while circumventing their undesirable properties.

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