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Revisiting the Bloch Equation through Averaging

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***Abstract*— A novel approach to finding an approximate analytic solution to the Bloch equation is developed in this paper. The method is based on time scaling and averaging of the Bloch equation after transformation to a rotating frame of reference. In order to accomplish the scaling, a novel time scaled magnetisation vector is introduced. The resultant time scaled system is subsequently approximated through averaging, a technique that to the best of our knowledge, has not previously been applied in the nuclear magnetic resonance context. Our proposed method of approximating the solution to the Bloch equation is valid for continuous wave excitation as well as the traditional pulse excitation with an arbitrary envelope, making this a widely applicable technique unlike previously proposed methods. Comparison of the approximate analytic solution and simulation results clearly indicates that the error is negligible when the field inhomogeneities are small compared to the excitation field amplitude. Extremum seeking techniques may be applied to determine the optimal excitation, given the form of the approximate solution. This result is applicable to a range of research areas including nuclear magnetic resonance, magnetic resonance imaging and optical resonance problems.**

I. INTRODUCTION

The Bloch equation, developed in 1946, describes the empirical behaviour of an ensemble of spins in the presence of an external magnetic field [1]. This equation is applicable to nuclear magnetic resonance, magnetic resonance imaging (MRI) and optical resonance problems.

In 1949 Torrey presented an analytic solution for a long-lived constant pulse excitation by adopting the Laplace transform [2]. Madhu and Kumar provided an analytic solution for the Bloch equation in response to the application of a constant radio frequency field [3]. In their approach, the Bloch equation corresponding to each magnetisation component is written as a third order differential equation, with the solution following from these equations.

Solving the Bloch equation during the period in which a time-varying external magnetic field is applied, termed the excitation period, is of crucial importance for slice selection in MRI. The bilinear form of the Bloch equation makes it very hard to find a closed form solution for an arbitrary excitation pattern. Several approximate solutions to the Bloch equation under restrictive limitations have been proposed [4], [5]. These approaches are limited by the excitation pattern they consider. In [6] an approximate solution is proposed for a rectangular pulse excitation when the relaxation terms are ignored. It has been generally accepted that an analytic solution does not exist for an arbitrary pulse excitation [6].

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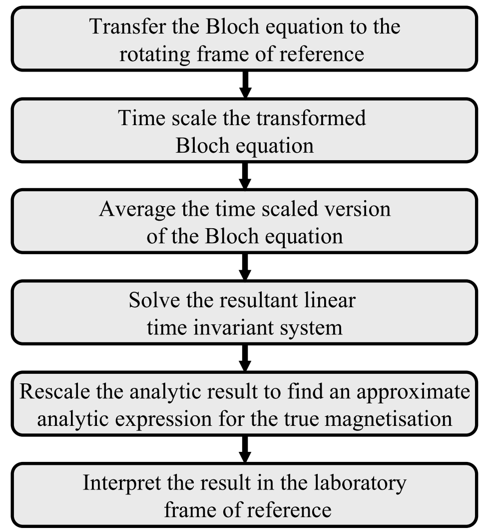

Fig. 1. The main steps of the approach presented in this paper to find an approximate analytic solution to the Bloch equation.

Several different numerical techniques have been used to solve the Bloch equation [7], [8]. Most MRI simulators implement approximate numerical solutions to the Bloch equation based on rotation matrices [9], [10]. The Shinnar-Le Roux method, used universally in MRI machines to selectively excite a slice, is based on a discrete approximation to the Bloch equation which simplifies the solution of the optimal slice selective pulse to the design of two polynomials [11].

We present a novel technique for finding an approximate analytic solution to the Bloch equation that retains important features of the Bloch equation, and can therefore be applied to the design of improved MRI pulse sequences. Our approach is based on a combination of time scaling and averaging methods from dynamical systems theory [12], [13]. We verify the success of the averaging method in simulations without formally establishing its validity. The steps of the method to find an approximate solution in the laboratory frame of reference are shown in Fig. 1. Since the magnetic resonance signal is demodulated after being received, it is sufficient to determine the solution in the rotating frame of reference. In this paper, all analytic solutions and simulation results represent the spin system response as observed from a frame of reference rotating at the Larmor frequency of the static magnetic field.

In Section II we present an overview of the proposed approach including the transferral of the Bloch equation to the rotating frame of reference, and the novel application of time scaling and averaging to the resultant system. Section

III contains numerical validation of the proposed method's ability to provide an approximate, yet accurate, solution to the Bloch equation.

II. Theory

The general form of the Bloch equation, without considering spin diffusion, may be written as

$$\begin{bmatrix}\dot{M}_x\\ \dot{M}_y\\ \dot{M}_z\end{bmatrix}=\begin{bmatrix}-\frac{1}{T_2} & \gamma B_z & \gamma B_y(t)\\ -\gamma B_z & -\frac{1}{T_2} & \gamma B_x(t)\\ -\gamma B_y(t) & -\gamma B_x(t) & -\frac{1}{T_1}\end{bmatrix}\begin{bmatrix}M_x\\ M_y\\ M_z\end{bmatrix}+\frac{1}{T_1}\begin{bmatrix}0\\ 0\\ M_0\end{bmatrix}. \quad (1)$$

Here, γ is the gyromagnetic ratio. T_1 and T_2 represent longitudinal and transverse relaxation time constants, respectively. M_x, M_y, and M_z are components of the magnetisation vector, dependent on both position and time. B_x, B_y, and B_z represent the external applied magnetic field components. M_0 is the thermal equilibrium magnetisation created by an ideally-uniform static field oriented in the z-direction which is aligned with the static external field.

A. Transformation to the Rotating Frame of Reference

Consider an external electromagnetic field with the following general form,

$$\begin{aligned}\mathbf{B}_{ext} &= B_x(t)\ \mathbf{e}_x - B_y(t)\ \mathbf{e}_y + B_z\mathbf{e}_z\\ &= B_1^e(t)\cos(\omega_{rf}t+\phi)\ \mathbf{e}_x - B_1^e(t)\sin(\omega_{rf}t+\phi)\ \mathbf{e}_y\\ &\quad + B_0\ \mathbf{e}_z,\end{aligned} \quad (2)$$

representing the rotation of the transverse component of the electromagnetic field in the clock-wise direction with rotational frequency ω_{rf}. If we assume both that the RF field is rotating at the Larmor frequency and that $\phi = 0$, the Bloch equation in a frame of reference rotating at the Larmor frequency may be written as

$$\begin{bmatrix}\dot{M}_{x'}\\ \dot{M}_{y'}\\ \dot{M}_{z'}\end{bmatrix}=\begin{bmatrix}-\frac{1}{T_2} & \Delta\omega & 0\\ -\Delta\omega & -\frac{1}{T_2} & u(t)\\ 0 & -u(t) & -\frac{1}{T_1}\end{bmatrix}\begin{bmatrix}M_{x'}\\ M_{y'}\\ M_{z'}\end{bmatrix}+\frac{1}{T_1}\begin{bmatrix}0\\ 0\\ M_0\end{bmatrix}, \quad (3)$$

where

$$u(t) \equiv \omega_1(t) = \gamma B_1^e(t), \quad (4)$$

is the Rabi frequency and $\Delta\omega$ represents any deviation from the main magnet Larmor frequency including field inhomogeneities, gradient fields, and off-resonance excitation [4].

B. Time Scaling the Bloch Equation

In order to scale the Bloch equation[1] we define $s(t)$ and $\boldsymbol{\sigma}(t)$ to be

$$s(t) \triangleq \int_0^t u(\tau)d\tau, \quad (5)$$

and

$$\boldsymbol{\sigma}(t) \triangleq \begin{bmatrix}1 & 0 & 0\\ 0 & \cos s(t) & -\sin s(t)\\ 0 & \sin s(t) & \cos s(t)\end{bmatrix}. \quad (6)$$

[1]The time scaling we introduce may be interpreted as transferring the Bloch equation to the excitation dependent rotating frame of reference [14].

As a result we may write

$$\dot{\boldsymbol{\sigma}}(t) = \mathbf{u}(t)\ \boldsymbol{\sigma}(t), \quad (7)$$

in which

$$\mathbf{u}(t) = u(t)\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{bmatrix}. \quad (8)$$

We define $\mathbf{N}'$ to be

$$\mathbf{N}'(t) \equiv \begin{bmatrix}N_{x'}(t)\\ N_{y'}(t)\\ N_{z'}(t)\end{bmatrix} \triangleq \boldsymbol{\sigma}(t)\mathbf{M}'(t), \quad (9)$$

which we refer to as the *time scaled magnetisation vector* throughout. This results in

$$\begin{aligned}\dot{\mathbf{N}}' &= \dot{\boldsymbol{\sigma}}(t)\mathbf{M}' + \boldsymbol{\sigma}(t)\dot{\mathbf{M}}'\\ &= \mathbf{u}(t)\boldsymbol{\sigma}(t)\mathbf{M}' + \boldsymbol{\sigma}(t)\boldsymbol{\Omega}'\mathbf{M}' + \frac{1}{T_1}\boldsymbol{\sigma}(t)\mathbf{M}_0,\end{aligned} \quad (10)$$

or equivalently

$$\begin{aligned}\dot{\mathbf{N}}' &= \mathbf{u}(t)\boldsymbol{\sigma}(t)\mathbf{M}' + \boldsymbol{\sigma}(t)\boldsymbol{\Omega}'\boldsymbol{\sigma}^{-1}(t)\underbrace{\boldsymbol{\sigma}(t)\mathbf{M}'}_{\mathbf{N}'(t)} + \frac{1}{T_1}\underbrace{\boldsymbol{\sigma}(t)\mathbf{M}_0}_{\mathbf{N}_0(t)}\\ &= \boldsymbol{\sigma}(t)\begin{bmatrix}-1/T_2 & \Delta\omega & 0\\ -\Delta\omega & -1/T_2 & 0\\ 0 & 0 & -1/T_1\end{bmatrix}\boldsymbol{\sigma}^{-1}(t)\mathbf{N}' + \frac{\mathbf{N}_0}{T_1}.\end{aligned} \quad (11)$$

Finally the Bloch equation in terms of the time scaled magnetisation vector may be written as

$$\dot{\mathbf{N}}'(t) = \boldsymbol{\Omega}'_N(t)\mathbf{N}'(t) + \frac{1}{T_1}\mathbf{N}_0(t), \quad (12)$$

where

$$\boldsymbol{\Omega}'_N(t) = \begin{bmatrix}-\frac{1}{T_2} & \Delta\omega\cos s(t) & \Delta\omega\sin s(t)\\ -\Delta\omega\cos s(t) & -\frac{\cos^2 s(t)}{T_2}-\frac{\sin^2 s(t)}{T_1} & (\frac{1}{2T_1}-\frac{1}{2T_2})\sin 2s(t)\\ -\Delta\omega\sin s(t) & (\frac{1}{2T_1}-\frac{1}{2T_2})\sin 2s(t) & -\frac{\cos^2 s(t)}{T_1}-\frac{\sin^2 s(t)}{T_2}\end{bmatrix} \quad (13)$$

and

$$\mathbf{N}_0(t) = M_0\begin{bmatrix}0\\ -\sin s(t)\\ \cos s(t)\end{bmatrix}. \quad (14)$$

The initial condition of (12) is

$$\mathbf{N}'(0) \equiv \begin{bmatrix}N_{x'}(0)\\ N_{y'}(0)\\ N_{z'}(0)\end{bmatrix} = \boldsymbol{\sigma}(0)\mathbf{M}'(0) = \begin{bmatrix}0\\ 0\\ M_0\end{bmatrix}. \quad (15)$$

After solving (12) with the initial condition (15), the true magnetisation, $\mathbf{M}'(t)$, may be found from

$$\mathbf{M}'(t) = \begin{bmatrix} M_{x'}(t) \\ M_{y'}(t) \\ M_{z'}(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos s(t) & \sin s(t) \\ 0 & -\sin s(t) & \cos s(t) \end{bmatrix} \begin{bmatrix} N_{x'}(t) \\ N_{y'}(t) \\ N_{z'}(t) \end{bmatrix}$$

$$= \begin{bmatrix} N_{x'}(t) \\ N_{y'}(t)\cos s(t) + N_{z'}(t)\sin s(t) \\ -N_{y'}(t)\sin s(t) + N_{z'}(t)\cos s(t) \end{bmatrix}. \qquad (16)$$

The advantage of (12) over (3) despite the perhaps more cumbersome expressions in (13) is that (12) is much better behaved from a numerical point of view, leading to significant computational time savings in simulation studies [14]. When a continuous-wave excitation is applied, this time efficiency becomes considerable [14].

C. Averaging the Time Scaled Bloch Equation

We use first order averaging [12], [13] to approximate (12), resulting in the following linear time invariant system

$$\dot{\mathbf{N}}'_{avg}(t) = \mathbf{\Omega}'_{N_{avg}}\mathbf{N}'_{avg}(t) + \frac{1}{T_1}\mathbf{N}_{0_{avg}}. \qquad (17)$$

Here[2]

$$\mathbf{\Omega}'_{N_{avg}} = \begin{bmatrix} -\frac{1}{T_2} & \Delta\omega\overline{\cos s(t)} & \Delta\omega\overline{\sin s(t)} \\ -\Delta\omega\overline{\cos s(t)} & -\frac{\overline{\cos^2 s(t)}}{T_2} - \frac{\overline{\sin^2 s(t)}}{T_1} & (\frac{1}{2T_1} - \frac{1}{2T_2})\overline{\sin 2s(t)} \\ -\Delta\omega\overline{\sin s(t)} & (\frac{1}{2T_1} - \frac{1}{2T_2})\overline{\sin 2s(t)} & -\frac{\overline{\cos^2 s(t)}}{T_1} - \frac{\overline{\sin^2 s(t)}}{T_2} \end{bmatrix} \qquad (18)$$

and

$$\mathbf{N}_{0_{avg}} = M_0 \begin{bmatrix} 0 \\ -\overline{\sin s(t)} \\ \overline{\cos s(t)} \end{bmatrix}, \qquad (19)$$

with the initial condition (15). In these expressions, the bar notation deontes the average of a function. The solution to (17) is found from linear systems theory to be

$$\mathbf{N}'_{avg}(t) = \exp(\mathbf{\Omega}'_{N_{avg}}t)\mathbf{N}'(0) + \frac{\mathbf{N}_{0_{avg}}}{T_1}\int_0^t \exp\left(\mathbf{\Omega}'_{N_{avg}}(t-\tau)\right)d\tau \qquad (20)$$

which can subsequently be used to find the magnetisation vector from (16). The resultant magnetisation vector is termed the *averaged magnetisation* in this paper.

III. Examples and Simulation Results

To investigate the validity of the averaging method presented in Section II, we solve the Bloch equation for different types of excitation, including pulse and continuous wave excitations.

[2]We assume that the field is spatially inhomogeneous. If the field is also inhomogeneous over time, this dependency must be taken into account in the averaging procedure.

A. $\pi/2$ Pulse Excitation

For a $\pi/2$ pulse excitation with an arbitrary envelope, the averaged equation may be written as

$$\begin{bmatrix} \dot{N}_{x'_{avg}} \\ \dot{N}_{y'_{avg}} \\ \dot{N}_{z'_{avg}} \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{1}{T_2} & 0 & \Delta\omega \\ 0 & -\frac{1}{T_1} & 0 \\ -\Delta\omega & 0 & -\frac{1}{T_2} \end{bmatrix}}_{\mathbf{\Omega}'_{N_{avg}}} \begin{bmatrix} N_{x'_{avg}} \\ N_{y'_{avg}} \\ N_{z'_{avg}} \end{bmatrix} + \frac{1}{T_1} \underbrace{\begin{bmatrix} 0 \\ -M_0 \\ 0 \end{bmatrix}}_{\mathbf{N}_{0_{avg}}} \qquad (21)$$

with initial condition (15). By substituting $\mathbf{\Omega}'_{N_{avg}}$ and $\mathbf{N}_{0_{avg}}$ in (20),

$$\mathbf{N}'_{avg} = \begin{bmatrix} M_0 e^{-\frac{t}{T_2}}\sin(\Delta\omega t) \\ -M_0(1-e^{-\frac{t}{T_1}}) \\ M_0 e^{-\frac{t}{T_2}}\cos(\Delta\omega t) \end{bmatrix}. \qquad (22)$$

After applying transformation (16), the magnetisation vector is

$$\frac{\mathbf{M}'_{avg}}{M_0} = \begin{bmatrix} e^{-\frac{t}{T2}}\sin(\Delta\omega\, t) \\ -\cos s(t) + \cos s(t)\, e^{-\frac{t}{T1}} + \sin s(t)\, e^{-\frac{t}{T2}}\cos(\Delta\omega\, t) \\ \sin s(t) - \sin s(t)\, e^{-\frac{t}{T1}} + \cos s(t)\, e^{-\frac{t}{T2}}\cos(\Delta\omega\, t) \end{bmatrix}. \qquad (23)$$

Clearly $\mathbf{N}'_{avg}$ is independent of the envelope of the pulse excitation while $\mathbf{M}_{avg}$ is not ($s(t)$ in (23)), demonstrating that the approximate analytic solution is dependent on the envelope shape of the excitation. For a $\pi/2$ rectangular pulse excitation with duration T_p,

$$s(t) = \omega_1 t\, \Pi\left(\frac{t - T_p/2}{T_p}\right) + \frac{\pi}{2} H(t - T_p), \qquad (24)$$

where $\Pi(t)$ represents the unit width rectangular window function, and $H(t)$ is the heaviside step function. Through substitution of $s(t)$ in (23), it is possible to find an analytical solution for this type of excitation which includes the magnetisation behaviour both during the excitation and relaxation periods. This is an advance on previous solutions, eg. [6]. During the excitation period,

$$\begin{bmatrix} M_{x'_{avg}}(t) \\ M_{y'_{avg}}(t) \\ M_{z'_{avg}}(t) \end{bmatrix} = M_0 \begin{bmatrix} e^{-\frac{t}{T2}}\sin(\Delta\omega\, t) \\ -\cos\omega_1 t + \cos\omega_1 t\, e^{-\frac{t}{T1}} + \sin\omega_1 t\, e^{-\frac{t}{T2}}\cos(\Delta\omega\, t) \\ \sin\omega_1 t - \sin\omega_1 t\, e^{-\frac{t}{T1}} + \cos\omega_1 t\, e^{-\frac{t}{T2}}\cos(\Delta\omega\, t) \end{bmatrix}, \quad 0 \le t < T_p. \qquad (25)$$

During relaxation, (23) reduces to

$$\begin{bmatrix} M_{x'_{avg}}(t) \\ M_{y'_{avg}}(t) \\ M_{z'_{avg}}(t) \end{bmatrix} = M_0 \begin{bmatrix} \sin(\Delta\omega\, t)\, e^{-\frac{t}{T2}} \\ \cos(\Delta\omega t) e^{-\frac{t}{T2}} \\ 1 - e^{-\frac{t}{T1}} \end{bmatrix}, \quad t > T_p. \qquad (26)$$

Consider a $\pi/2$ rectangular pulse excitation with duration 100μs, for which $\omega_1 = \gamma B_1 = 15712$rad/s, and one isochromat with $\Delta\omega = 10$rad/s. The results during the

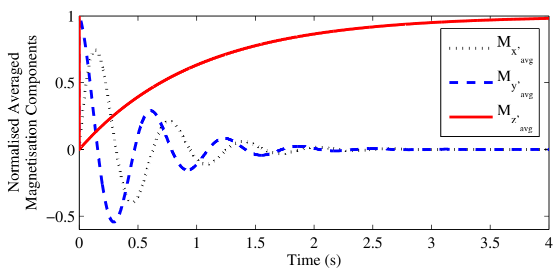

Fig. 2. Approximation for a rectangular pulse excitation through averaging, when $\Delta\omega = 10$rad/s, for a spin system with $T_1 = 1$s, and $T_2 = 0.5$s.

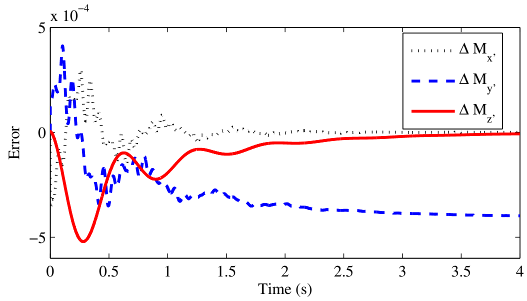

Fig. 3. The error between the exact solution and the averaged solution for a rectangular pulse excitation, for the spin system in Fig. 2, when $\Delta\omega = 10$rad/s.

excitation and relaxation periods for the averaged equation and the error between the exact solution based on simulating the Bloch equation and the approximate analytic solution are shown in Figs. 2, and 3. For field inhomogeneities sampled from a Lorentzian or a Gaussian distribution, (26) must be integrated over all isochromats. A T_2^* decay rate much faster than the T_2 process itself will result, as is to be expected.

Since the magnetisation behaviour during the excitation period is of fundamental importance to the problem of slice selection in MRI, consider an increase in $\Delta\omega$ to 100rad/s, for the same spin system as Fig. 2 with excitation period 100μs. The approximate analytic solution and the error for this system are depicted in Figs. 4, and 5.

For a $\pi/2$ pulse having a sinc envelop with duration T_p,

$$s(t) = \omega_1 \int_0^t \mathrm{sinc}\left(\frac{\tau - T_p/2}{T_p}\right) d\tau \; \Pi\left(\frac{t - T_p/2}{T_p}\right) + \frac{\pi}{2} H(t - T_p). \tag{27}$$

In this case, during the excitation period,

$$s(t) = \omega_1 \int_0^t \mathrm{sinc}\left(\frac{\tau - T_p/2}{T_p}\right) d\tau, \quad 0 \le t < T_p, \tag{28}$$

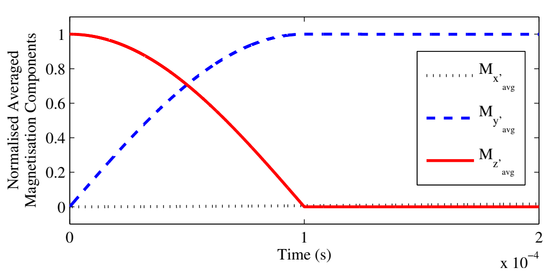

Fig. 4. Averaged result for a rectangular pulse during the excitation period, when $\Delta\omega = 100$rad/s, for the spin system in Fig. 2. The duration of the pulse is 100μs.

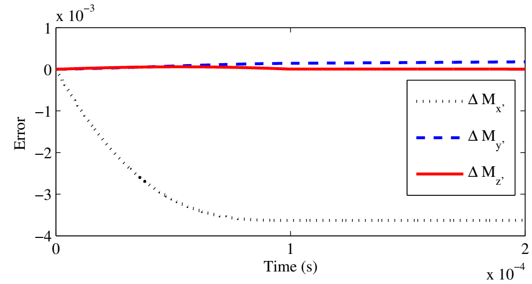

Fig. 5. The error between the exact solution and the averaged solution for a rectangular pulse during the excitation period, when $\Delta\omega = 100$rad/s, for the spin system in Fig. 2. The duration of the pulse is 100μs.

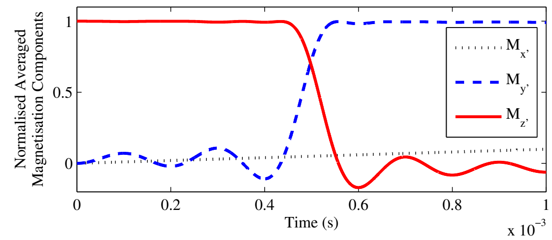

Fig. 6. Averaged result for a sinc pulse during the excitation period, when $\Delta\omega = 100$rad/s, for the spin system in Fig. 2. The pulse duration is 1ms.

which after substitution in (23) results in the true magnetisation. It is important to mention that even if the relaxation effects are ignored during the excitation period, it is not possible to find an analytic solution for the pulse excitation with a sinc envelope. But here we have found an analytic answer based on time scaling and averaging of the Bloch equation. Figs. 6, and 7 show the results of a sinc pulse during the excitation period for a single isochromat with $\Delta\omega = 100$rad/s. The duration of the pulse is 1ms. The error between simulation of the Bloch equation and the approximate analytic solution is demonstrated to be extremely small, indicating the predictive power of the proposed averaging technique. The result is beyond what classical averaging theory predicts, and this warrants further investigation.

B. Continuous Wave Excitation

Based on the laser candle idea in photonics [15], [16], we consider application of an on-resonance continuous wave excitation, modulated at the Rabi frequency, to the Bloch equation. For an initial phase angle of zero, the excitation in the laboratory frame of reference is

$$\mathbf{B}_{xy}(t) \equiv \mathbf{B}_1(t) = B_1^e(t)\cos(\omega_0 t)\mathbf{e}_x - B_1^e(t)\sin(\omega_0 t)\mathbf{e}_y, \tag{29}$$

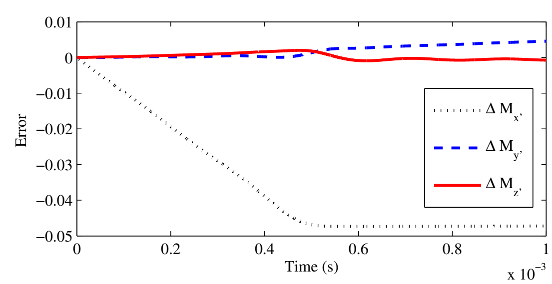

Fig. 7. The error between the exact solution and the averaged solution for a sinc pulse during the excitation period, when $\Delta\omega = 100$rad/s, for the spin system in Fig. 2. The pulse duration is 1ms.

where $\omega_0 = \gamma B_0$ is the Larmor frequency of the main static field (for simplicity we have ignored the gradient fields), and

$$B_1^e(t) = B_1\big(1+\alpha\cos(\gamma B_1 t)\big), \tag{30}$$

in which α is a constant modulation factor, γ is the gyromagnetic ratio, and B_1 is the constant amplitude of the rotating field with zero modulation factor. After transforming the Bloch equation to the rotating frame of reference, the Rabi frequency of precession will be about the x'-direction and is given by

$$\begin{aligned} u(t) \equiv \omega_{x'}(t) \equiv \omega_1(t) &= \gamma B_1\Big(1+\alpha\cos(\gamma B_1 t)\Big) \\ &= \omega_1\Big(1+\alpha\cos(\omega_1 t)\Big), \end{aligned} \tag{31}$$

which clearly indicates that the Rabi frequency is time dependent. In [17], [18] we presented a periodic solution for the steady state response, for this type of excitation based on Galerkin's procedure for nonlinear periodic systems. In the current paper we present an approximate analytic solution based on the method described in Section II, which contains both the transient and steady state response of the spin system.

For the continuous wave excitation,

$$s(t) = \int_0^t u(\tau)\,d\tau = \omega_1 t + \alpha\sin(\omega_1 t). \tag{32}$$

To perform periodic averaging, it is necessary to determine the averages of $\sin s(t)$, $\sin 2s(t)$, $\cos s(t)$, $\sin^2 s(t)$, and $\cos^2 s(t)$. The average of the sine function may be written as

$$\overline{\sin s(t)} = \frac{1}{T_p}\int_{T_p} \sin\Big(\omega_1\tau + \alpha\sin(\omega_1\tau)\Big)d\tau, \tag{33}$$

where $T_p = \frac{2\pi}{\omega_1}$. Since the function inside the integral is an odd function, it is concluded that the integral is zero. Similarly, $\overline{\sin 2s(t)}$ is also zero. The average of $\cos s(t)$, given by

$$\overline{\cos s(t)} = \frac{1}{T_p}\int_{T_p} \cos\big(\omega_1\tau + \alpha\sin(\omega_1\tau)\big)d\tau, \tag{34}$$

does not have a closed form solution, and similarly for the averages of $\cos^2 s(t)$ and $\sin^2 s(t)$. However, if α is known, it is always possible to find the average values of $\cos s(t)$, $\sin^2 s(t)$, and $\cos^2 s(t)$ numerically, independent of the Rabi frequency, ω_1.

When there are no field inhomogeneities, or alternatively when the excitation field is strong enough, based on the above discussion $\mathbf{\Omega}'_{N_{avg}}$ in (17) becomes

$$\mathbf{\Omega}'_{N_{avg}} = \begin{bmatrix} -\frac{1}{T_2} & 0 & 0 \\ 0 & -\frac{\overline{\cos^2 s(t)}}{T_2} - \frac{\overline{\sin^2 s(t)}}{T_1} & 0 \\ 0 & 0 & -\frac{\overline{\cos^2 s(t)}}{T_1} - \frac{\overline{\sin^2 s(t)}}{T_2} \end{bmatrix}, \tag{35}$$

and

$$\mathbf{N}_{0_{avg}} = M_0 \begin{bmatrix} 0 \\ 0 \\ \overline{\cos s(t)} \end{bmatrix}. \tag{36}$$

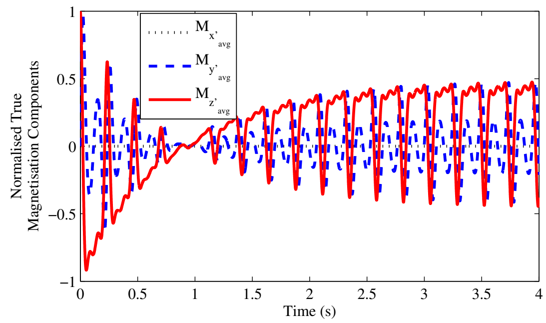

Fig. 8. Exact solution based on simulating the Bloch equation for an on-resonance continuous-wave excitation (31), for the spin system in Fig. 2, when $\omega_1 = 27$rad/s.

From (20) it is found that

$$\begin{bmatrix} N_{x'_{avg}} \\ N_{y'_{avg}} \\ N_{z'_{avg}} \end{bmatrix} = M_0 \begin{bmatrix} 0 \\ 0 \\ \Xi(T_1,T_2,\alpha) \end{bmatrix}, \tag{37}$$

where

$$\begin{aligned} &\Xi(T_1,T_2,\alpha) = \\ &\Big(1-\Phi(T_1,T_2,\alpha)\Big)e^{-\left(\frac{\overline{\cos^2 s(t)}}{T_1}+\frac{\overline{\sin^2 s(t)}}{T_2}\right)t} + \Phi(T_1,T_2,\alpha), \end{aligned} \tag{38}$$

in which

$$\Phi(T_1,T_2,\alpha) = \frac{\overline{\cos s(t)}}{\overline{\cos^2 s(t)} + \frac{T_1}{T_2}\overline{\sin^2 s(t)}}. \tag{39}$$

As a result the true magnetisation in the rotating frame of reference is well approximated by

$$\begin{bmatrix} M_{x'_{avg}} \\ M_{y'_{avg}} \\ M_{z'_{avg}} \end{bmatrix} = M_0 \begin{bmatrix} 0 \\ \Xi(T_1,T_2,\alpha)\sin(\omega_1 t + \alpha\sin\omega_1 t) \\ \Xi(T_1,T_2,\alpha)\cos(\omega_1 t + \alpha\sin\omega_1 t) \end{bmatrix}. \tag{40}$$

The peak of the steady state response of the spin system depends only on the maximum value of $\Phi(T_1,T_2,\alpha)$. For different values of T_1/T_2 it is possible to find the maximising α by applying the trapezoid or Simpson methods to calculate the integrals numerically. Since in practice the precise values of T_1, and T_2 are unknown, it appers feasible to apply global extremum seeking techniques [19] to adjust the value of the optimal α. We will pursue this in future work.

For $\alpha = \sqrt{3}$, which maximises the peak value of the steady state response when $T_1 = 1$s, and $T_2 = 0.5$s,

$$\begin{aligned} &\overline{\cos s(t)} = -0.5793, \qquad \overline{\cos^2 s(t)} = 0.7315, \\ &\overline{\sin^2 s(t)} = 0.2685. \end{aligned} \tag{41}$$

The numerically evaluated Bloch equation, approximate analytic solution, and error are depicted in Figs. 8, 9, and 10, when $\omega_1 = 27$rad/s. It is interesting to note that for this continuous wave type of excitation, the evolution of the time scaled magnetisation vector is, unlike the pulse case, different from the true magnetisation vector. Fig. 11 represents the time scaled magnetisation vector for the Rabi-modulated excitation waveform.

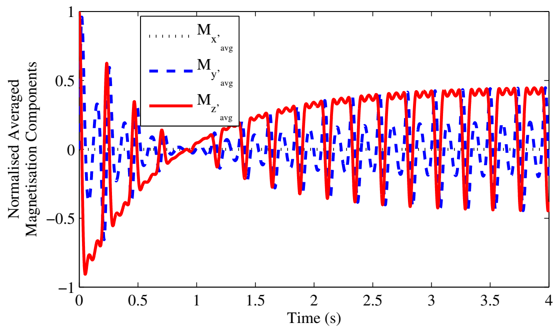

Fig. 9. Averaged result for an on-resonance continuous-wave excitation (31), for the spin system in Fig. 2, when $\omega_1 = 27$rad/s.

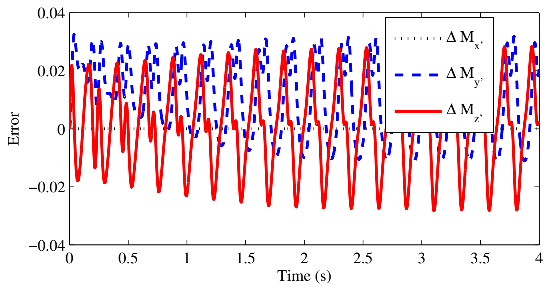

Fig. 10. The error between the exact solution and the averaged solution for an on-resonance continuous-wave excitation (31), for the spin system in Fig. 2, when $\omega_1 = 27$rad/s.

IV. Conclusion

A novel procedure for deriving approximate analytic solutions to the Bloch equation was presented. The approximation method's accuracy requires further investigation as it appears to be better than would be expected from classical theory. Its advantages are, besides providing much faster simulation times, excellent analytic insight into the form of the Bloch equation solution enabling optimisation methods, and even suggesting on-line adaptive optimisation ideas to be applied. Given that the time scaled version of the Bloch equation developed here-in is computationally more efficient than simulating directly from the Bloch equation, it has the potential to be used in magnetic resonance simulators that desire a level of accuracy and generalisability not achievable using rotation matrices.

We have shown that the combination of time scaling and averaging results in pleasing approximate solutions to the Bloch equation. For off-resonance scenarios involving large deviations from the Larmor frequency, it appears that the accuracy of the approximation solution decreases, nevertheless it remains possible to design improved excitation patterns. To further understand these issues higher order averaging may be explored. It is important to note that averaging cannot be applied to the Bloch equation in its original form, revealing that time scaling plays a key role in bringing the equation in a form where averaging ideas can be applied.

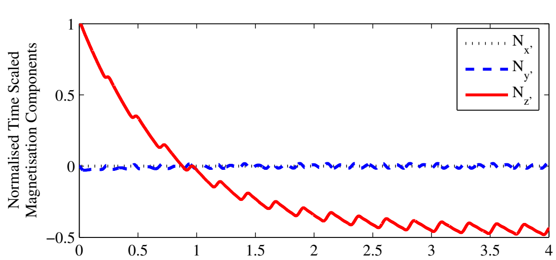

Fig. 11. Evolution of the time scaled magnetisation vector for an on-resonance continuous-wave excitation (31), for the spin system in Fig. 2, when $\omega_1 = 27$rad/s.

We expect that our ideas will allow researchers to revisit the pulse design question, for example to achieve better slice selectivity at a theoretical level or to control inhomogeneous ensemble of spins [20]. This has the potential to result in valuable practical performance measure. In future work, we will apply the results derived here to the magnetic resonance signal optimisation problem proposed in [17], [18].

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