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Passive path following control for port-Hamiltonian systems

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Abstract— **This paper is devoted to path following control for port-Hamiltonian systems. The control law presented here is extension of an existing passive velocity field controller for fully actuated mechanical systems. The proposed method employs vector fields on the phase (co-tangent) spaces instead of those on the velocity (tangent) spaces. Since port-Hamiltonian systems can describe a wider class of systems than conventional mechanical ones, the proposed method is applicable to various systems. Furthermore, by making use of the port-Hamiltonian structure of the closed loop system, we can obtain a novel controller to assign the desired total energy. Moreover, a numerical simulation of a simple nonholonomic system exhibits the effectiveness of the proposed method.**

Index Terms— **Hamiltonian systems, path following control, virtual potential field control, passivity**

I. INTRODUCTION

Trajectory tracking control is an important task for control of mechanical systems. There are several methods proposed by many authors for this problem. See e.g. [1], [2], [3], [4] and the references therein. Typical trajectory tracking control problem is to let the state of the plant to track the desired time-varying signal. However, for some applications such as control of welding robots, this problem setting is not appropriate. Instead of using a time-varying desired trajectory, the plant should track a desired path which does not depend on time.

To take care of this problem, various approaches are proposed. Salisbury [5] and Hogan [6] proposed a method employing virtual potential functions which take their minima on the desired paths. Li et al. [7], [8] proposed a method called passive velocity field control (PVFC) to design vector fields to track a desired path directly. This method employs a virtual potential energy like function but it does not have intuitive meaning to the control system. Inspired by the idea of PVFC, Duindam et al. [9], [10] proposed a method to design vector fields directly with a natural potential function which takes its minimum on the desired path. This method employs the structure of fully actuated mechanical systems without friction described by a simple equation with a covariant derivative. Although this approach gives a smart answer to the above mentioned path following problem, there are some defects. Since we cannot control the (virtual) total energy, the velocity of the plant to proceed the desired path is not assignable. Also, the plant systems in the real world always have unknown dissipative elements which can cause a serious problem in this approach.

This paper proposes a method which generalizes the result [9], [10]. First of all, we adopt port-Hamiltonian models for controller design which can describe mechanical systems with a class of nonholonomic constraints as well as the conventional ones. We re-formulate the existing results for this class of Hamiltonian systems and derive a path following method applicable to a wider class of systems. Furthermore, by utilizing the structure of port-Hamiltonian form of the closed loop system, a novel controller to assign the total energy is proposed. Consequently, we can easily assign the desired velocity and derive a control system robustly stabilized against parameter variation such as modelling error for friction. Moreover, a numerical example of a simple nonholonomic system demonstrates the effectiveness of the proposed method.

II. PROBLEM SETTING

Let us consider the following port-Hamiltonian system. [3], [11]

$$\begin{pmatrix}\dot q\\ \dot p\end{pmatrix} = \begin{pmatrix}0 & J_{12}(q)\\ -J_{12}(q)^{\mathrm T} & J_{22}(q,p)\end{pmatrix}\begin{pmatrix}\frac{\partial H}{\partial q}^{\mathrm T}\\ \frac{\partial H}{\partial p}^{\mathrm T}\end{pmatrix} + \begin{pmatrix}0\\ G(q)\end{pmatrix}u$$
$$H(q,p) = \frac{1}{2}p^{\mathrm T}M(q)^{-1}p \tag{1}$$

Here $x = (q,p) \in \mathbb{R}^l \times \mathbb{R}^m (l \geq m)$. The Hamiltonian function $H(q,p) \in \mathbb{R}$ describes the kinetic energy of the system. It is supposed that the matrix $G(q) \in \mathbb{R}^{m\times m}$ is nonsingular, that the matrix $J_{12}(q) \in \mathbb{R}^{l\times m}$ is column full rank and that the matrix $J_{22}(q,p) \in \mathbb{R}^{m\times m}$ is skew-symmetric. This dynamics is a generalized version of conventional mechanical systems. It can describe a mechanical system with a class of nonholonomic constraints with respect to the velocity $\dot q$. In such a case, the matrices $J_{12}(q)$ and $J_{22}(q,p)$ are determined by the constraints. The system reduces to a conventional simple mechanical system when the parameters are selected as $l = m$, $J_{12} = I$, $J_{22} = 0$ and $G = I$.

In this paper, the inner product on the phase space is defined by

$$\langle p_u, p_v\rangle := p_u^{\mathrm T}M(q)^{-1}p_v \tag{2}$$

for $p_u, p_v \in \mathbb{R}^m$. Accordingly, the norm is defined by

$$\|p_u\| := \sqrt{\langle p_u, p_u\rangle} \geq 0. \tag{3}$$

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A scalar valued function $\mathrm{sgn}(x) \in \mathbb{R}$ returns the sign of the argument $x \in \mathbb{R}$ as

$$\mathrm{sgn}(x) := \begin{cases} 1 & (x \geq 0) \\ -1 & (x < 0) \end{cases}. \tag{4}$$

III. Main results

This section gives the main results of the present paper path following control of port-Hamiltonian system (1) by designing the vector fields directly as a generalization of the results by Duindam et al. [9], [10] for a simple mechanical system.

As discussed in the results on trajectory tracking control of port-Hamiltonian systems [4], [12], the desired trajectory of the whole state x can be characterized either by the configuration state q or by the phase state p. Here we design the desired path on the configuration space q. Let us consider a potential function $U(q)$ which takes its minimum values on the desired path. More precisely, the potential function $U(q)$ is chosen in such a way that the following assumptions hold.

Assumption 1: The scalar function $U : \mathbb{R}^l \to \mathbb{R}$ satisfies the following conditions.

- $U(q) \geq 0$.
- $U(q)$ takes its minimum value 0 if and only if q is on the desired path.

Furthermore, let us define the desired vector $p_w(q)$ on the phase space.

Assumption 2: The vector valued function $p_w : \mathbb{R}^l \to \mathbb{R}^m$ satisfies the following conditions.

- $p_w(q) \neq 0$ if q is on the desired path.
- $\langle J_{12}(q)^{\mathrm{T}}(\partial U/\partial q)^{\mathrm{T}}, p_w(q)\rangle = 0$.
- The measure of the set $\{q | p_w(q) = 0\}$ is 0.
- If there exists a scalar $k(t) \neq 0$ satisfying $p(t) \equiv k(t)p_w(q(t))$ and $J_{12}(q(t))^{\mathrm{T}}(\partial U/\partial q)^{\mathrm{T}}(q(t)) \equiv 0$, then q stays on the desired path.

In what follows, we consider a region close to the desired path on which $p_w(q) \neq 0$ holds and the semi-global stability of the desired path within this region is discussed. Let $p_{we}(q)$ denote the normalized version of $p_w(q)$, that is,

$$p_{we}(q) := \frac{p_w(q)}{\|p_w(q)\|}.$$

Due to Assumptions 1 and 2, if the state is on the desired path at $t = 0$, and if p is parallel to $p_{we}(q)$, i.e., there exists a scalar $k(t)$ satisfying $p(t) = k(t)p_{we}(q(t))$, then the state will stay on the desired path for all $t \geq 0$.

Next let us decompose p into two elements: one is linearly dependent on $p_{we}(q)$ and the other is orthogonal to $p_{we}(q)$ which is denoted by $p_{\tilde{w}}(q,p)$. That is, p is decomposed as

$$p = \alpha(q,p)\, p_{we}(q) + p_{\tilde{w}}(q,p), \quad \langle p_{we}(q), p_{\tilde{w}}(q,p)\rangle = 0 \tag{5}$$

where

$$\begin{aligned} \alpha(q,p) &:= \langle p, p_{we}(q)\rangle \\ p_{\tilde{w}}(q,p) &:= p - \alpha(q,p)\, p_{we}(q). \end{aligned}$$

According to the decomposition (5), we can decompose the Hamiltonian function as

$$\begin{aligned} H(q,p) &= H_{k,w}(q,p) + H_{k,\tilde{w}}(q,p) \\ H_{k,w}(q,p) &= \frac{1}{2}\alpha(q,p)^2 \\ H_{k,\tilde{w}}(q,p) &= \frac{1}{2}\langle p_{\tilde{w}}, p_{\tilde{w}}\rangle. \end{aligned}$$

Here $H_{k,w}(q,p)$ and $H_{k,\tilde{w}}(q,p)$ denote the kinetic energy with respect to the desired direction $p_{we}(q)$ and that with respect to the undesired one $p_{\tilde{w}}(q,p)$.

As stated above, Assumptions 1 and 2 suggest that path following control is achieved if the potential energy $U(q)$ takes its minimum value and if the phase state p is parallel to $p_{we}(q)$. This means that we need to reduce the undesired kinetic energy $H_{k,\tilde{w}}(q,p)$ with respect to $p_{\tilde{w}}(q,p)$ direction.

In the sequel, a path following controller is designed by four steps. It is noted that the first three steps are generalized versions of those proposed in [9], [10] but the last one is a novel procedure proposed in the present paper.

In the first step, a controller called *nominal controller* is applied to make the desired and undesired kinetic energies $H_{k,w}(q,p)$ and $H_{k,\tilde{w}}(q,p)$ independently controllable by decoupling $H_{k,w}(q,p)$ and $H_{k,\tilde{w}}(q,p)$.

In the second step, another controller called *asymptotic controller* is added to reduce the undesired kinetic energy $H_{k,\tilde{w}}(q,p)$ down to 0 and also to keep the total kinetic energy $H(q,p) = H_{k,w}(q,p) + H_{k,\tilde{w}}(q,p)$ constant simultaneously. Consequently, $H(q,p) = H_{k,w}(q,p)$ will be achieved asymptotically.

In the third step, a controller called *gradient controller* is added to let the total energy $H(q,p)+U(q)$ time invariant. In this control system, since the asymptotic controller reduces $H_{k,\tilde{w}}(q,p)$ and $U(q)$ down to 0, the condition $H(q,p) + U(q) = H_{k,w}(q,p)$ (implying $U(q) = 0$ and $H_{k,\tilde{w}}(q,p) = 0$) is achieved asymptotically.

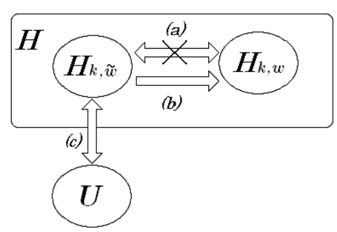

Fig. 1. Energy flow

The energy flow produced by these three controllers is depicted in Figure 1. The arrow (a) denotes the role the *nominal controller* which lets both of the kinetic energies $H_{k,w}(q,p)$ and $H_{k,\tilde{w}}(q,p)$ constant simultaneously. The arrow (b) denotes that of the *asymptotic controller* which reduces the undesired kinetic energy $H_{k,\tilde{w}}(q,p)$. The arrow (c) denotes that of the *gradient controller* which preserves the total energy $H(q,p) + U(q)$. Consequently, both $H_{k,\tilde{w}}(q,p)$ and $U(q)$ converge to 0 asymptotically.

Although the resulting control system achieves path following control smoothly, there are two major defects: (i)

Since the potential energy $U(q)$ takes its minimum on the desired path, the kinetic energy takes its maximum on it, that is, the plant moves fast on the desired path and moves slow outside it. This behavior is unavoidable. (ii) The plant system in the real world always have dissipative elements although the plant model in Equation (1) does not have it. The modelling error with respect to this term very often reduces the total energy of the above control system and let $H(q,p)+U(q)$ converge to 0 eventually. To overcome these problems, we propose an additional controller called *energy controller* to let the total energy to track its desired value, denoted by H_R, in the last step.

A. Nominal controller

This subsection gives the nominal controller. The objective of this controller is to make $H_{k,w}(q,p)$ and $H_{k,\tilde{w}}(q,p)$ independently controllable.

Theorem 1: Consider a port-Hamiltonian system (1) with a vector $p_w(q)$. Suppose that Assumption 2 holds. Then the kinetic energies $H_{k,w}(q,p)$ and $H_{k,\tilde{w}}(q,p)$ are constant along the path of the closed loop system derived by the nominal controller defined by

$$u_n = \langle p_{we}, p\rangle G^{-1}\eta - \langle \eta, p\rangle G^{-1} p_{we}. \tag{6}$$

Here $\eta(q,p) \in \mathbb{R}^m$ is a vector satisfying

$$\langle \eta, p_{we}\rangle = 0 \tag{7}$$

$$\alpha\langle \eta, p_{\tilde{w}}\rangle = \frac{\mathrm{d}H_{k,w}}{\mathrm{d}t} = -\frac{\mathrm{d}H_{k,\tilde{w}}}{\mathrm{d}t} \tag{8}$$

along the path of the plant system (1) with input $u=0$. It is described by

$$\begin{aligned}\eta \equiv & -\frac{1}{2}M\left[\tfrac{\partial M^{-1}}{\partial q_1}p,\ \dots\ ,\tfrac{\partial M^{-1}}{\partial q_l}p\right]J_{12}M^{-1}p_{we}\\ & +\frac{1}{2}M\left[\tfrac{\partial M^{-1}}{\partial q_1}p_{we},\ \dots\ ,\tfrac{\partial M^{-1}}{\partial q_l}p_{we}\right]J_{12}M^{-1}p\\ & +\frac{1}{2}J_{12}^{\mathrm{T}}\begin{bmatrix} p^{\mathrm{T}}\frac{\partial M^{-1}}{\partial q_1}\\ \dots \\ p^{\mathrm{T}}\frac{\partial M^{-1}}{\partial q_l}\end{bmatrix}p_{we} + \frac{\partial p_{we}}{\partial q}J_{12}M^{-1}p\\ & - J_{22}M^{-1}p_{we}\end{aligned}. \tag{9}$$

Proof: Let us consider a feedback system with the port-Hamiltonian system (1) with the feedback $u=u_n$. In order to prove the time invariance of $H_{k,w}$ and $H_{k,\tilde{w}}$, it is proven that the total kinetic energy $H = H_{k,w}+H_{k,\tilde{w}}$ is time invariant along the state trajectory of the feedback system first. The time derivative of H is calculated as

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \langle Gu, p\rangle = \langle p_{we}, p\rangle\langle \eta, p\rangle - \langle \eta, p\rangle\langle p_{we}, p\rangle = 0. \tag{10}$$

This proves that H is constant.

Next we prove that $H_{k,\tilde{w}}$ is constant. The time derivative of $H_{k,\tilde{w}}$ along the closed loop system is calculated as

$$\begin{aligned}\frac{\mathrm{d}H_{k,\tilde{w}}}{\mathrm{d}t} &= \langle Gu - \alpha\eta, p_{\tilde{w}}\rangle\\ &= -\langle \eta, p\rangle\langle p_{we}, p_{\tilde{w}}\rangle\\ &= 0\end{aligned} \tag{11}$$

which implies that $H_{k,\tilde{w}}$ is constant. This also suggests that $H_{k,w} = H - H_{k,\tilde{w}}$ is constant as well. ■

B. Asymptotic controller

Next let us introduce the asymptotic controller. The objective of this controller is to reduce the undesired kinetic energy $H_{k,\tilde{w}}(q,p)$ and to let the total kinetic energy $H_{k,w}(q,p) + H_{k,\tilde{w}}(q,p)$ constant simultaneously.

Theorem 2: Consider a port-Hamiltonian system (1) with a vector $p_w(q)$. Suppose that Assumption 2 holds and define the asymptotic controller by

$$u_a = -\beta\ \mathrm{sgn}\langle p_{we}, p\rangle\left(\langle p_{we}, p\rangle G^{-1}p_{\tilde{w}} - \langle p_{\tilde{w}}, p\rangle G^{-1}p_{we}\right) \tag{12}$$

where $\beta(q,p) > 0 \in \mathbb{R}$ is a design parameter. Then the following properties hold along the state trajectory of the closed loop system derived by the feedback $u = u_n + u_a$: The total kinetic energy $H(q,p)$ is constant. The undesired kinetic energy $H_{k,\tilde{w}}$ monotonically decreases and converges to 0 as $t\to\infty$.

Proof: Consider a feedback system with the port-Hamiltonian system (1) and the controller $u = u_n + u_a$. First of all, it is proven that the total kinetic energy $H = H_{k,w} + H_{k,\tilde{w}}$ is constant along the closed loop system.

$$\begin{aligned}\frac{\mathrm{d}H}{\mathrm{d}t} &= \langle Gu, p\rangle = \langle Gu_a, p\rangle\\ &= -\beta sgn\langle p_{we}, p\rangle\left(\langle p_{we}, p\rangle\langle p_{\tilde{w}}, p\rangle - \langle p_{\tilde{w}}, p\rangle\langle p_{we}, p\rangle\right)\\ &= 0\end{aligned} \tag{13}$$

which implies that H is constant. The second equality follows from Equation (10).

Next let us calculate the time derivative of $H_{k,\tilde{w}}$ along the state trajectory of the closed loop system as

$$\begin{aligned}\frac{\mathrm{d}H_{k,\tilde{w}}}{\mathrm{d}t} =& \langle Gu - \alpha\eta, p_{\tilde{w}}\rangle = \langle Gu_a, p_{\tilde{w}}\rangle\\ =& -\beta sgn\langle p_{we}, p\rangle\left(\langle p_{we}, p\rangle\langle p_{\tilde{w}}, p_{\tilde{w}}\rangle\right.\\ & \left. - \langle p_{\tilde{w}}, p\rangle\langle p_{we}, p_{\tilde{w}}\rangle\right)\\ =& -\beta|\langle p_{we}, p\rangle|\langle p_{\tilde{w}}, p_{\tilde{w}}\rangle\end{aligned}. \tag{14}$$

The second equality follows from Equation (11). It is obvious that $\mathrm{d}H_{k,\tilde{w}}/\mathrm{d}t < 0$ holds if $p_{\tilde{w}} \neq 0$, $\beta > 0$ and $\langle p_{we}, p\rangle \neq 0$. In order to prove that $\mathrm{d}H_{k,\tilde{w}}/\mathrm{d}t = 0$ implies $p_{\tilde{w}} = 0$, i.e., $H_{k,\tilde{w}} = 0$, we first prove that a set satisfying $p_{\tilde{w}} \neq 0$ and $\langle p_{we}, p\rangle = 0$ is not an invariant set. Take the time derivative of $\langle p_{we}, p\rangle$ as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle p_{we}, p\rangle =& \langle Gu, p_{we}\rangle + \langle \eta, p\rangle\\ =& \langle p_{we}, p\rangle\langle \eta, p_{we}\rangle - \langle \eta, p\rangle\langle p_{we}, p_{we}\rangle\\ & - \beta sgn\langle p_{we}, p\rangle\left(\langle p_{we}, p\rangle\langle p_{\tilde{w}}, p_{we}\rangle\right.\\ & \left. - \langle p_{\tilde{w}}, p\rangle\langle p_{we}, p_{we}\rangle\right) + \langle \eta, p\rangle\\ =& \beta\langle p_{\tilde{w}}, p_{\tilde{w}}\rangle \mathrm{sgn}\langle p_{we}, p\rangle\end{aligned} \tag{15}$$

The third equality is implied by Equation (7). This equation does not dismiss if $p_{\tilde{w}} \neq 0$. Therefore the set satisfying $p_{\tilde{w}} \neq 0$ and $\langle p_{we}, p\rangle = 0$ is not an invariant set. Therefore $H_{k,\tilde{w}}$ converges to the set $p_{\tilde{w}} = 0$, i.e., $H_{k,\tilde{w}} = 0$, as $t\to\infty$. This completes the proof. ■

Remark 1: The asymptotic controller given in Equation (12) can be generalized to

$$\begin{aligned}u_a =& -\operatorname{sgn}\langle p_{we}, p\rangle \{\beta\left(\langle p_{we}, p\rangle G^{-1}p_{\tilde{w}} - \langle p_{\tilde{w}}, p\rangle G^{-1}p_{we}\right)\\ &+ \langle p_{we}, p\rangle K_{a1}G^{\mathrm{T}}M^{-1}p_{\tilde{w}}\\ &- \langle GK_{a1}G^{\mathrm{T}}M^{-1}p_{\tilde{w}}, p\rangle G^{-1}p_{we}\\ &+ \langle GK_{a2}G^{\mathrm{T}}M^{-1}p_{we}, p\rangle G^{-1}p_{\tilde{w}}\\ &-\langle p_{\tilde{w}}, p\rangle K_{a2}G^{\mathrm{T}}M^{-1}p_{we}\}\end{aligned} \tag{16}$$

with additional free parameters $K_{a1}(q,p), K_{a2}(q,p) \succeq 0 \in \mathbb{R}^{m\times m}$. We can prove the same statements as in Theorem 2 for this controller.

C. Gradient controller

Next let us introduce the gradient controller to take care of the potential function and reduce it to achieve the path following control.

Theorem 3: Consider a port-Hamiltonian system (1) with a scalar function $U(q)$ and a vector $p_w(q)$. Suppose that Assumptions 1 and 2 hold and define the gradient controller by

$$u_p = -G^{-1}J_{12}^{\mathrm{T}}\frac{\partial U}{\partial q}^{\mathrm{T}}. \tag{17}$$

Then the following properties hold along the state trajectory of the closed loop system derived by the feedback $u = u_n + u_a + u_p$: The total energy $H(q,p) + U(q)$ is constant. The state converges to the desired path as $t \to \infty$ for all initial conditions except the measure zero set $\{(q,p)|p = 0, J_{12}^{\mathrm{T}}(\partial U/\partial q)^{\mathrm{T}} = 0\}$.

Proof: Consider a feedback system with the port-Hamiltonian system (1) with the controller $u = u_n+u_a+u_p$. First of all, it is proven that $H + U$ is constant along this dynamics. As a preparation, compute the time derivative of U as

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \frac{\partial U}{\partial q}\dot{q} = \frac{\partial U}{\partial q}J_{12}M^{-1}p = \langle J_{12}^{\mathrm{T}}\frac{\partial U}{\partial q}^{\mathrm{T}}, p\rangle. \tag{18}$$

Using this relation, we can calculate the time derivative of $H + U$ as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(H+U) =& \langle Gu + J_{12}^{\mathrm{T}}\frac{\partial U}{\partial q}^{\mathrm{T}}, p\rangle\\ =& \langle Gu_n, p\rangle + \langle Gu_a, p\rangle.\\ =& 0\end{aligned} \tag{19}$$

It is proven that $H + U$ is constant. Here the third equality is implied by Equations (10) and (13).

Next, in order to prove the convergence of the state to the desired path, let us define a Lyapunov like function

$$V = H_{k,\tilde{w}} + U.$$

Computing its time derivative, we obtain

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} =& \langle Gu - \alpha\eta, p_{\tilde{w}}\rangle + \langle J_{12}^{\mathrm{T}}\frac{\partial U}{\partial q}^{\mathrm{T}}, p\rangle\\ =& -\beta|\langle p_{we}, p\rangle|\langle p_{\tilde{w}}, p_{\tilde{w}}\rangle - \langle J_{12}^{\mathrm{T}}\frac{\partial U}{\partial q}^{\mathrm{T}}, p_{\tilde{w}}\rangle\\ &+ \alpha\langle J_{12}^{\mathrm{T}}\frac{\partial U}{\partial q}^{\mathrm{T}}, p_{we}\rangle + \langle J_{12}^{\mathrm{T}}\frac{\partial U}{\partial q}^{\mathrm{T}}, p_{\tilde{w}}\rangle\\ =& -\beta|\langle p_{we}, p\rangle|\langle p_{\tilde{w}}, p_{\tilde{w}}\rangle\\ \le& 0.\end{aligned} \tag{20}$$

Here the second and third equalities follow from Equation (14) and Assumption 2, respectively. The equality in Equation (20) holds if (q,p) is contained in the following set

$$E = \Big\{(q,p)|\langle p_{we}, p\rangle\langle p_{\tilde{w}}, p_{\tilde{w}}\rangle = 0\Big\}.$$

La Salle's invariance principle implies that the state converges to the maximum invariant set contained in E. Hence the maximum invariant set is investigated in what follows.

First of all, the behavior of the states starting from the subset $\Big\{(q,p)|\langle p_{we}, p\rangle = 0, p_{\tilde{w}} \neq 0\Big\}$ of the set E is examined. Take the time derivative of $\langle p_{we}, p\rangle$ as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle p_{we}, p\rangle =& \langle Gu, p_{we}\rangle + \langle \eta, p\rangle\\ =& \beta\langle p_{\tilde{w}}, p_{\tilde{w}}\rangle \operatorname{sgn}\langle p_{we}, p\rangle - \langle J_{12}^{\mathrm{T}}\frac{\partial U}{\partial q}^{\mathrm{T}}, p_{we}\rangle\\ =& \beta\langle p_{\tilde{w}}, p_{\tilde{w}}\rangle \operatorname{sgn}\langle p_{we}, p\rangle\\ &\begin{cases} > 0 & (\langle p_{we}, p\rangle \ge 0, p_{\tilde{w}} \neq 0\ )\\ = 0 & (p_{\tilde{w}} = 0\ )\\ < 0 & (\langle p_{we}, p\rangle < 0, p_{\tilde{w}} \neq 0\ )\end{cases}.\end{aligned} \tag{21}$$

The second and third equalities follow from Equation (15) and Assumption 2, respectively. This equation implies that $\mathrm{d}\langle p_{we}, p\rangle/\mathrm{d}t \neq 0$ holds if $p_{\tilde{w}} \neq 0$. Hence a subset of E satisfying $\langle p_{we}, p\rangle = 0$ and $p_{\tilde{w}} \neq 0$ is not invariant. It also implies that $|\langle p_{we}, p\rangle|$ is monotonically nondecreasing.

Next, the behavior of the states starting from the subset $\Big\{(q,p)|p_{\tilde{w}} = 0\Big\}$ of E is investigated. The time derivative of $p_{\tilde{w}}$ along the dynamics with the constraint $p_{\tilde{w}} = 0$ ($p = \langle p_{we}, p\rangle p_{we}$) can be calculated as

$$\begin{aligned}\frac{\mathrm{d}p_{\tilde{w}}}{\mathrm{d}t} =& Gu - \langle Gu, p_{we}\rangle p_{we} - \langle \eta, p\rangle p_{we} - \langle p_{we}, p\rangle\eta\\ =& -\langle p_{we}, p\rangle\langle \eta, p_{we}\rangle p_{we} - \langle \eta, p\rangle p_{we}\\ &- \beta sgn\langle p_{we}, p\rangle\left(\langle p_{we}, p\rangle p_{\tilde{w}} - \langle p_{\tilde{w}}, p\rangle p_{we}\right.\\ &\left.-\langle p_{we}, p\rangle\langle p_{\tilde{w}}, p_{we}\rangle p_{we} + \langle p_{\tilde{w}}, p\rangle\langle p_{we}, p_{we}\rangle p_{we}\right)\\ &- J_{12}^{\mathrm{T}}\frac{\partial U}{\partial q}^{\mathrm{T}} - \langle J_{12}^{\mathrm{T}}\frac{\partial U}{\partial q}^{\mathrm{T}}, p_{we}\rangle p_{we}\\ =& -2\langle p_{we}, p\rangle\langle \eta, p_{we}\rangle p_{we} - J_{12}^{\mathrm{T}}\frac{\partial U}{\partial q}^{\mathrm{T}}\\ =& -J_{12}^{\mathrm{T}}\frac{\partial U}{\partial q}^{\mathrm{T}}.\end{aligned} \tag{22}$$

Here the first and the third equalities are proven using $p_{\tilde{w}} = 0$. The third one is also implied by $p = \langle p, p_{we}\rangle p_{we}$ and

Assumption 2. The fourth equality is implied by Equation (7). Equation (22) implies that the subset of E satisfying $p_{\tilde{w}} = 0$ is invariant only if $J_{12}^{\mathrm{T}}(\partial U/\partial q)^{\mathrm{T}} = 0$. This suggests that the maximum invariant set M contained in E is characterized by

$$M = \left\{(q,p)|J_{12}\frac{\partial U}{\partial q} = 0, p_{\tilde{w}} = 0\right\}.$$

Assumption 2 implies that if the state stays in the set M then it tracks the desired path or $p(t) \equiv 0$ holds. Let us consider the possibility of the latter case $p(t) \equiv 0$. The condition $p = \langle p_{we}, p\rangle p_{we} + p_{\tilde{w}} = 0$ holds if the initial state satisfy $\langle p_{we}, p\rangle = 0$ and if $p_{\tilde{w}}(q(t), p(t)) \equiv 0$ holds. Further the latter condition $p_{\tilde{w}}(q(t), p(t)) \equiv 0$ holds if $J_{12}^{\mathrm{T}}(\partial U/\partial q)^{\mathrm{T}} \equiv 0$ due to Equation (22). Hence $p(t) \equiv 0$ does not hold unless the initial state is contained in the measure zero set $\{(q,p)|p = 0, J_{12}^{\mathrm{T}}(\partial U/\partial q)^{\mathrm{T}} = 0\}$. Then La Salle's invariance principle proves that the state converges to the desired path as $t \to \infty$ unless the initial state is in the set $\{(q,p)|p = 0, J_{12}^{\mathrm{T}}(\partial U/\partial q)^{\mathrm{T}} = 0\}$. This completes the proof. ∎

The resulting closed loop system with the feedback $u = u_n + u_a + u_p$ is described again by a port-Hamiltonian system

$$\begin{pmatrix}\dot{q}\\ \dot{p}\end{pmatrix} = \begin{pmatrix}0 & J_{12}\\ -J_{12}^{\mathrm{T}} & J_{22} + \bar{J}_{22}\end{pmatrix}\begin{pmatrix}\frac{\partial \bar{H}}{\partial q}^{\mathrm{T}}\\ \frac{\partial \bar{H}}{\partial p}^{\mathrm{T}}\end{pmatrix}. \qquad (23)$$
$$\bar{H} = H + U$$

Here $\bar{J}_{22}(q,p) \in \mathbb{R}^{m\times m}$ (with respect to the asymptotic controller (16)) is described by the following skew-symmetric matrix

$$\begin{aligned}\bar{J}_{22} &= L - L^{\mathrm{T}}\\ L &= \eta p_{we}^{\mathrm{T}} + \mathrm{sgn}\langle p_{we}, p\rangle \left\{\beta p_{we} p_{\tilde{w}}^{\mathrm{T}}\right. \qquad (24)\\ &\quad \left. + p_{we}\left(GK_{a1}G^{\mathrm{T}}M^{-1}p_{\tilde{w}}\right)^{\mathrm{T}} + GK_{a2}G^{\mathrm{T}}M^{-1}p_{we}p_{\tilde{w}}^{\mathrm{T}}\right\}.\end{aligned}$$

The structure of port-Hamiltonian systems thus obtained will be utilized for assigning the desired energy in the following section.

D. Energy controller

The controller proposed in the previous section is to let the total energy $H(q,p) + U(q)$ constant. This subsection proposes an additional controller called energy controller to regulate the energy to its desired value H_R.

Theorem 4: Consider a port-Hamiltonian system (1) with a scalar function $U(q)$ and a vector $p_w(q)$. Suppose that Assumptions 1 and 2 hold and define the energy controller by

$$u_r = -\gamma\,\langle p_{we}, p\rangle(H + U - H_R)G^{-1}p_{we} \qquad (25)$$

with the desired energy H_R and a design parameter $\gamma(q,p) > 0 \in \mathbb{R}$. Then the following properties hold along the state trajectory of the closed loop system derived by the feedback $u = u_n + u_a + u_p + u_r$: The total energy $H(q,p)+U(q)$ converges to the desired value H_R as $t \to \infty$. The state converges to the desired path as $t \to \infty$ for all initial conditions except the measure zero set $\{(q,p)|p = 0, J_{12}^{\mathrm{T}}(\partial U/\partial q)^{\mathrm{T}} = 0\}$.

Proof: Consider a feedback system with the port-Hamiltonian system (1) with the input $u = u_n + u_a + u_p + u_r$. First of all, it will be proven that the total energy $H + U$ converges to its desired value H_R. The time derivative of $H + U$ along this dynamics is given by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(H + U - H_R) &= \langle Gu + J_{12}^{\mathrm{T}}\frac{\partial U}{\partial q}^{\mathrm{T}}, p\rangle = \langle Gu_r, p\rangle\\ &= -\langle p_{we}, p\rangle^2\gamma(H + U - H_R). \qquad (26)\end{aligned}$$

Since $\gamma > 0$, $H+U$ converges to H_R. Here the third equality is implied by Equations (10) , (13) and (19).

Next it will be proven that the state converges to the desired path. As in the proof of Theorem 3, take the Lyapunov function candidate

$$V = H_{k,\tilde{w}} + U.$$

Then the Equations (20), (22) hold and Equation (21) holds if $\langle p_{we}, p\rangle = 0$ for this dynamics as well. Therefore we can prove the same conclusion as stated in Theorem 3. This completes the proof. ∎

IV. Numerical example

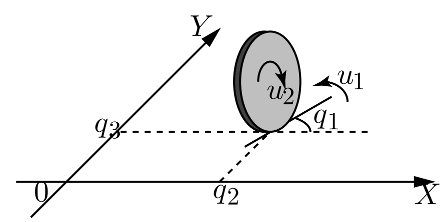

Fig. 2. A rolling coin

The proposed method given in the previous section is applied to a rolling coin on a horizontal plane depicted in Figure 2. Let X-Y denote the Cartesian coordinates on the horizontal plane. The position of the coin on these coordinates is denoted by (q_2, q_3). The angle between the X axis and the direction of travel is denoted by q_1. The angular momentum with respect to the rolling in the direction of travel is denoted by p_1 and that with respect to changing the direction of travel is denoted by p_2. The input torques corresponding to p_1 and p_2 are denoted by u_1 and u_2, respectively. See [3], [11] for detail. The other parameters are set to 1. Then the dynamics of this system is described by a port-Hamiltonian system in the form of (1) as follows.

$$\begin{pmatrix}\dot{q}_1\\ \dot{q}_2\\ \dot{q}_3\\ \dot{p}_1\\ \dot{p}_2\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & \cos q_1\\ 0 & 0 & 0 & 0 & \sin q_1\\ -1 & 0 & 0 & 0 & 0\\ 0 & -\cos q_1 & -\sin q_1 & 0 & 0\end{pmatrix}\begin{pmatrix}0\\ 0\\ 0\\ p_1\\ p_2\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & 0\\ 0 & 0\\ 1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}$$
$$H = \frac{1}{2}\begin{pmatrix}p_1 & p_2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}p_1\\ p_2\end{pmatrix} \qquad (27)$$

The desired path of the coin on the coordinates X-Y is a circle centered at $(q_2, q_3) = (0, 0)$ with a radius of r_0. The potential function $U(q)$ satisfying Assumption 1 is selected as

$$U = \frac{1}{8}(q_2^2 + q_3^2 - r_0^2)^2 + (q_2^2 + q_3^2)\cos^2(q_1 - \mathrm{atan2}(q_3, q_2)).$$

The desired phase vector $p_w(q)$ satisfying Assumption 2 is chosen as

$$p_w(q) = \begin{pmatrix} \frac{1}{2}(q_2^2 + q_3^2 - r_0^2) + 2 \\ -2(-q_2 \sin q_1 + q_3 \cos q_1) \end{pmatrix}.$$

Figures 3 and 4 depict the responses of the control system with the design parameters as follows: the radius $r_0 = 1$, the desired total energy $H_R = 5$ $(0 \le t < 5)$, and $H_R = 2$ $(5 \le t \le 10)$, the initial state $x(0) = (0, 1, -1, 0, 2)^{\mathrm{T}}$, the design parameters $\beta = 1$, $K_{a1} = 0$, $K_{a2} = 0$, and $\gamma = 0.5$. Figure 3 shows the locus of the coin on the X-Y plane. The coin starting from $(q_2(0), q_3(0)) = (1, -1)$ tracks the desired circle centered at the origin with a radius of 1. Figure 4 shows the time response of the energies U, $H + U$ and $H_{k,w}$. The solid (red) line denotes the response of the total energy $H(q, p) + U(q)$. The dashed (blue) one denotes that of the desired kinetic energy $H_{k,w}(q)$. The dashed-dotted (green) one denotes that of the potential energy $U(q)$. The dotted (purple) one denotes the time history of the desired energy H_R. This figure shows that both the total energy $H(q, p) + U(q)$ and the desired kinetic energy $H_{k,w}(q, p)$ converge to their desired value H_R smoothly. Both figures reveal the effectiveness of the proposed method.

V. Conclusion

This paper was devoted to path following method for a class of port-Hamiltonian systems by generalizing the idea of Duindam et al. [9], [10]. Using the phase space (co-tangent space) to characterize the desired and the undesired vector fields, we can derive a control method to design vector fields directly for a wider class of systems including nonholonomic ones. Furthermore, by making use of the port-Hamiltonian structure of the closed loop system, we can obtain a novel controller to assign the desired total energy. Moreover, a numerical simulation using a rolling coin has exhibited the effectiveness of the proposed method.

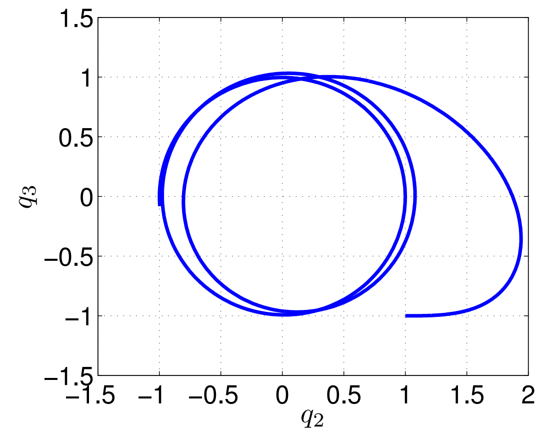

Fig. 3. Locus of the coin on the X-Y plane

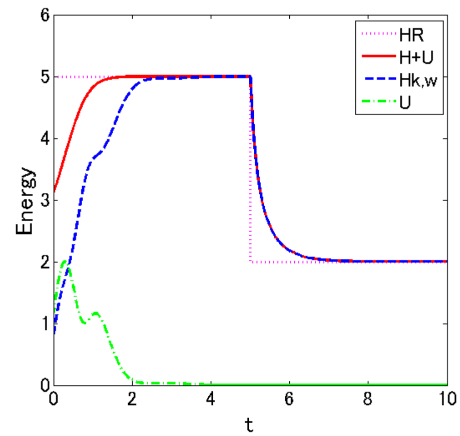

Fig. 4. Energy variation

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