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Stability of Zeno Equilibria in Lagrangian Hybrid Systems

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***Abstract*— This paper presents both necessary and sufficient conditions for the stability of Zeno equilibria in Lagrangian hybrid systems, i.e., hybrid systems that model mechanical systems undergoing impacts. These conditions for stability are motivated by the sufficient conditions for Zeno behavior in Lagrangian hybrid systems obtained in [10]—we show that the same conditions that imply the existence of Zeno behavior near Zeno equilibria imply the stability of the Zeno equilibria. This paper, therefore, not only presents conditions for the stability of Zeno equilibria, but directly relates the stability of Zeno equilibria to the existence of Zeno behavior.**

I. INTRODUCTION

Zeno behavior occurs in a hybrid system when an infinite number of discrete transitions occur in a finite amount of time. Despite the simplicity of the definition of Zeno behavior, understanding this behavior on a fundamental level presents difficult and intriguing problems in hybrid systems. Can simple conditions for the existence of Zeno behavior be obtained? How does the existence of Zeno behavior relate to the convergence properties, or stability, of hybrid systems? In order to obtain an intuitive understanding of this phenomena, and help to answer some of the fundamental questions that arise when studying Zeno behavior, it is useful to study it in the context of hybrid systems that model real world systems.

In this paper, we study hybrid systems modeling mechanical systems undergoing impacts: *Lagrangian hybrid systems*. In particular, we consider a configuration space, a Lagrangian modeling a mechanical systems, and a *unilateral constraint function* that gives the set of admissible configurations for this system. From this data, we obtain a Lagrangian hybrid system. Moreover, hybrid systems of this form commonly display Zeno behavior (when an infinite number of collisions occur in a finite amount of time), and therefore provide the ideal class of systems in which to gain an intuitive understanding of Zeno behavior.

In [10], sufficient conditions for the existence of Zeno behavior in Lagrangian hybrid systems were presented. These conditions were obtained by considering *Zeno equilibria*—subsets of the continuous domains of a hybrid system that are fixed points of the discrete dynamics but not the continuous dynamics. It was shown that one only needs to check the sign of the second derivative of the unilateral constraint function evaluated at a Zeno equilibrium point to verify the existence of Zeno behavior. These conditions, and the framework in which they were presented, naturally raises the question: can similar conditions for the stability of Zeno equilibria in Lagrangian hybrid systems be obtained?

The main result of this paper are both necessary and sufficient conditions for the stability of Zeno equilibria in Lagrangian hybrid systems. Moreover, the sufficient conditions that we obtain are *exactly* the same as the conditions for the existence of Zeno behavior presented in [10]. That is, given a Zeno equilibrium point of a Lagrangian hybrid system, if the second derivative of the unilateral constraint function evaluated as this point is negative, then this point is stable and the hybrid system is Zeno. This result is appealing not only because it presents conditions for the stability of Zeno equilibria, but also because it relates the stability of such equilibria to Zeno behavior and vice versa. That is, this paper allows for a deeper insight into the relationship between stability of Zeno equilibria and Zeno behavior in hybrid systems modeling mechanical systems undergoing impacts.

Due to the subtle and complex nature of Zeno behavior, it has been studied in many forms and from many different perspectives. Most of the conditions for Zeno behavior are necessary and tend to be very conservative; see [20] for general hybrid systems, and [6], [19] for linear complementarity systems. Until recently, sufficient conditions for Zeno behavior were more rare [2]. Necessary and sufficient conditions for Zeno behavior in a significantly different class of controlled hybrid systems were found in [8]. We also note that this paper studies Zeno behavior in Lagrangian hybrid systems, which were studied in [1], [3] and [4] as motivated by [5].

II. LAGRANGIAN HYBRID SYSTEMS

In this section, we introduce the notion of a hybrid Lagrangian and the associated Lagrangian hybrid system. Hybrid Lagrangians of this form have been studied in the context of Zeno behavior and reduction; see [1], [3], [4] and [9]. First, we review the notion of a simple hybrid system.

Definition 1: A *simple hybrid system* is a tuple:

$$\mathscr{H} = (D, G, R, f),$$

where

- D is a smooth manifold called the *domain*,
- G is an embedded submanifold of D called the *guard*,
- R is a smooth map $R : G \to D$ called the *reset map*,
- f is a vector field on the manifold D.

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This paper focuses on *simple* hybrid systems, having a single domain, guard and reset map. A general hybrid system (see [12]), which is not discussed here, consists of a collection of domains, guards, reset maps and vector fields as indexed by an oriented graph.

Hybrid executions. An *execution* of a simple hybrid system $\mathscr{H}$ is a tuple $\chi^{\mathscr{H}} = (\Lambda, \mathcal{I}, \mathcal{C})$, where

- $\Lambda = \{0, 1, 2, \ldots\} \subseteq \mathbb{N}$ is an indexing set.
- $\mathcal{I} = \{I_i\}_{i\in\Lambda}$ is a *hybrid interval* where $I_i = [\tau_i, \tau_{i+1}]$ if $i, i+1 \in \Lambda$ and $I_{N-1} = [\tau_{N-1}, \tau_N]$ or $[\tau_{N-1}, \tau_N)$ or $[\tau_{N-1}, \infty)$ if $|\Lambda| = N$, N finite. Here, $\tau_i, \tau_{i+1}, \tau_N \in \mathbb{R}$ and $\tau_i \leq \tau_{i+1}$.
- $\mathcal{C} = \{c_i\}_{i\in\Lambda}$ is a collection of integral curves of f, i.e., $\dot{c}_i(t) = f(c_i(t))$ for $t \in I_i$, $i \in \Lambda$,

And the following conditions hold for every $i, i+1 \in \Lambda$:

(i) $c_i(\tau_{i+1}) \in G$,
(ii) $R(c_i(\tau_{i+1})) = c_{i+1}(\tau_{i+1})$,
(iii) $\tau_{i+1} = \min\{t \in I_i : c_i(t) \in G\}$.

The *initial condition* for the execution is $c_0(\tau_0)$.

Lagrangians. Let Q be the n-dimensional *configuration space* for a mechanical system (assumed to be a smooth manifold) and TQ the tangent bundle of Q. In this paper, we will consider Lagrangians, $L : TQ \to \mathbb{R}$, describing mechanical, or robotic, systems, which are Lagrangians of the form

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - V(q), \tag{1}$$

where $M(q)$ is the (positive definite) inertial matrix, $\frac{1}{2}\dot{q}^T M(q)\dot{q}$ is the kinetic energy and $V(q)$ is the potential energy. In this case, the Euler-Lagrange equations yield the (unconstrained) equations of motion for the system:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + N(q) = 0, \tag{2}$$

where $C(q, \dot{q})$ is the *Coriolis matrix* (cf. [14]) and $N(q) = \frac{\partial V}{\partial q}(q)$. Setting $x = (q, \dot{q})$, the Lagrangian vector field, f_L, associated to L takes the familiar form:

$$\dot{x} = f_L(x) = \begin{pmatrix} \dot{q} \\ M(q)^{-1}(-C(q, \dot{q})\dot{q} - N(q)) \end{pmatrix}. \tag{3}$$

This process of associating a dynamical system to a Lagrangian will be mirrored in the setting of hybrid systems. First, we introduce the notion of a hybrid Lagrangian.

Definition 2: A *simple hybrid Lagrangian* is defined to be a tuple

$$\mathbf{L} = (Q, L, h),$$

where

- Q is the configuration space,
- $L : TQ \to \mathbb{R}$ is a hyperregular Lagrangian,
- $h : Q \to \mathbb{R}$ provides a unilateral constraint on the configuration space; we assume that the level set $h^{-1}(0)$ is a smooth manifold.

Simple Lagrangian hybrid systems. For a Lagrangian (1), there is an associated dynamical system (3). Similarly, given a hybrid Lagrangian $\mathbf{L} = (Q, L, h)$ the *simple Lagrangian hybrid system* (**SLHS**) associated to $\mathbf{L}$ is the simple hybrid system:

$$\mathscr{H}_{\mathbf{L}} = (D_{\mathbf{L}}, G_{\mathbf{L}}, R_{\mathbf{L}}, f_{\mathbf{L}}).$$

First, we define

$$\begin{aligned} D_{\mathbf{L}} &= \{(q, \dot{q}) \in TQ : h(q) \geq 0\}, \\ G_{\mathbf{L}} &= \{(q, \dot{q}) \in TQ : h(q) = 0 \text{ and } dh(q)\dot{q} \leq 0\}, \end{aligned}$$

where $dh(q) = \left(\frac{\partial h}{\partial q}(q)\right)^T = \begin{pmatrix} \frac{\partial h}{\partial q_1}(q) & \cdots & \frac{\partial h}{\partial q_n}(q) \end{pmatrix}$.

In this paper, we adopt the reset map ([5]):

$$R_{\mathbf{L}}(q, \dot{q}) = (q, P_{\mathbf{L}}(q, \dot{q})),$$

which based on the *impact equation*

$$P_{\mathbf{L}}(q, \dot{q}) = \dot{q} - (1+e)\frac{dh(q)\dot{q}}{dh(q)M(q)^{-1}dh(q)^T} M(q)^{-1}dh(q)^T, \tag{4}$$

where $0 \leq e \leq 1$ is the *coefficient of restitution*, which is a measure of the energy dissipated through impact. This reset map corresponds to rigid-body collision law under the assumption of *frictionless impact* [5]. Examples of more complicated collision laws that account for friction can be found in [5], [7].

Finally, $f_{\mathbf{L}} = f_L$ is the Lagrangian vector field associated to L in (3).

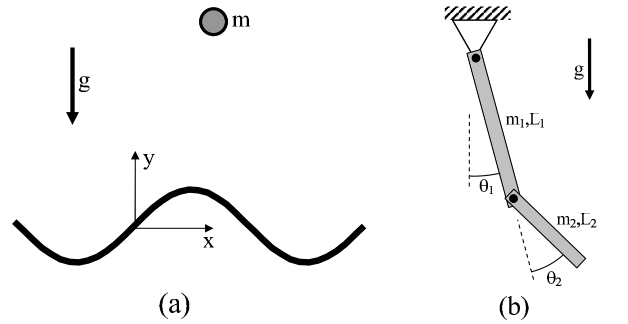

Fig. 1. (a) The bouncing ball on a sinusoidal surface (b) The double pendulum

Example 1 (Ball): The first running example of this paper is a planar model of a ball bouncing on a sinusoidal surface (cf. Fig. 1(a)). The ball is modelled as a point mass m. In this case

$$\mathbf{B} = (Q_{\mathbf{B}}, L_{\mathbf{B}}, h_{\mathbf{B}}),$$

where $Q_{\mathbf{B}} = \mathbb{R}^2$, and the configuration is the position of the ball $q = (x, y)$,

$$L_{\mathbf{B}}(x, \dot{x}) = \frac{1}{2}m\|\dot{q}\|^2 - mgy.$$

Finally, we make the problem interesting by considering the sinusoidal constraint function

$$h_{\mathbf{B}}(q) = y - \sin(x) \geq 0.$$

So, for this example, there are trivial dynamics and a nontrivial constraint function.

Example 2 (Double Pendulum): Our second running example is a constrained double pendulum with a mechanical stop (cf. Fig. 1(b)). The double pendulum consists of two rigid links of masses m_1, m_2, lengths L_1, L_2, and uniform mass distribution, which are attached by passive joints, while a mechanical stop dictates the range of motion of the second link. The example serves as a simplified model of a leg with a passive knee and a mechanical stop, which is widely investigated in the robotics literature in the context of passive dynamics of bipedal walkers (cf. [13],[18]). In this case

$$\mathbf{P} = (Q_{\mathbf{P}}, L_{\mathbf{P}}, h_{\mathbf{P}}),$$

where $Q_{\mathbf{P}} = \mathbb{S}^1 \times \mathbb{S}^1$, $q = (\theta_1, \theta_2)$, and

$$\begin{aligned} L_{\mathbf{P}}(q, \dot{q}) \quad = \quad & \tfrac{1}{2}\dot{q}^T M(q)\dot{q} + (\tfrac{1}{2}m_1 L_1 + m_2 L_1) g \cos\theta_1 \\ & + \tfrac{1}{2} m_2 L_2 g \cos(\theta_1 + \theta_2), \end{aligned}$$

with the elements of the 2×2 inertia matrix $M(q)$ given by

$$\begin{aligned} M_{11} &= m_1 L_1^2/3 + m_2(L_1^2 + L_2^2/3 + L_1 L_2 \cos\theta_2) \\ M_{12} &= M_{21} = m_2(3L_1 L_2 \cos\theta_2 + 2L_2^2)/6 \\ M_{22} &= m_2 L_2^2/3. \end{aligned}$$

Finally, the constraint that represents the mechanical stop is given by $h_{\mathbf{P}}(q) = \theta_2 \geq 0$. So, for this example, there are nontrivial dynamics and a trivial constraint function.

III. Zeno Behavior and Zeno Equilibria

This section discusses Zeno behavior and the corresponding notion of Zeno equilibria. More importantly, we state the sufficient conditions for Zeno behavior that will motivate the main result of this paper in that our sufficient conditions for the stability of Zeno equilibria utilize exactly the same conditions; that is, in Lagrangian hybrid systems, the existence of Zeno behavior and the stability of Zeno equilibria can be detected with the same simple and easily verifiable conditions.

Zeno behavior. An execution $\chi^{\mathscr{H}}$ is *Zeno* if $\Lambda = \mathbb{N}$ and

$$\lim_{i \to \infty} \tau_i = \tau_\infty < \infty.$$

Here τ_∞ is called the *Zeno time*. If $\chi^{\mathscr{H}_{\mathbf{L}}}$ is a Zeno execution of a Lagrangian hybrid system $\mathscr{H}_{\mathbf{L}}$, then its *Zeno point* is defined to be

$$x_\infty = (q_\infty, \dot{q}_\infty) = \lim_{i \to \infty} c_i(\tau_i) = \lim_{i \to \infty} (q_i(\tau_i), \dot{q}_i(\tau_i)).$$

These limit points are intricately related to a type of equilibrium point that are unique to hybrid systems: Zeno equilibria.

Definition 3: A Zeno equilibrium point of a simple hybrid system $\mathscr{H}$ is a point $x^* \in G$ such that

- $R(x^*) = x^*$,
- $f(x^*) \neq 0$.

Zeno equilibria. If $\mathscr{H}_{\mathbf{L}}$ is a Lagrangian hybrid system, then due to the special form of these systems we find that the point $(q^*, \dot{q}^*)$ is a Zeno equilibria iff $\dot{q}^* = P_{\mathbf{L}}(q, \dot{q}^*)$, with $P_{\mathbf{L}}$ given in (4). In particular, the special form of $P_{\mathbf{L}}$ implies that this hold iff $dh(q^*)\dot{q}^* = 0$. Therefore the set of all Zeno equilibria for a Lagrangian hybrid system is given by the hypersurfaces in $G_{\mathbf{L}}$:

$$Z = \{(q, \dot{q}) \in G_{\mathbf{L}} : dh(q)\dot{q} = 0\}.$$

Note that if $\dim(Q) > 1$, the Zeno equilibria in Lagrangian hybrid systems are always non-isolated (see [9])—this motivates the study of such equilibria.

Sufficient conditions for Zeno behavior. Let $\ddot{h}(q, \dot{q})$ be the acceleration of $h(q(t))$ along trajectories of the unconstrained dynamics (2), which is given by:

$$\ddot{h}(q, \dot{q}) = \dot{q}^T H(q)\dot{q} + dh(q)M(q)^{-1}(-C(q, \dot{q})\dot{q} - N(q)), \tag{5}$$

where $H(q)$ is the Hessian of h at q. The following theorem, which was proven in [10], provides sufficient conditions for existence of Zeno executions in the vicinity of a Zeno equilibrium point.

Theorem 1 ([10]): *Let $\mathscr{H}_{\mathbf{L}}$ be a simple Lagrangian hybrid system and Let $(q^*, \dot{q}^*)$ be a Zeno equilibrium point of $\mathscr{H}_{\mathbf{L}}$. Then if $e < 1$ and $\ddot{h}(q^*, \dot{q}^*) < 0$, there exists a neighborhood $W \subset D_{\mathbf{L}}$ of $(q^*, \dot{q}^*)$ such that for every $(q_0, \dot{q}_0) \in W$, there is a unique Zeno execution $\chi^{\mathscr{H}_{\mathbf{L}}}$ of $\mathscr{H}_{\mathbf{L}}$ with $c_0(\tau_0) = (q_0, \dot{q}_0)$.*

IV. Stability of Zeno Equilibria

In this section, we present and prove the main result of this paper: sufficient conditions for the stability of Zeno equilibria. In particular, we introduce a type of stability that Zeno equilibria in **SLHS** can display: bounded-time local stability (**BTLS**). We show that the same conditions on the coefficient of restitution and the second derivative of the unilateral constraint function implies this type of stability. Conversely, if these conditions are not satisfied, the Zeno equilibrium point is *not* **BTLS**.

Definition 4: Let $x^* = (q^*, \dot{q}^*)$ be a Zeno equilibrium point of a simple Lagrangian hybrid system $\mathscr{H}_{\mathbf{L}}$. Then x^* is defined as *bounded-time locally stable* if for each open neighborhood $U \subseteq TQ$ of x^* and $\epsilon_t > 0$, there exists another open neighborhood W of x^*, such that for every initial conditions $c_0(\tau_0) \in W \cap D_{\mathbf{L}}$, the corresponding execution $\chi^{\mathscr{H}_{\mathbf{L}}}$ is Zeno, and satisfies $c_i(t) \in U$ for all $t \in I_i$ and $i \in \Lambda$, while its Zeno time satisfies $\tau_\infty - \tau_0 < \epsilon_t$.

A. Statement of Main Result

We now present the main result of the paper: conditions for **BTLS** of Zeno equilibria of **SLHS**.

Theorem 2: *Let $x^* = (q^*, \dot{q}^*)$ be a Zeno equilibrium point of a simple Lagrangian hybrid system $\mathscr{H}_{\mathbf{L}}$. Then the following two conditions hold:*

(i) *If $e < 1$ and $\ddot{h}(q^*, \dot{q}^*) < 0$, then x^* is* **BTLS**.
(ii) *If $\ddot{h}(q^*, \dot{q}^*) > 0$, then x^* is not* **BTLS**.

For part (i), we not only prove the existence of the neighborhood W for given U, but also provide an explicit relation between W and U. For the sake of concreteness and simplicity, we use a *local coordinate chart* for small neighborhoods of x^*. Therefore, we can identify both q and $\dot{q}$ with elements of $\mathbb{R}^n$, and use the induced Euclidean norm $\|\cdot\|$ to define neighborhoods of $x^* = (q^*, \dot{q}^*)$ as

$$N(\epsilon_q, \epsilon_v) = \{(q, \dot{q}) \in D_{\mathbf{L}} : \|q - q^*\| < \epsilon_q, \ \|\dot{q} - \dot{q}^*\| < \epsilon_v\}$$

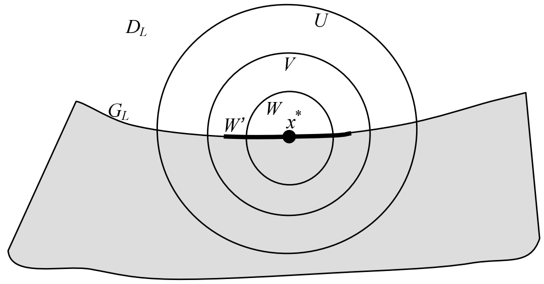

Fig. 2. Illustration of the neighborhoods U, V, W and W' of x^*.

Using this notation, for a given U there exist ϵ_g and ϵ_v such that $U \subseteq N(\epsilon_q, \epsilon_v)$. Assuming that $e < 1$ and $\ddot{h}(q^*, \dot{q}^*) < 0$, our goal is to construct a neighborhood $W = N(\delta_q, \delta_v)$ that satisfies the requirements given in Definition 4.

B. Proof of Main Result

The rest of this section proves Theorem 2 in stages through a series of lemmas. Before presenting these lemmas, we will first give a general outline of the proof. In particular, the proof of part (i) of Theorem 2 is divided into three steps:

1) We define an intermediate neighborhood $V \subset U$, such that any execution that stays within V at all times is guaranteed to be Zeno.
2) We define another neighborhood $W' \subset G_{\mathbf{L}} \cap V$, which lies *on the guard* $G_{\mathbf{L}}$, such that any execution whose first discrete event $c_0(\tau_1)$ lies within W' is guaranteed to stay within V.
3) We construct the neighborhood W, such that any execution with initial conditions within W is guaranteed to pass through a point of W' at time τ_1, and thus it is a Zeno execution that stays within U, as required. An illustration of these neighborhoods appears in Fig. 2.

We now formally proceed through these steps in order to establish the main result of the paper. Due to space limitations, the detailed proofs of the lemmas are relegated to [16].

Step 1. We begin by defining the intermediate neighborhood $V = N(\epsilon'_q, \epsilon'_v)$, where $\epsilon'_q < \epsilon_q$ and $\epsilon'_v < \epsilon_v$ are chosen so that for

$$\begin{aligned} a_{min} &= -\max_{(q,\dot{q})\in V} \ddot{h}(q,\dot{q}), \\ a_{max} &= -\min_{(q,\dot{q})\in V} \ddot{h}(q,\dot{q}), \end{aligned}$$

The following conditions hold:

$$a_{max} > a_{min} > 0 \qquad \text{and} \qquad e\frac{a_{max}}{a_{min}} < 1. \tag{6}$$

Note that the fact that $e < 1$ and $\ddot{h}(q^*, \dot{q}^*) < 0$, along with the continuity of $\ddot{h}(q, \dot{q})$, imply that such ϵ'_q, ϵ'_v exist. This definition of V implies that when $(q(t), \dot{q}(t)) \in V$, $h(q(t))$ satisfies the second-order differential inclusion

$$\ddot{h}(q(t), \dot{q}(t)) \in [-a_{max}, -a_{min}]. \tag{7}$$

For simplicity of notation, for an execution $\chi^{\mathscr{H}_{\mathbf{L}}}$, define

$$\begin{aligned} v_i^- &= dh(q_{i-1}(\tau_i))\dot{q}_{i-1}(\tau_i), \\ v_i^+ &= dh(q_i(\tau_i))\dot{q}_i(\tau_i), \end{aligned}$$

which are the pre- and post- collision velocities at the time τ_i. Note that (4) implies that $v_i^+ = -ev_i^-$. Also, let $T_i = \tau_i - \tau_{i-1}$, which is the time difference between consecutive collisions. The following lemma states that any execution which is bounded within V is guaranteed to be Zeno.

Lemma 1 ([16]): *Let $x^* = (q^*, \dot{q}^*)$ be a Zeno equilibrium point of a simple Lagrangian hybrid system such that $\ddot{h}(q^*, \dot{q}^*) < 0$ and $e < 1$, and let $V = N(\epsilon'_q, \epsilon'_v)$ be a neighborhood of x^* that satisfies (6). Then for any execution $\chi^{\mathscr{H}_{\mathbf{L}}}$ such that $c_i(t) \in V$ for all $t \in I_i$ and $i \in \Lambda$, the discrete-time series of v_i^+ and T_i satisfy:*

$$e\sqrt{\tfrac{a_{min}}{a_{max}}} \le \tfrac{v_{i+1}^+}{v_i^+} \le e\sqrt{\tfrac{a_{max}}{a_{min}}}, \tag{8}$$

$$\tfrac{T_{i+1}}{T_i} \le e\tfrac{a_{max}}{a_{min}}. \tag{9}$$

Therefore, $\chi^{\mathscr{H}_{\mathbf{L}}}$ is Zeno.

The proof of this lemma, which appears in [16], utilizes techniques of optimal control to establish bounds on solutions of the differential inclusion (7), in a way similar to the work of Liberzon and Margaliot [11].

Step 2. As the next step towards computing the neighborhood W, we compute the neighborhood $W' \subset G_{\mathbf{L}} \cap V$, of initial conditions *on the guard* $G_{\mathbf{L}}$ (i.e. corresponding to a collision), such that any execution with initial conditions in W' stays within V.

In order to construct W' for given neighborhoods U and V, we first define the following scalars:

$$\begin{aligned} e' &= e\frac{a_{max}}{a_{min}} \\ e'' &= e\sqrt{\frac{a_{max}}{a_{min}}} \\ \beta &= \|\dot{q}^*\| + \epsilon'_v \\ \eta &= \max_{(q,\dot{q})\in V} \frac{\|M^{-1}(q)dh(q)^T\|}{dh(q)M(q)dh(q)^T} \\ \zeta &= \max_{(q,\dot{q})\in V} \left\|M^{-1}(q)\left(C(q,\dot{q})\dot{q} + N(q)\right)\right\|. \end{aligned} \tag{10}$$

The following lemma completes the definition of W'.

Lemma 2 ([16]): *Let $x^* = (q^*, \dot{q}^*)$ be a Zeno equilibrium point of a simple Lagrangian hybrid system $\mathscr{H}_{\mathbf{L}}$ such that $\ddot{h}(q^*, \ddot{q}^*) < 0$ and $e < 1$, and let $V = N(\epsilon'_q, \epsilon'_v)$ be a neighborhood of x^* that satisfies (6). For a given $\epsilon'_t > 0$, let W' be the neighborhood defined as follows:*

$$\begin{aligned} W' \;=\; &\{(q,\dot{q}) \in TQ : \;\; h(q) = 0,\ \|q - q^*\| < \delta'_q, \\ &\|\dot{q} - \dot{q}^*\| < \delta'_v \text{ and } 0 > dh(q)\dot{q} > -v_{1max}\}. \end{aligned} \tag{11}$$

such that δ'_q, δ'_v and v_{1max} satisfy the conditions:

$$\delta'_q < \epsilon'_q,\ \delta'_v < \epsilon'_v \text{ and } v_{1max} < \min\{c_1, c_2, c_3\} \tag{12}$$

$$\begin{aligned} \textit{where}\quad c_1 &= \tfrac{a_{min}(1-e')}{2e}\epsilon'_t \\ c_2 &= \tfrac{a_{min}\left(1-e'\right)}{2e\beta}(\epsilon'_q - \delta'_q) \\ c_3 &= \left(\epsilon'_v - \delta'_v\right) / \left(\tfrac{(1+e)\eta}{1-e''} + \tfrac{2e\zeta}{a_{min}(1-e')}\right). \end{aligned}$$

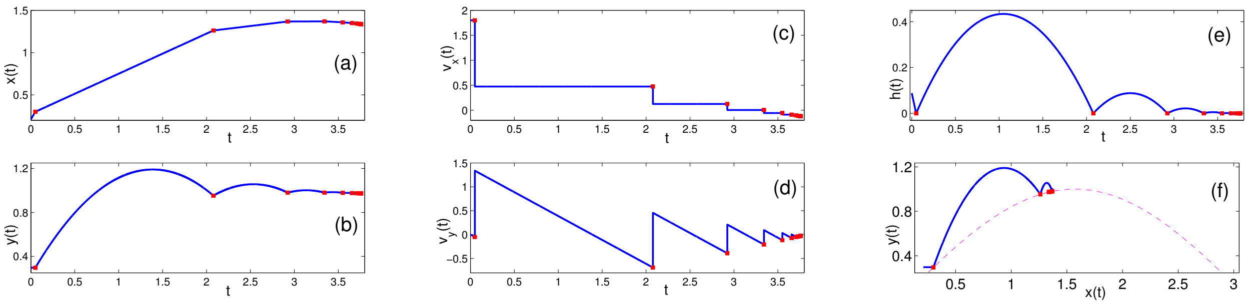

Fig. 3. Simulation results for the ball example with initial velocities $v_x(0) = 1.8$ and $v_y(0) = 0$.

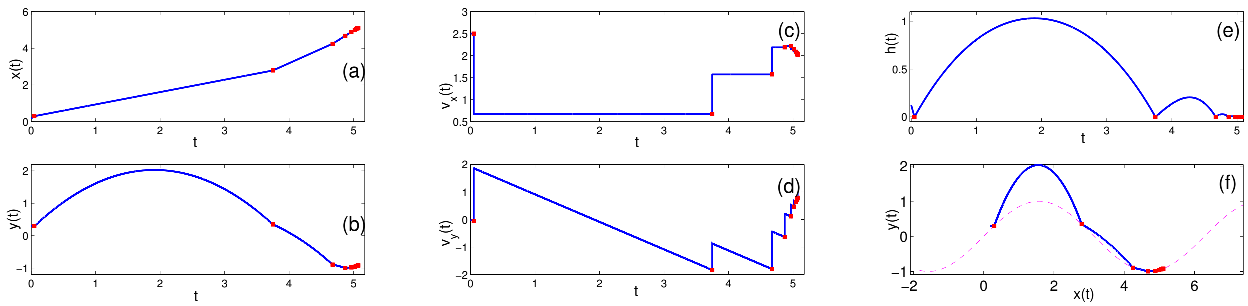

Fig. 4. Simulation results for the ball example with initial velocities $v_x(0) = 2.5$ and $v_y(0) = 0$.

Then each execution $\chi^{\mathscr{H}_{\mathbf{L}}}$ such that $c_0(\tau_1) \in W'$ is Zeno and satisfies $c_i(t) \in V$ for all $t \in I_i$ and $i \geq 1$. Moreover, the corresponding Zeno time satisfies

$$\tau_\infty - \tau_1 < \epsilon_t'. \tag{13}$$

Step 3. At this final stage, for a given $\epsilon_t'' > 0$, we define the neighborhood W as

$$W = N(\delta_q, \delta_v),$$

where $\delta_q < \delta_q'$ and $\delta_q < \delta_q'$ satisfy:

$$\text{(i)}\ \ \frac{dh(q)\dot{q} + \sqrt{(dh(q)\dot{q})^2 - a_{min}h(q)}}{a_{min}} < \min\{\frac{\delta_q' - \delta_q}{\beta},\ \frac{\delta_v' - \delta_v}{\zeta},\ \epsilon_t''\}$$
$$\text{(ii)}\ \ \left(2h(q) + \frac{(dh(q)\dot{q})^2}{a_{min}}\right) a_{max} < (v_{1max})^2 \tag{14}$$

for all $(q, \dot{q}) \in N(\delta_q, \delta_v) \cap D_{\mathbf{L}}$.

Note that since $h(q^*) = 0$ and $dh(q^*)\dot{q}^* = 0$, continuity of $h(q)$ and $dh(q)$ imply that such δ_q, δ_v exist. The following lemma states that if the initial condition are within W, then at the first collision time τ_1, $(q, \dot{q})$ are within W'.

Lemma 3 ([16]): *Let $x^* = (q^*, \dot{q}^*)$ be a Zeno equilibrium point of a simple Lagrangian hybrid system $\mathscr{H}_{\mathbf{L}}$ such that $\ddot{h}(q^*, \ddot{q}^*) < 0$ and $e < 1$, and let V, W' and W be the neighborhoods of x^* defined in (6), (11) and (14) respectively. Then each execution $\chi^{\mathscr{H}_{\mathbf{L}}}$ such that $c_0(\tau_0) \in W \cap D_{\mathbf{L}}$ satisfies $c_0(t) \in V$ for $t \in I_0$, and $c_0(\tau_1) \in W'$ and $\tau_1 - \tau_0 < \epsilon_t''$.*

Finally, the proof of part (i) of Theorem 2 is a straightforward application of Lemmas 1, 2, and 3. The details of the proof, as well as the proof of part (ii), appear in [16].

V. SIMULATION RESULTS

In this section, we present numerical simulations of the examples considered at the beginning of this paper.

Example 3 (Ball): Continuing with Example 1, by direct computation the condition for stability of a Zeno equilibrium point $(q, \dot{q})$ in this system as given in Theorem 2 is:

$$\ddot{h}(q, \ddot{q}) = v_x^2 \sin(x) - g < 0$$

where we denote $\dot{q} = (v_x, v_y)$. This indicates that Zeno equilibrium points that satisfy $\sin(x) < 0$ (i.e. near the minima) are more likely to attract Zeno executions. Moreover, setting the horizontal velocity v_x sufficiently small increases the chances of exhibiting Zeno convergence even at points such that $\sin(x) > 0$ (i.e. near the maxima). For the sake of simplicity, we take $m = 1$, $g = 1$ and $e = 0.5$.

We simulate this system under two different sets of initial conditions, where in both cases the initial conditions at $t = 0$ are chosen such that at $t_1 = 0.05$, a first collision occurs at $x(t_1) = 0.3$, $y(t_1) = \sin(0.3)$. In the first case, the initial velocities are chosen as $v_x(0) = 1.8$ and $v_y(0) = 0$. The execution was simulated until a collision time τ_k at which the collision velocity $dh(q(\tau_k))\dot{q}(\tau_k)$ is less than 10^{-10}. Figures 3(a)-(f) show the simulation results of this running example. Figures 3(a),(b),(c),(d),(e) show the time plots of $x(t), y(t), v_x(t), v_y(t)$ and $h(q(t))$, respectively. The points of collision events are marked with squares ("■"). Figure 3(f) plots $x(t)$ vs. $y(t)$, with the constraint surface $y = \sin(x)$ appearing as a dashed curve. This simulation results in a Zeno execution that converges at a Zeno time $t_\infty = 3.761$ to the Zeno equilibrium point $q^* = (1.337, 0.973)$ and $\dot{q}^* = (-0.121, -0.028)$. This Zeno point is close to a maximum point of the surface; note that the horizontal velocity v_x is significantly decreased from its initial value, so that $\ddot{h}(q^*, \ddot{q}^*) = -0.986 < 0$ and the stability condition is satisfied. Note, too, that the motion of $h(q(t))$ in the vicinity of the Zeno point is remarkably similar to that of a simple bouncing ball (cf. Figure 3(e)).

In the second case, the initial velocities are chosen as $v_x(0) = 2.5$ and $v_y(0) = 0$. Figures 4(a)-(f) show the simulation results under these initial conditions. This simulation results in a Zeno execution that converges at a Zeno time $t_\infty = 5.0731$ to the Zeno equilibrium point $q^* = (5.114, -0.920)$ and $\dot{q}^* = (2.023, 0.791)$. One can see that the trajectory is initially "repelled" from the maximum point due to the large horizontal velocity, and attracted towards the next minimum point, while the horizontal velocity is *increased*, such that $\ddot{h}(q^*, \ddot{q}^*) = -4.766$ satisfies the stability condition in Theorem 2.

Example 4 (Double Pendulum): In the second running example (Example 2) consisting of a double pendulum with a mechanical stop, the condition for stability of Zeno equilibria given in Theorem 2 is

$$\ddot{h}(q, \dot{q}) = \frac{g \sin\theta_1}{\tilde{L}} < 0, \text{ where } \tilde{L} = \frac{(4m_1+3m_2)L_1L_2}{3(m_1(L_1+2L_2)m_2L_2)}.$$

This indicates that only points at which $\sin\theta_1 < 0$ (i.e. the link L_1 is inclined to the left) can be stable Zeno equilibria. Simulation results of this system, which are not shown here due to space limitations, are quite similar to those of the ball example. The reader is referred to [15] for simulation results of the *completed double-pendulum system* (i.e. executions are also carried *beyond* the Zeno points).

VI. Conclusion

In this paper we analyzed the stability of Zeno equilibria of simple Lagrangian hybrid systems, and derived sufficient conditions for stability and for instability of such equilibria. The stability conditions presented are analogous to determining the local stability of equilibrium points of a nonlinear continuous system by computing the eigenvalues of its linearization. This paper provides *almost necessary and sufficient conditions* for stability of Zeno equilibria, where the exceptional intermediate case of $\ddot{h}(q^*, \dot{q}^*) = 0$ is analogous to the case where the linearization of a continuous system has eigenvalues on the imaginary axis, and stability cannot be determined via linearization. This analogy motivates future investigation of techniques for *global* stability analysis of Zeno equilibria, where a promising direction is the use of Lyapunov-like functions as was already done in the analysis of isolated Zeno equilibrium points [9].

The fact that Zeno behavior is fundamentally a modeling phenomena indicates that the conditions used to detect Zeno behavior can be used to "complete" the hybrid system model. That is, carry an execution past the Zeno point by switching to a holonomically constrained dynamical system. Although this has been studied to a limited degree in [4], the result presented in this paper can be used to complete hybrid systems in a formal manner, which is the subject of our future work [15].

Finally, the paper analyzes stability only for *simple* Lagrangian hybrid systems, i.e. systems with a single domain and a single guard. The extension to mechanical systems with multiple unilateral constraints is still a challenging open problem, although preliminary results for stability of a specific two-constraint mechanical system were obtained in [17]. This extension, along with the completion process described above, will enable the analysis of complex mechanical and robotic systems with intermittent contacts, such as bipedal walkers with knees (e.g. [18] and [13]), under a unified framework of Lagrangian hybrid systems.

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