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Consensus of Multiple Nonholonomic Systems

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Abstract— **This paper considers the consensus of multiple nonholonomic systems such that a group of systems converges to a desired trajectory. Cooperative control laws are proposed and analyzed with the aid of results from graph theory and Lyapunov analysis. The proposed control laws are decentralized. Robustness of the proposed control laws to communication delays is also considered. As an application of the proposed results, formation control of wheeled mobile robots is discussed. Simulation results show effectiveness of the proposed results.**

I. Introduction

A consensus problem is one in which a set of entities converges to a common value. The consensus problem is closely related to the coordination of multiple systems. Many research works in the consensus problem are motivated by Viscek's model [1] which is a special case of a distributed behavioral model proposed in [2]. The authors of [3] gave a theoretical explanation for consensus of the heading angles of a group of agents using nearest neighbor rules under undirected switching information exchange topologies. The stability of the consensus algorithms were analyzed with the aid of results from graph theory. It is shown that consensus is achieved asymptotically if the union of the information exchange graphs for the team is connected most of the time as the system evolves. In [4], the consensus algorithms were extended to the case where the information exchange graphs were directed. In [5], the authors considered the problem of information consensus among multiple agents in the presence of limited and unreliable information exchange with dynamically switching topologies. Algorithms were proposed to achieve information consensus between agents in both the discrete and continuous cases. The consensus problem for networks of dynamic agents with fixed and switching topologies was discussed in [6]. Consensus protocols for networks with or without time-delays were proposed in different communication scenarios. In contrast to the algebraic graph approach in [3, 5], nonlinear tools are applied to study consensus problems in [7]. A set-value function was used to study the stability of the consensus algorithms.

Most consensus results in the literature are obtained for linear agents in low dimensions. In [8], the consensus problem was considered for multiple chained systems where each system has three dimension. Consensus is achieved but the consensus value cannot be assigned in advance. In [9], consensus of multiple chained form systems was considered where each system has dimension n (≥ 3). Consensus is achieved and the consensus value can be assigned in advance. In this paper we also consider the consensus of multiple chained form systems. New cooperative control laws are proposed. Since communication delays are inevitable between neighboring systems, we analyze the effects of time-delays on the proposed cooperative control laws. It is shown that our proposed cooperative control laws are robust to constant communication delays under suitable assumptions on the communication graph. As an application of the proposed results, we show that formation control of multiple mobile robots can be solved by our proposed results. To verify effectiveness of the proposed cooperative control laws, simulation results are included for this application. Compared with the results in [9], the proposed control laws in this paper have larger stability margins with respect to communication delays.

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II. Problem Statement

Consider m nonholonomic systems, indexed by j ($1 \leq j \leq m$), the kinematics of the j-th system is the so-called chained form (see [10])

$$\dot{q}_{1j} = u_{1j}, \quad \dot{q}_{2j} = u_{2j}, \quad \dot{q}_{ij} = q_{i-1,j}u_{1j}, \quad 3 \leq i \leq n \quad (1)$$

where $q_{*j} = [q_{1j}, \ldots, q_{nj}]^\top$ and $u_{*j} = [u_{1j}, u_{2j}]^\top$ are the state and input of the j-th system, respectively. The communication between the systems can be described by the edges $\mathcal{E}$ of the graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ where the m systems are represented by the m nodes in $\mathcal{V}$. The existence of an edge $(l, j) \in \mathcal{E}$ means that the state q_{*l} of system l is available to system j for control and vice versa. The symbol $\mathcal{N}_j$ denotes the neighbors of node j and is the set of indices of systems whose state is available to system j. The information available to system j for the controller design is the $j-$th system's own state, the desired trajectory, and the state of each system l such that $l \in \mathcal{N}_j$. Due to sensor range limitations and bounded communication bandwidth between systems, $\mathcal{N}_j$ may change with time, which means that the edge set $\mathcal{E}$ may be time-varying and consequently the Laplacian matrix L corresponding to $\mathcal{G}$ will be time-varying. In this paper, we only discuss the fixed communication case.

The problem discussed in this paper is defined as follows.

Consensus Problem: Given a desired trajectory $q^d = [q_1^d, \ldots, q_n^d]^\top$ which satisfies

$$\dot{q}_1^d = w_1, \quad \dot{q}_2^d = w_2, \quad \dot{q}_i^d = q_{i-1}^d w_1, \quad 3 \leq i \leq n \quad (2)$$

where w_1 and w_2 are known time-varying functions, design a decentralized control law u_{*j} for system j using q_{*j}, q^d,

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and q_{*l} for $l \in \mathcal{N}_j$ such that

$$\lim_{t\to\infty}(q_{*j} - q_{*l}) = 0, \quad 1 \le j, l \le m. \tag{3}$$

$$\lim_{t\to\infty}\left(\frac{1}{m}\sum_{j=1}^{m} q_{*j} - q^d\right) = 0. \tag{4}$$

Remark 1: In the consensus problem, eqn. (3) means that consensus is achieved for m systems. Eqn. (4) means that the consensus value tracks the desired trajectory.

III. Controller Design

To facilitate the controller design, the following assumptions are made on the desired trajectory q^d in eqn. (2).

Assumption 1: Trajectories q_i^d for $2 \le i \le n-1$ are bounded.

Assumption 2: The integral $\int_t^{t+T} w_1^{2n-4}(\tau)d\tau \ge \delta$ for some $T, \delta > 0$ and all $t \ge 0$.

To facilitate the controller design, we introduce the following variables:

$$z_{ij} = q_{ij} - q_i^d - \phi_{ij}, \quad 1 \le i \le n, \ 1 \le j \le m \tag{5}$$

where

$$\begin{cases} \phi_{1j} = 0, \quad \phi_{nj} = 0, \quad \phi_{n-1,j} = -\alpha_n z_{nj} w_1^{2n-5} \\ \phi_{n-2,j} = -\alpha_{n-1} z_{n-1,j} w_1^{2n-5} - z_{nj} + \dfrac{\partial \phi_{n-1,j}}{\partial w_1}\dfrac{\dot{w}_1}{w_1} \\ \quad + \dfrac{\partial \phi_{n-1,j}}{\partial z_{nj}}\left(-\alpha_n w_1^{2n-5} z_{nj} + z_{n-1,j}\right) \\ \phi_{ij} = -\alpha_{i+1} z_{i+1,j} w_1^{2n-5} - z_{i+2,j} + \displaystyle\sum_{l=0}^{n-i-2} \dfrac{\partial \phi_{i+1,j}}{\partial w_1^{[l]}}\dfrac{w_1^{[l+1]}}{w_1} \\ \quad + \dfrac{\partial \phi_{i+1,j}}{\partial z_{nj}}\left(-\alpha_n w_1^{2n-5} z_{nj} + z_{n-1,j}\right) \\ \quad + \displaystyle\sum_{l=i+2}^{n-1} \dfrac{\partial \phi_{i+1,j}}{\partial z_{lj}}\left(-\alpha_l w_1^{2n-5} z_{lj} - z_{l+1,j} + z_{l-1,j}\right), \\ \quad i = n-3, \ldots, 2; \end{cases} \tag{6}$$

the constants $\alpha_i > 0$ for $3 \le i \le n$, and the notation $w_1^{[i]}$ indicates the i-th derivative of w_1 (i.e. $w_1^{[i]} = \frac{d^i w_1}{dt^i}$). These definitions yield the following dynamic equations,

$$\begin{cases} \dot{z}_{1j} = u_{1j} - w_1 \\ \dot{z}_{2j} = u_{2j} - w_2 - \displaystyle\sum_{l=0}^{n-3} \dfrac{\partial \phi_{2j}}{\partial w_1^{[l]}} w_1^{[l+1]} - \dfrac{\partial \phi_{2j}}{\partial z_{nj}}(w_1 z_{n-1,j} \\ \quad - \alpha_n w_1^{2n-4} z_{nj}) - \displaystyle\sum_{l=3}^{n-1} \dfrac{\partial \phi_{2j}}{\partial z_{lj}}(-\alpha_l w_1^{2n-4} z_{lj} \\ \quad - w_1 z_{l+1,j} + w_1 z_{l-1,j}) - (u_{1j} - w_1)\displaystyle\sum_{l=3}^{n} \dfrac{\partial \phi_{2j}}{\partial z_{lj}} e_{lj} \\ \dot{z}_{3j} = -\alpha_3 z_{3j} w_1^{2n-4} - z_{4j} w_1 + z_{2j} w_1 + (u_{1j} - w_1) e_{3j} \\ \quad \vdots \\ \dot{z}_{n-1,j} = -\alpha_{n-1} z_{n-1,j} w_1^{2n-4} - z_{nj} w_1 + z_{n-2,j} w_1 \\ \quad + (u_{1j} - w_1) e_{n-1,j} \\ \dot{z}_{nj} = -\alpha_n w_1^{2n-4} z_{nj} + w_1 z_{n-1,j} + (u_{1j} - w_1) e_{nj} \end{cases} \tag{7}$$

where for $1 \le j \le m$

$$e_{nj} = q_{n-1,j}, \ e_{ij} = q_{i-1,j} - \sum_{l=i+1}^{n} \frac{\partial \phi_{ij}}{\partial z_{lj}} e_{lj} \tag{8}$$

for $i = n-1, \ldots, 2$. Eqn. (7) is obtained by the backstepping procedure in [9].

Lemma 1: For the variables defined in eqn. (5), under Assumption 1, if $\lim_{t\to\infty}(z_{kj} - c_k) = 0$ for $1 \le k \le n$ and $1 \le j \le m$, then $\lim_{t\to\infty}(q_{kl} - q_{kj}) = 0$ for $1 \le k \le n$, $1 \le l \ne j \le m$. Furthermore, if

$$\lim_{t\to\infty} z_{kj} = 0, \tag{9}$$

then $\lim_{t\to\infty}(q_{kj} - q_{kl}) = 0$ and $\lim_{t\to\infty}(q_{kj} - q_k^d) = 0$ where c_k are constants or bounded time-varying functions.

Proof: Noting the definition of ϕ_{ij}, $\phi_{*j} = \Lambda(t) z_{*j}$ where $\phi_{*j} = [\phi_{1j}, \ldots, \phi_{nj}]^\top$, $z_{*j} = [z_{1j}, \ldots, z_{nj}]^\top$, and $\Lambda(t)$ is a bounded matrix function of $w_1^{[i]}(t)$ for $i \in [0, n-3]$. So, $\lim_{t\to\infty}(q_{*j} - q_{*l}) = \lim_{t\to\infty}(z_{*j} - z_{*l} + \phi_{*j} - \phi_{*l}) = \lim_{t\to\infty}(I - \Lambda(t))(z_{*j} - z_{*l})$. If $w_1^{[i]}(t)$ ($i \in [0, n-3]$) are bounded, then $\Lambda(t)$ is bounded. Therefore, $\lim_{t\to\infty}(q_{*j} - q_{*l}) = 0$ if $\lim_{t\to\infty}(z_{kj} - c_k) = 0$. Furthermore, if $\lim_{t\to\infty} z_{*j} = 0$, we have $\lim_{t\to\infty} \phi_{*j} = 0$. So, $\lim_{t\to\infty}(q_{kj} - q_{kl}) = 0$ and $\lim_{t\to\infty}(q_{kj} - q_k^d) = 0$. ■

By Lemma 1, it is possible to solve the stated problem by designing control laws such that eqn. (9) holds.

Given a symmetric $m \times m$ constant matrix $B = [b_{ji}]$ with $b_{ji} > 0$, let $\mathcal{G}$ be the communication graph among m systems, the Laplacian matrix $L = [L_{ji}]$ of the graph $\mathcal{G}$ with weight matrix B on the communication links is defined as follows,

$$L_{ji} = \begin{cases} -b_{ji}, & \text{if } j \ne i, \quad i \in \mathcal{N}_j \\ \displaystyle\sum_{\substack{l=1 \\ l \ne j}}^{m} b_{jl}, & \text{if } j = i \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, L is a symmetric matrix and has real eigenvalues. Without loss of generality, we assume its eigenvalues $\lambda_l(L)$ ($1 \le l \le m$) satisfy $\lambda_1(L) \le \lambda_2(L) \le \cdots \le \lambda_m(L)$.

Lemma 2: Given a symmetric constant matrix $B = [b_{ji}]$ with $b_{ji} > 0$, if the communication graph $\mathcal{G}$ is fixed and strongly connected, the eigenvalues $\lambda_l(L)$ ($1 \le l \le m$) of the Laplacian matrix L of the graph $\mathcal{G}$ with the weight matrix B satisfy $\lambda_m(L) \ge \lambda_{m-1}(L) \ge \cdots \ge \lambda_2(L) > \lambda_1(L) = 0$. Furthermore, for any bounded function vector $\xi(t) \in R^m$, if $\lim_{t\to\infty} \xi^\top(t) L \xi(t) = 0$, then

$$\lim_{t\to\infty}\left(\xi(t) - \left(\sum_{l=1}^{m}\frac{\xi_l(t)}{m}\right)\mathbf{1}\right) = \mathbf{0}$$

where $\mathbf{1} = [1, \ldots, 1]^\top$ and $\mathbf{0} = [0, \ldots, 0]^\top$.

Proof: Noting the definition of L, by the Gerschgorin Circle Theorem, each $\lambda_i(L)$ is contained in the union of the m Gerschgorin circles $|z - L_{jj}| \le L_{jj}$ for $1 \le j \le m$. Therefore, either $\lambda_j(L) > 0$ or $\lambda_j(L) = 0$ for $1 \le j \le m$. Since $\mathcal{G}$ is strongly connected, there is exactly one zero eigenvalue [6], i.e., $\lambda_1 = 0$ and $\lambda_m \ge \cdots \ge \lambda_3 \ge \lambda_2 > 0$.

Since L is symmetric and $\lambda_1 = 0$, there exists an orthogonal matrix $Q = [Q_{ij}]$ with its first column being $\mathbf{1}/\sqrt{m}$ such that $Q^\top LQ = \mathrm{diag}[0, \lambda_2, \ldots, \lambda_m]$. So,

$$\lim_{t\to\infty} \xi^\top L\xi = \lim_{t\to\infty} (Q^\top \xi)^\top \mathrm{diag}[0, \lambda_2, \ldots, \lambda_m](Q^\top \xi) = 0.$$

Let $y = [y_1, y_2, \ldots, y_m]^\top = Q^\top \xi$, then $\lim_{t\to\infty} y_i = 0$ for $2 \le i \le m$. Noting $y_1 = \frac{1}{\sqrt{m}} \sum_{l=1}^m \xi_l$ and

$$\lim_{t\to\infty} \left(\xi - \frac{1}{m}\sum_{l=1}^m \xi_l \mathbf{1}\right) = \lim_{t\to\infty} \left(Qy - \frac{1}{\sqrt{m}} y_1 \mathbf{1}\right)$$
$$= \lim_{t\to\infty} \left[\sum_{l=2}^m Q_{1l} y_l, \cdots, \sum_{l=2}^m Q_{ml} y_l\right]^\top = [0, \cdots, 0]^\top . \quad (10)$$

Therefore, the lemma is proved. ■

Noting the special structure of (7), we have the following results.

Theorem 1: Consider the system (1), under Assumptions 1 and 2, if the communication graph is fixed and is strongly connected, controllers

$$u_{1j} = -\sum_{i\in\mathcal{N}_j} b_{ji}(z_{1j} - z_{1i} + \Delta_j - \Delta_i) + w_1 \quad (11)$$

$$\begin{aligned} u_{2j} = & -\sum_{i\in\mathcal{N}_j} b_{ji}(z_{2j} - z_{2i}) + w_2 + \sum_{l=0}^{n-3} \frac{\partial \phi_{2j}}{\partial w_1^{[l]}} w_1^{[l+1]} \\ & + \frac{\partial \phi_{2j}}{\partial z_{nj}} \left(-\alpha_n w_1^{2n-4} z_{nj} + w_1 z_{n-1,j}\right) \\ & + \sum_{l=3}^{n-1} \frac{\partial \phi_{2j}}{\partial z_{lj}} (w_1 z_{l-1,j} - \alpha_l w_1^{2n-4} z_{lj} \\ & - w_1 z_{l+1,j}) \end{aligned} \quad (12)$$

make (3) hold, where constant $b_{ji} = b_{ij} > 0$, $\alpha_i > 0$, and

$$\Delta_j(t) = -z_{2j}(t) \sum_{l=3}^n \frac{\partial \phi_{2j}(t)}{\partial z_{lj}(t)} e_{lj}(t) + \sum_{l=3}^n z_{lj}(t) e_{lj}(t). \quad (13)$$

Proof: Apply the control laws (11)-(12) to (7), we have

$$\begin{cases} \dot z_{1j} = u_{1j} - w_1 \\ \dot z_{2j} = -\sum_{i\in\mathcal{N}_j} b_{ji}(z_{2j} - z_{2i}) - (u_{1j} - w_1) \sum_{l=3}^n \frac{\partial \phi_{2j}}{\partial z_{lj}} e_{lj} \\ \dot z_{3j} = -\alpha_3 z_{3j} w_1^{2n-4} - z_{4j} w_1 + z_{2j} w_1 + (u_{1j} - w_1) e_{3j} \\ \quad \vdots \\ \dot z_{n-1,j} = -\alpha_{n-1} z_{n-1,j} w_1^{2n-4} - z_{nj} w_1 + z_{n-2,j} w_1 \\ \qquad + (u_{1j} - w_1) e_{n-1,j} \\ \dot z_{nj} = -\alpha_n w_1^{2n-4} z_{nj} + w_1 z_{n-1,j} + (u_{1j} - w_1) e_{nj} \end{cases} \quad (14)$$

where $u_{1j} - w_1 = -\sum_{i\in\mathcal{N}_j} b_{ji}(z_{1j} - z_{1i} + \Delta_j - \Delta_i)$. Define the positive definite Lyapunov function

$$V = \frac{1}{2} \sum_{j=1}^m \sum_{i=1}^n z_{ij}^2. \quad (15)$$

Differentiating it along (14), we have

$$\begin{aligned} \dot V = & -\sum_{j=1}^m \sum_{i=3}^n \alpha_i w_1^{2n-4} z_{ij}^2 - z_{2*}^\top L z_{2*} \\ & -(z_{1*} + \Delta)^\top L (z_{1*} + \Delta) \le 0 \end{aligned}$$

where $z_{2*} = [z_{21}, \ldots, z_{2m}]^\top$, $z_{1*} = [z_{11}, \ldots, z_{1m}]^\top$, $\Delta = [\Delta_1, \ldots, \Delta_m]^\top$. Therefore, V is bounded. Furthermore, z_{ij} is bounded. By Barbalat's Lemma [11], $\lim_{t\to\infty} \dot V = 0$ and

$$\lim_{t\to\infty} \alpha_i w_1^{2n-4} z_{ij}^2 = 0, \quad 3 \le i \le n \quad (16)$$
$$\lim_{t\to\infty} z_{2*}^\top L z_{2*} = 0, \quad (17)$$
$$\lim_{t\to\infty} (z_{1*} + \Delta)^\top L (z_{1*} + \Delta) = 0. \quad (18)$$

By Assumption 2, we see $\lim_{t\to\infty} z_{ij} = 0$ for $3 \le i \le n$ and $1 \le j \le m$. By Lemma 2, we have $\lim_{t\to\infty}(z_{2*}(t) - c_2(t)\mathbf{1}) = 0$ and $\lim_{t\to\infty}(z_{1*}(t) + \Delta(t) - c_1(t)\mathbf{1}) = 0$ where c_1 and c_2 are bounded and are defined as $c_2 = \frac{1}{m}\sum_{l=1}^m z_{2l}$ and $c_1 = \frac{1}{m}\sum_{l=1}^m (z_{1l} + \Delta_l)$. By the definitions of ϕ_{ij}, e_{ij}, and Δ_l, it can be proved that $\lim_{t\to\infty}(z_{1j} - z_{1l}) = 0$ and $\lim_{t\to\infty}(z_{2j} - z_{2l}) = 0$ for $1 \le j, l \le m$. By Lemma 1, eqn. (3) holds. ■

Remark 2: In (11)-(12), the first term is a weighted sum of the relative state information between system j and its neighbors, the second term is the desired input, and the other terms are used to cancel the terms induced by the variable transform. The motion of the system is driven by the relative information between neighbors (see (14)), which distinguishes the cooperative control laws (11)-(12) from the tracking control laws for a single nonholonomic system in the literature [12, 13].

Remark 3: In the control laws, the control parameters are b_{jl} and α_i. Generally, increasing α_i will increase the convergence rate of z_{ij} for $3 \le i \le n$. The value of b_{ji} and the topology of the communication graph determines $\lambda_2(L)$. The parameter $\lambda_2(\mathcal{G})$ is known as the algebraic connectivity [14]. Increasing $\lambda_2(L)$ causes $(z_{1j} - z_{1i})$ and $(z_{2j} - z_{2i})$ to converge to zero faster. The values $\lambda_2(L)$ and α_i affect the convergence rate of q_{*j} to q^d. The rate at which $(z_{*j} - z_{*i})$ converges to zero is called the *cohesion rate*. Generally, a dense interconnection of $\mathcal{G}$ means a larger value of $\lambda_2(L)$. Therefore, more interconnections facilitates the cooperative performance. However, increasing the number of interconnections does not necessarily imply a larger value of $\lambda_2(L)$. Under the same topology of the communication graph $\mathcal{G}$, different weights b_{jl} may lead to different $\lambda_2(L)$.

Remark 4: The control laws (11)-(12) are decentralized because for each system its controller depends only on its own state, its neighbors' states, and the desired trajectory.

The controllers in Theorem 1 cannot make (4) hold. To make (4) hold, we introduce damping terms in the controllers and have the following result.

Theorem 2: Consider the system (1), under Assumptions 1-2, if the communication graph is fixed and is strongly connected, the controllers

$$
\begin{aligned}
u_{1j} &= -\sum_{i\in\mathcal{N}_j} b_{ji}(z_{1j}-z_{1i}+\Delta_j-\Delta_i) \\
&\quad -\mu_j(z_{1j}+\Delta_j)+w_1 \qquad (19)\\
u_{2j} &= -\sum_{i\in\mathcal{N}_j} b_{ji}(z_{2j}-z_{2i})-\mu_j z_{2j}+w_2 \\
&\quad +\sum_{l=0}^{n-3}\frac{\partial\phi_{2j}}{\partial w_1^{[l]}}w_1^{[l+1]}+\frac{\partial\phi_{2j}}{\partial z_{nj}}(w_1 z_{n-1,j} \\
&\quad -\alpha_n w_1^{2n-4}z_{nj})+\sum_{l=3}^{n-1}\frac{\partial\phi_{2j}}{\partial z_{lj}}(-\alpha_l w_1^{2n-4}z_{lj} \\
&\quad -w_1 z_{l+1,j}+w_1 z_{l-1,j}) \qquad (20)
\end{aligned}
$$

make (3)-(4) hold, where constant $b_{ji}=b_{ij}>0$, $\alpha_i>0$, $\mu_j\geq 0$ and $\sum_{l=1}^{m}\mu_l>0$, and Δ_j is defined in (13).

Proof: Let the positive definite Lyapunov function V be defined as in (15). Differentiate it along the solution of the closed-loop system (7) with the control laws (11)-(12), we have

$$
\begin{aligned}
\dot V = &-\sum_{j=1}^{m}\sum_{i=3}^{n}\alpha_i w_1^{2n-4}z_{ij}^2-\sum_{j=1}^{m}\mu_j[z_{2j}^2+(z_{1j}+\Delta_j)^2] \\
&-z_{2*}^\top L z_{2*}-(z_{1*}+\Delta)^\top L(z_{1*}+\Delta)\leq 0 \qquad (21)
\end{aligned}
$$

Therefore, V is bounded. Furthermore, z_{ij} are bounded. By Barbalat's Lemma [11], $\lim_{t\to\infty}\dot V=0$ and

$$\lim_{t\to\infty} w_1^{2n-4}z_{ij}^2=0,\quad 3\leq i\leq n \qquad (22)$$

$$\lim_{t\to\infty} z_{2*}^\top L z_{2*}=0, \qquad (23)$$

$$\lim_{t\to\infty}(z_{1*}+\Delta)^\top L(z_{1*}+\Delta)=0, \qquad (24)$$

$$\lim_{t\to\infty}\sum_{j=1}^{m}\mu_j z_{2j}^2=0,\ \lim_{t\to\infty}\sum_{j=1}^{m}\mu_j(z_{1j}+\Delta_j)^2=0 \qquad (25)$$

By Assumption 2, $\lim_{t\to\infty}z_{ij}=0$ for $3\leq i\leq n$ and $1\leq j\leq m$. By Lemma 2, $\lim_{t\to\infty}(z_{2*}(t)-c_2(t)\mathbf{1})=0$ and $\lim_{t\to\infty}(z_{1*}(t)+\Delta(t)-c_1(t)\mathbf{1})=0$ where c_1 and c_2 are bounded and are defined as $c_2=\frac{1}{m}\sum_{l=1}^{m}z_{2l}$, $c_1=\frac{1}{m}\sum_{l=1}^{m}(z_{1l}+\Delta_l)$. Since at least one of μ_j is greater than zero, say $\mu_p>0$, then by (25) we have $\lim_{t\to\infty}z_{2p}=0$ and $\lim_{t\to\infty}(z_{1p}+\Delta_p)=0$. Noting z_{2j} converges to c_2, we see $c_2=0$. Since $(z_{1j}+\Delta_j)$ converges to c_1, we see $c_1=0$. Noting the definitions of ϕ_{ij} and Δ_j, we can prove that z_{1*} converges to zero with the aid of Assumption 1. By Lemma 1, eqns. (3)-(4) hold. ■

Remark 5: Terms $\mu_j(z_{1j}-\Delta_j)$ and $\mu_j z_{2j}$ in (19)-(20) are called the damping terms which are used to make $\mu_j(z_{1j}-\Delta_j)$ and $\mu_j z_{2j}$ converge to zero. Large μ_j means that z_{1j} and z_{2j} converge to zero fast. The rate at which z_{ij} converges to zero is called the *tracking rate*. It can be adjusted by the control parameters μ_j and α_i. Therefore, we can adjust the cohesion rate and the tracking rate by choosing suitable control parameters. In Theorem 1, we can change the cohesion rate. While in Theorem 2, we can change both the cohesion rate and the tracking rate.

Remark 6: If the communication graph $\mathcal{G}$ is strongly connected, z_{1*} and z_{2*} converge to zero if one of μ_j is greater than zero. In fact, if $\mathcal{G}$ is not strongly connected, we can make z_{1*} and z_{2*} converge to zero by letting some μ_j be positive. In the worst case, if there is no communication between any two systems, we can make z_{1*} and z_{2*} converge to zero by choosing $\mu_j>0$ for all j. Actually, in this case, there is not any cooperation between the systems. The control law for each system degenerates into the control law proposed for a single nonholonomic system as in [12, 13].

IV. Closed-loop System Stability With Communication Delays

Next, we consider the effects of communication delays on the proposed results in the last section. For simplicity, in this paper we assume that communication delays only appear in the neighbors' states and are constant. Corresponding to Theorem 2, we have the following result.

Theorem 3: Consider system (1), under Assumptions 1-2, if the communication graph is fixed and is strongly connected, the controllers

$$
\begin{aligned}
u_{1j}(t) &= w_1(t)-\mu_j(z_{1j}(t)+\Delta_j(t))-\sum_{i\in\mathcal{N}_j}b_{ji}(z_{1j}(t) \\
&\quad -z_{1i}(t-\delta_i)+\Delta_j(t)-\Delta_i(t-\delta_i)) \qquad (26)\\
u_{2j}(t) &= -\mu_j z_{2j(t)}-\sum_{i\in\mathcal{N}_j}b_{ji}(z_{2j}(t)-z_{2i}(t-\delta_i)) \\
&\quad +w_2(t)+\sum_{l=0}^{n-3}\frac{\partial\phi_{2j}(t)}{\partial w_1^{[l]}}(t)w_1^{[l+1]}(t) \\
&\quad +\frac{\partial\phi_{2j}(t)}{\partial z_{nj}(t)}(w_1(t)z_{n-1,j}(t) \\
&\quad -\alpha_n w_1^{2n-4}(t)z_{nj}(t)) \\
&\quad +\sum_{l=3}^{n-1}\frac{\partial\phi_{2j}(t)}{\partial z_{lj}(t)}(-\alpha_l w_1^{2n-4}(t)z_{lj}(t) \\
&\quad -w_1(t)z_{l+1,j}(t)+w_1(t)z_{l-1,j}(t)) \qquad (27)
\end{aligned}
$$

make (3)-(4) hold, where constant $b_{ji}=b_{ij}>0$, $\alpha_i>0$, $\mu_j\geq 0$ and $\sum_{l=1}^{m}\mu_l>0$, $\Delta_j(t)$ is defined in (13), and the constants $\delta_j>0$.

Proof: Define the nonnegative function

$$
\begin{aligned}
V(t) &= \frac{1}{2}\sum_{j=1}^{m}[\sum_{i=1}^{n}z_{ij}^2(t)+\sum_{i\in\mathcal{N}_j}\int_{t-\delta_i}^{t}b_{ji}((z_{1i}(s) \\
&\quad +\Delta_i(s))^2+z_{2i}^2(s))ds]. \qquad (28)
\end{aligned}
$$

Differentiating (28) along the solution of (7) with the control laws (11)-(12), we have

$$
\begin{aligned}
\dot V &= -\sum_{j=1}^{m}\sum_{i=3}^{n}\alpha_i z_{ij}^2 w_1^{2n-4}-\sum_{j=1}^{m}\mu_j(z_{1j}(t)+\Delta_j(t))^2 \\
&\quad -\sum_{j=1}^{m}\mu_j z_{2j}^2(t)-\frac{1}{2}\sum_{j=1}^{m}\sum_{i\in\mathcal{N}_j}b_{ji}\left[(\overline{\Delta}_j(t)\right. \\
&\quad \left.-\overline{\Delta}_i(t-\delta_i))^2+(z_{2j}(t)-z_{2i}(t-\delta_i))^2\right] \qquad (29)
\end{aligned}
$$

where we have used the fact that the communication graph $\mathcal{G}$ is bidirectional, and $\overline{\Delta}(t) = [\overline{\Delta}_1(t), \ldots, \overline{\Delta}_m(t)]^\top = [z_{11}(t) + \Delta_1(t), \ldots, z_{1m}(t) + \Delta_m(t)]^\top$. Therefore, V is bounded. Furthermore, z_{ij} are bounded for $1 \le i \le n$ and $1 \le j \le m$. By Barbalat's Lemma [11], we have

$$\lim_{t\to\infty} z_{lj} w_1^{n-2} = 0, \quad (3 \le l \le n), \tag{30}$$

$$\lim_{t\to\infty} \sum_{j=1}^{m} \mu_j \overline{\Delta}_j^2(t) = 0, \qquad \lim_{t\to\infty} \sum_{j=1}^{m} \mu_j z_{2j}^2 = 0, \tag{31}$$

$$\lim_{t\to\infty} (\overline{\Delta}_j(t) - \overline{\Delta}_i(t - \delta_i)) = 0, i \in \mathcal{N}_j, 1 \le j \le m \tag{32}$$

$$\lim_{t\to\infty} (z_{2j}(t) - z_{2i}(t - \delta_i)) = 0, i \in \mathcal{N}_j, 1 \le j \le m. \tag{33}$$

By Assumption 2 and (30), we see $\lim_{t\to\infty} z_{ij} = 0$ for $(3 \le i \le n)$. Since at least one of μ_j, say μ_p, is greater than zero, we see from (31) that $\lim_{t\to\infty} \overline{\Delta}p = 0$ and $\lim_{t\to\infty} z_{2p} = 0$. Since the graph $\mathcal{G}$ is strongly connected, from (33) we can prove that $\lim_{t\to\infty} \overline{\Delta}_j = 0$ and $\lim_{t\to\infty} z_{2j} = 0$ $(1 \le j \le m)$. By the definition of ϕ_{ij}, we can prove that $\lim_{t\to\infty} z_{1j} = 0$ for $1 \le j \le m$. By Lemma 1, eqns. (3)-(4) hold. ■

Remark 7: In [9], the cooperative controllers for system (1) were proposed with the aid of graph theory. However, the robust stability margin of the controllers in [9] with respect to communication delays is smaller that of the controllers in this paper.

V. Applications

Consider a set of m wheeled mobile robots which move on a plane. Throughout this section, without loss of generality, the mobile robots will be indexed by $1 \le j \le m$. The kinematics of each robot are as follows [15]:

$$\dot{x}_j = v_{1j} \cos\theta_j, \quad \dot{y}_j = v_{1j} \sin\theta_j, \quad \dot{\theta}_j = v_{2j}, \tag{34}$$

where (x_j, y_j) are the coordinates of the center point of the front wheels of robot j in the fixed coordinate frame O-XY, θ_j is the orientation of robot j with respect to the X-axis of the coordinate frame O-XY, v_{1j} and v_{2j} are the speed and angular rate of robot j, respectively.

Assume that the communication graph among the m robots is $\mathcal{G}$. The desired formation $\mathcal{F}$ is described by constant centroid offset vectors (p_{jx}, p_{jy}) and the desired trajectory (x^d, y^d, θ^d) is generated by

$$\dot{x}^d = v_1^d \cos\theta^d, \quad \dot{y}^d = v_1^d \sin\theta^d, \quad \dot{\theta}^d = v_2^d. \tag{35}$$

We consider the following problem.

Formation Control with a Desired Trajectory: Design control laws for each robot, based on its own state, the relative state information between its neighbors, and the desired trajectory, such that the group of robots come into formation $\mathcal{F}$ and move along the desired trajectory, i.e., design control laws for system (34) such that

$$\lim_{t\to\infty} \left[\begin{bmatrix} x_l - x_j \\ y_l - y_j \end{bmatrix} - \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} p_{lx} - p_{jx} \\ p_{ly} - p_{jy} \end{bmatrix} \right] = 0, \tag{36}$$

$$\lim_{t\to\infty} \left(\sum_{i=1}^{m} \frac{x_i}{m} - x^d \right) = 0, \lim_{t\to\infty} \left(\sum_{i=1}^{m} \frac{y_i}{m} - y^d \right) = 0 \tag{37}$$

for $1 \le l \ne j \le m$, where ψ is a variable.

Next, we show how this control problem can be solved by the results proposed in the previous sections. Let

$$\begin{cases} q_{1j} = -\theta_j, \\ q_{2j} = \left(x_j - p_{jx} + \dfrac{1}{m} \displaystyle\sum_{i=1}^{m} p_{ix} \right) \cos\theta_j \\ \qquad + \left(y_j - p_{jy} + \dfrac{1}{m} \displaystyle\sum_{i=1}^{m} p_{iy} \right) \sin\theta_j \\ q_{3j} = -\left(x_j - p_{jx} + \dfrac{1}{m} \displaystyle\sum_{i=1}^{m} p_{ix} \right) \sin\theta_j \\ \qquad + \left(y_j - p_{jy} + \dfrac{1}{m} \displaystyle\sum_{i=1}^{m} p_{iy} \right) \cos\theta_j \end{cases} \tag{38}$$

and

$$u_{1j} = -v_{2j}, \qquad u_{2j} = v_{1j} + q_{3j} v_{2j} \tag{39}$$

we have eqn. (1) with $n = 3$. Letting

$$\begin{cases} q_1^d = -\theta^d, \quad q_2^d = x^d \cos\theta^d + y^d \sin\theta^d, \\ q_3^d = -x^d \sin\theta^d + y^d \cos\theta^d, \quad w_1 = -v_2^d, \\ w_2 = (-x^d \sin\theta^d + y^d \cos\theta^d) v_2^d - v_1^d \end{cases} \tag{40}$$

we have eqn. (2) with $n = 3$. Simple calculation derives the following result.

Lemma 3: By the transformations in eqns. (39), (38), and (40), under Assumptions 1-2 with $n = 3$, if $\lim_{t\to\infty}(q_{ij} - q_i^d) = 0$ for $1 \le i \le 3$ and $1 \le j \le m$, then eqns. (36)-(37) are satisfied.

By Lemma 3, the formation control problem with a desired trajectory can be solved by the controllers proposed in the previous sections.

VI. Simulations

To verify the effectiveness of the proposed results, we present some simulation results.

Consider the application discussed in Section V. Let $m = 5$ and the initial conditions (x, y, θ) of the five robots be $(11.5, -36, -0.2)$, $(-10.3, -45.9, 0.3)$, $(3.8, -42.3, 0.2)$, $(35.3, 16.7, 2)$, and $(-32.7, 17.9, -2)$. Assume the desired formation $\mathcal{P}$ is defined by $(p_{1x}, p_{1y}) = (1.24, 3.8)$, $(p_{2x}, p_{2y}) = (-3.24, 2.35)$, $(p_{3x}, p_{3y}) = (-3.24, -2.35)$, $(p_{4x}, p_{4y}) = (1.24, -3.8)$, and $(p_{5x}, p_{5y}) = (4, 0)$ (see Fig. 1). The desired trajectory is $(x^d, y^d, \theta^d) = (40 \sin\frac{t}{2}, -40 \cos\frac{t}{2}, \frac{t}{2})$. The desired trajectory satisfies Assumptions 1-2. By Lemma 3, the controllers can be obtained with the aid of the results in Sections III and IV. Assume the communication graph $\mathcal{G}$ is shown in Fig. 2. The cooperative controllers can be obtained by Theorem 2. In the simulation, we choose the control parameters $b_{ji} = 2$ and $\alpha_3 = 10$. Fig. 3 shows the path of the centroid of the five robots versus time and the geometric patterns of the five robots at several times. The five robots come into the desired formation and the centroid of the group of robots converges to the desired trajectory for the case of no communication delay. If there are constant communication delays in the control, according to Theorem 3, the control laws achieve the same objectives. Fig. 4 shows the path of the centroid of the five robots and

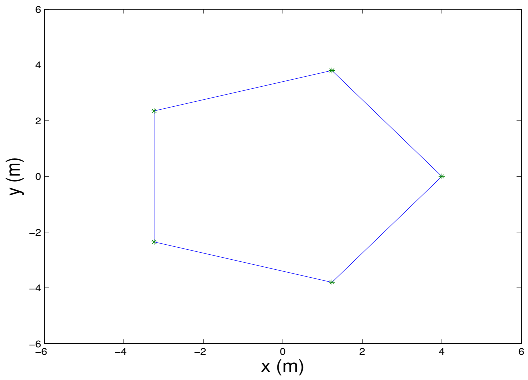

Fig. 1. Desired geometric pattern.

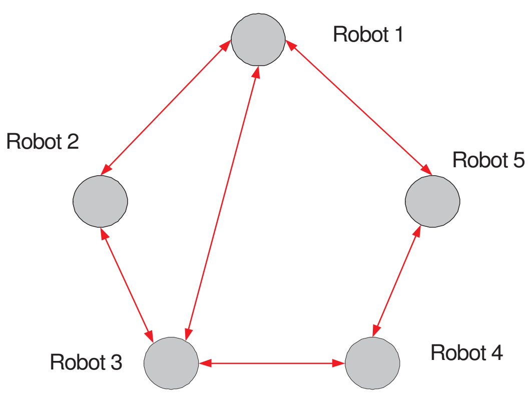

Fig. 2. Communication graph $\mathcal{G}$

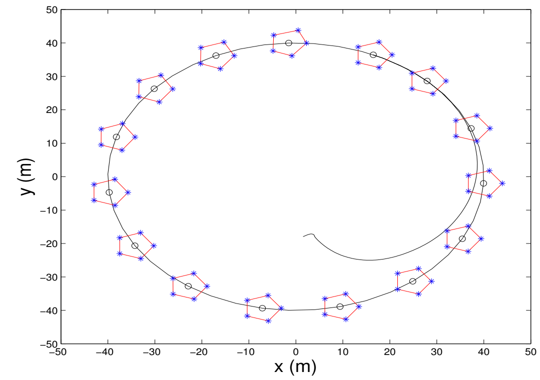

Fig. 3. Path of the centroid of the five robots and the geometric patterns at several times

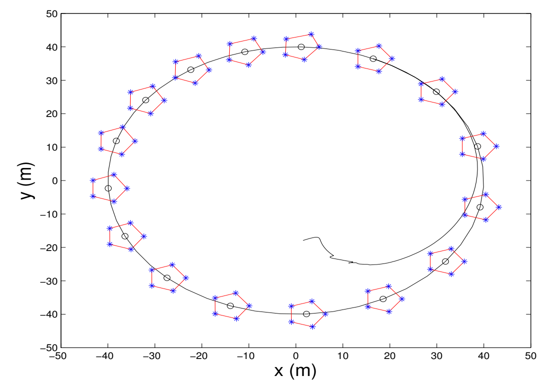

Fig. 4. Path of the centroid of the five robots and the geometric patterns at several times with delay $\delta_j = 0.1sec$

the geometric patterns of the five robots at several times for the case where we assume all the communication delays are the same and $\delta_j = 0.1sec$. It is shown that the five robots come into the desired formation and the centroid of the group of robots converges to the desired trajectory.

VII. Conclusion

This paper has discussed the consensus problem for multiple nonholonomic systems, with a fixed communication graph, converging to a desired trajectory. Cooperative control laws were proposed with the aid of the results from graph theory. The proposed control laws are robust to finite constant communication delays. An application of the proposed results to wheeled mobile robots was presented. Simulation results showed the effectiveness of the proposed control laws.

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