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Zeno behavior in homogeneous hybrid systems

Rafal Goebel and Andrew R. Teel

Abstract— **The link between Zeno behavior and homogeneity in hybrid systems is pursued. Zeno behavior is associated with homogeneous hybrid systems having negative degree. Zeno behavior is typically ruled out in homogeneous hybrid systems with nonnegative degree. Next, asymptotic stability in homogeneous systems is shown to be robust to homogeneous perturbations. In turn, homogeneous perturbations are used to characterize local, approximate homogeneity. For systems that are locally, approximately homogeneous with negative (respectively, nonnegative) degree, Zeno behavior can be established (respectively, ruled out typically). In addition, stability results based on linear and conical approximations are established.**

I. INTRODUCTION

In hybrid systems, Zeno solutions are those that, roughly speaking, experience an infinite number of jumps in a finite amount of ordinary time. Researchers have investigated Zeno behavior, studying how to simulate systems with Zeno solutions [20], [21], [3], how to continue solutions past Zeno times [33], [9], [6], and giving conditions that guarantee or rule out Zeno behavior [11], [19], [30], [4], [1], [5], [2], [23]. Among this last group of references, [19] and [2] are most closely related to the results in this paper. Zeno equilibria are those that attract Zeno solutions. Thus, solutions converge toward Zeno equilibria by taking an infinite number of jumps while taking a finite amount of ordinary time.

In differential and difference equations/inclusions, homogeneity is a property of a vector field, map, or set-valued mapping that has been employed frequently in the context of stability theory. Standard homogeneity appears in stability studies in [25] and [16, Section 57]. Homogeneity with respect to non-standard dilations is used in stability studies in [18], [28], [17], [26], and in several control design results including [22], [17], [15] and [31]. Homogeneity has also been extended to hybrid systems in [32]. Among other things, homogeneity allows asserting global asymptotic stability from local asymptotic stability [28, Proposition 1] and enables establishing convergence rates based on the degree of homogeneity [16, Theorem 57.1], [26, Theorem 11], [8, Theorem 7.1], [32, Proposition 3]. Non-exponential asymptotic convergence is associated with homogeneous systems of positive degree, exponential convergence is associated with homogeneous systems of degree zero, and finite-time convergence is associated with homogeneous systems having negative degree. Moreover, local approximate homogeneity implies that local asymptotic stability and the associated convergence rates are robust to "higher order" perturbations. This fact generalizes classical results that permit concluding local exponential stability for a continuously differentiable nonlinear system when the linearization is exponentially stable. A result related to linearization in the context of hybrid systems is given in [24, Theorem IV.2]. Results based on local, approximate homogeneity were provided in [25, Theorem 24*] for differential equations that are homogeneous in the standard sense, in [18, Theorem 3.3] for differential equations that are homogeneous with respect to non-standard dilations and have unique solutions, and in [28, Theorem 3] dropping the uniqueness assumption.

With the observations of the previous two paragraphs as background, it is natural to investigate the connection between homogeneity and the existence of Zeno equilibria or lack thereof in a hybrid system. (In the course of writing this paper, we were directed to a recent submission [29] which uses homogeneity, under the name "time-scaling symmetry", to study Zeno solutions in discontinuous differential equations with a focus on dimension reduction.) In this paper, we establish the link between homogeneity with negative degree and Zeno behavior in hybrid systems. In particular, through the notion of homogeneous perturbations and robustness, we establish the existence of Zeno behavior (or lack thereof) by considering a system's local, homogeneous approximation when one exists. These results extend the recent contribution in [2] where Zeno behavior is predicted by considering constant approximations, a special case of homogeneous approximations with negative degree. We also give results on determining asymptotic stability by establishing asymptotic stability for the "linearization" of the given hybrid. Our approach is complementary to the techniques used in [32] which focus on converse Lyapunov theorems for homogeneous hybrid systems, and do not address Zeno behavior nor linearizations explicitly.

II. PRELIMINARIES

A. Hybrid systems

Let $F, G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be set-valued mappings and $C, D \subset \mathbb{R}^n$ be sets. We consider hybrid systems of the form

$$\mathcal{H}: \quad \left\{ \begin{array}{cc} \dot{x} \in F(x) & x \in C \\ x^+ \in G(x) & x \in D \end{array} \right. \tag{1}$$

For more background on hybrid systems in this framework, see [7], [12], and [13].

A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid time domain* if $E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \ldots \leq t_J$. It is a *hybrid time domain* if

Rafal Goebel is with the Department of Mathematics and Statistics, Loyola University Chicago, Chicago, IL 60626 rgoebel1@luc.edu. Andrew R. Teel is with the ECE Department, University of California Santa Barbara, CA 93106-9560; teel@ece.ucsb.edu. His research was supported in part by NSF under Grant no. ECS-0622253 and CNS-0720842, and by AFOSR under Grant no. FA9550-06-1-0134.

for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, ...J\})$ is a compact hybrid time domain. Equivalently, E is a hybrid time domain if E is a union of a finite or infinite sequence of intervals $[t_j, t_{j+1}] \times \{j\}$, with the "last" interval possibly of the form $[t_j, T)$ with T finite or $T = +\infty$. A *hybrid arc* is a function ϕ whose domain $\operatorname{dom}\phi$ is a hybrid time domain and such that for each $j \in \mathbb{N}$, $t \to \phi(t, j)$ is locally absolutely continuous on $I_j := \{t \,|\, (t, j) \in \operatorname{dom}\phi\}$. A hybrid arc ϕ is a *solution to the hybrid system* $\mathcal{H}$ if $\phi(0, 0) \in C \cup D$ and

(S1) for all $j \in \mathbb{N}$ such that I_j has nonempty interior and for almost all $t \in I_j$,

$$\phi(t, j) \in C, \quad \dot{\phi}(t, j) \in F(\phi(t, j));$$

(S2) for all $(t, j) \in \operatorname{dom}\phi$ such that $(t, j + 1) \in \operatorname{dom}\phi$,

$$\phi(t, j) \in D, \quad \phi(t, j + 1) \in G(\phi(t, j)).$$

Results on structural properties of solutions to $\mathcal{H}$, like sequential compactness of the space of solutions and outer/upper semicontinuous dependence of solutions on initial conditions, were obtained in [13], under the following assumptions (simplified slightly for this paper):[1]

Assumption 2.1:

(A1) *C and D are closed subsets of $\mathbb{R}^n$;*

(A2) *$F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded, and $F(x)$ is nonempty and convex for all $x \in C$;*

(A3) *$G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded, and $G(x)$ is nonempty for all $x \in D$.*

B. Asymptotic, "USOT", and Zeno stability

The following asymptotic stability definition is taken from [10]. It follows standard definitions, however it does not insist on complete solutions from initial conditions near the attracting set. The advantage of this is that technical conditions for the local existence of solutions are avoided. To distinguish it from a stability notion where completeness is assumed, the prefix "pre" is added.

Definition 2.2: *[pre-AS] A compact set $\mathcal{A} \in \mathbb{R}^n$ is* pre-asymptotically stable *for the hybrid system* (1) *if:*

(a) *for each $\varepsilon > 0$ there exists $\delta > 0$ such that, for each solution ϕ to* (1) *with $|\phi(0, 0)|_{\mathcal{A}} \le \delta$ one has $|\phi(t, j)|_{\mathcal{A}} \le \varepsilon$ for all $(t, j) \in \operatorname{dom}\phi$;*

(b) *each solution ϕ to* (1) *is bounded, and if it is complete, then also $|\phi(t, j)|_{\mathcal{A}} \to 0$ as $t + j \to \infty$, $(t, j) \in \operatorname{dom}\phi$.*

Above and in what follows, with some abuse of notation we write $|\cdot|_{\mathcal{A}}$ for the distance from the set $\mathcal{A}$, in the Euclidean norm. That norm itself will be denoted by $|\cdot|$. The property above corresponds to *global* pre-asymptotic stability since the condition (b) is required to hold for *each* solution.

We also consider *uniformly small ordinary time pre-asymptotic stability* (USOT pre-AS), in which the "ordinary time" it takes a solution to reach the attractor decreases to zero with initial conditions approaching the attractor. This stability notion, defined below, was introduced in [14] where necessary and sufficiently Lyapunov-based conditions were given for it and it was related to Zeno behavior.

We will use the following object: given a set $X \subset \mathbb{R}^n$ and a hybrid arc ϕ, let

$$T_X(\phi) := \sup\{t \,|\, \exists j \text{ s.t. } (t, j) \in \operatorname{dom}\phi,\ \phi(t, j) \in X\}.$$

Definition 2.3: *[USOT pre-AS] A compact set $\mathcal{A} \subset \mathbb{R}^n$ is called* uniformly small ordinary time pre-asymptotically stable *for the hybrid system $\mathcal{H}$ if the following hold:*

(i) *$\mathcal{A}$ pre-asymptotically stable, and*

(ii) *for each $\varepsilon > 0$ there exists $\delta > 0$ such that every maximal solution ϕ to $\mathcal{H}$ with $|\phi(0, 0)|_{\mathcal{A}} \le \delta$ satisfies $T_{\mathbb{R}^n \setminus \mathcal{A}}(\phi) \le \varepsilon$.*

Zeno hybrid arcs ϕ are defined in terms of the quantity

$$\sup_t \operatorname{dom}\phi := \sup\{t \in \mathbb{R}_{\ge 0} \,|\, \exists j \text{ s.t. } (t, j) \in \operatorname{dom}\phi\}.$$

We use the following definition of a Zeno arc. For details, see [14]. Also, compare the definition to [5], [2], and [23].

Definition 2.4: *[Zeno arc] A hybrid arc ϕ is* Zeno *if*

(i) *ϕ is complete,*

(ii) *$\sup_t \operatorname{dom}\phi < \infty$,*

(iii) *there is no j such that $(\sup_t \operatorname{dom}\phi, j) \in \operatorname{dom}\phi$.*

It is straightforward, from (i) and (iii) above, that each Zeno arc ϕ satisfies $\sup_t \operatorname{dom}\phi > 0$. Note that $T_{\mathbb{R}^n}(\phi) = \sup_t \operatorname{dom}\phi$, and so any Zeno arc satisfies $0 < T_{\mathbb{R}^n}(\phi) < \infty$. Since Zeno arcs are complete, they experience infinitely many jumps in finite (ordinary) time. Item (iii) rules out such arcs for which the "tail", or even the whole arc itself, consists of infinitely many instantaneous jumps.

Example 2.5: Consider a hybrid system in $\mathbb{R}^2$ with data

$$F(x) = \begin{pmatrix} 0 \\ x_1 \end{pmatrix}, \quad G(x) = \begin{pmatrix} 0.5x_1 \\ 0 \end{pmatrix}$$
$$C = \left\{x : 0 \le x_2 \le x_1^3\right\}, \quad D = \left\{x : x_1 \ge 0,\ x_2 = x_1^3\right\}.$$

By inspection, it can be verified that the origin is pre-asymptotically stable for this system. (This system will be revisited in several subsequent examples, and pre-AS of the origin will be established via different tools.) It can be also verified that each maximal solution with nonzero initial condition in C is Zeno. Furthermore, a simple calculation shows that $T_{\mathbb{R}^2 \setminus \{0\}}(\phi) = 4\,(\phi_1(0))^2/3$ for each solution ϕ with $\phi_2(0) = 0$ and then $T_{\mathbb{R}^2 \setminus \{0\}}(\phi) \le 4\,(\phi_1(0))^2/3$ for all solutions ϕ. Thus, the origin is USOT pre-AS. □

A combination of pre-asymptotic stability and Zeno behavior suggests the following two definition, which were used in [14]. We add here that the system in Example 2.5 has the origin uniformly Zeno stable. (This conclusion was also shown via Lyapunov arguments in [14].)

Definition 2.6: *[Zeno stability] A compact set $\mathcal{A} \subset \mathbb{R}^n$ is called* Zeno asymptotically stable *for the hybrid system $\mathcal{H}$ if the following hold:*

(i) *$\mathcal{A}$ is pre-asymptotically stable, and*

[1] A set-valued map $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *outer semicontinuous* if for every convergent sequence of x_i's with and every convergent sequence of $y_i \in \Phi(x_i)$, $\lim y_i \in \Phi(\lim x_i)$. Φ is *locally bounded* if for every compact $K \subset \mathbb{R}^n$ there exists a compact $K' \subset \mathbb{R}^m$ such that $\Phi(K) \subset K'$.

(ii) *there exists $\varepsilon > 0$ such that every maximal solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{A}} \in (0, \varepsilon]$ is Zeno and*

$$T_{\mathbb{R}^n}(\phi) = T_{\mathbb{R}^n \setminus \mathcal{A}}(\phi).$$

Definition 2.7: *[Uniform Zeno stability] A compact set $\mathcal{A} \subset \mathbb{R}^n$ is called* uniformly Zeno asymptotically stable *for the hybrid system $\mathcal{H}$ if the following hold:*

(i) *$\mathcal{A}$ is Zeno asymptotically stable, and*
(ii) *$\mathcal{A}$ is USOT pre-AS.*

In Zeno asymptotic stability, all solutions starting near the stable set have time domains bounded in the ordinary time direction, while in uniform Zeno asymptotic stability the amount of ordinary time decreases to zero as the initial conditions approach the stable set.

In words, a uniformly Zeno asymptotically stable compact set $\mathcal{A}$ is one that is pre-asymptotically stable with trajectories that converge toward $\mathcal{A}$ using a finite amount of ordinary time (that decreases to zero with initial conditions approaching $\mathcal{A}$) but that never actually reaches a final ordinary time and never actually reaches $\mathcal{A}$ [14, Lemma 4.3]. The first condition in the definition can be verified by making sure that i) there are no solutions of $\dot{x} \in -F(x)$, $x \in C$ that start in $\mathcal{A}$ and leave $\mathcal{A}$, and ii) there are no solutions of $x^+ \in G(x)$, $x \in D$ that start outside of $\mathcal{A}$ and converge to $\mathcal{A}$ [14, Proposition 4.5].

To keep this paper self-contained, we focus on establishing (respectively, ruling out) the USOT pre-AS property, which appears in the uniform Zeno stability definition, based on approximate homogeneity properties of the hybrid system. A more complete discussion of Zeno behavior is in [14].

III. Homogeneity

In this section, we review the concept of homogeneity, apply it to hybrid systems, and derive some results for homogeneous, asymptotically stable systems. A generalized notion of homogeneity for hybrid systems was introduced in [32]. In this paper we stick to homogeneity with respect to dilations with nonzero exponents. For hybrid systems, this rules out systems with logic variables and timers taking values in a compact set. The results can be extended, though. In the last section, we briefly outline how systems with logic variables can be addressed.

A *dilation* is a map induced by the matrix

$$M(\lambda) = \operatorname{diag}\{\lambda^{r_1}, \lambda^{r_2}, \ldots, \lambda^{r_n}\}, \tag{2}$$

where $r_1, r_2, \ldots, r_n > 0$ and $\lambda > 0$, given by $x \mapsto M(\lambda)x$. By abuse of terminology, we will use $M(\lambda)$ to denote the dilation. The standard dilation refers to the case where $r_1 = r_2 = \cdots = r_n = 1$. A function $\omega : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is called a *homogeneous norm* if it is continuous, positive definite, and homogeneous (of degree 1) with respect to $M(\lambda)$, namely, for all $x \in \mathbb{R}^n$, $\omega(M(\lambda)x) = \lambda\omega(x)$. In the rest of the paper, ω will denote a homogeneous norm. Given such an ω, we define $\mathcal{B} := \{x \in \mathbb{R}^n \,|\, \omega(x) \leq 1\}$.

Definition 3.1: *[Homogeneity] The hybrid system* (1) *is* homogeneous of degree $d \in \mathbb{R}$ with respect to the dilation $M(\lambda)$ *if the following conditions are satisfied for all $\lambda > 0$:*

$$F(M(\lambda)x) = \lambda^d M(\lambda)F(x) \qquad \forall x \in C,$$
$$G(M(\lambda)x) = M(\lambda)G(x) \qquad \forall x \in D,$$
$$C = M(\lambda)C, \qquad D = M(\lambda)D.$$

Example 3.2: Consider a simple model of a ball bouncing on the floor. For $x \in \mathbb{R}^2$, let the first coordinate represent the position (height above the floor) and the second coordinate be the velocity. With $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, let

$$F(x) = \begin{pmatrix} x_2 \\ -g \end{pmatrix}, \qquad C = \left\{x \in \mathbb{R}^2 \,|\, x_1 \geq 0\right\},$$
$$G(x) = \begin{pmatrix} 0 \\ -\gamma x_2 \end{pmatrix}, \quad D = \left\{x \in \mathbb{R}^2 \,|\, x_1 = 0, x_2 \leq 0\right\}.$$

The hybrid system with such data is homogeneous of degree $d = -1$ with respect to $M(\lambda) = \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda \end{pmatrix}$. Indeed,

$$\begin{aligned} F\left(M(\lambda)x\right) &= F\begin{pmatrix} \lambda^2 x_1 \\ \lambda x_2 \end{pmatrix} = \begin{pmatrix} \lambda x_2 \\ -g \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_2 \\ -g \end{pmatrix} \\ &= \lambda^{-1}\begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} x_2 \\ -g \end{pmatrix} = \lambda^{-1}M(\lambda)F(x) \end{aligned}$$

and a similar calculation applies for G. Also, as $M(\lambda)x = \begin{pmatrix} \lambda^2 x_1 \\ \lambda x_2 \end{pmatrix}$, if $x \in C$ which amounts to $x_1 \geq 0$, then $\lambda^2 x_1 \geq 0$ which amounts to $M(\lambda)x \in C$. Thus $M(\lambda)C \subset C$, while a reverse inclusion follows from this one by considering λ^{-1}. Similar arguments can be made for D. □

Example 3.3: Recall the system from Example 2.5. We note here that it is homogeneous of degree $d = -2$ with respect to the dilation $M(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^3 \end{pmatrix}$. Indeed,

$$\begin{aligned} F(M(\lambda)x) &= F\begin{pmatrix} \lambda x_1 \\ \lambda^3 x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda x_1 \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} 0 \\ x_1 \end{pmatrix} \\ &= \lambda^{-2}\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^3 \end{pmatrix}\begin{pmatrix} 0 \\ x_1 \end{pmatrix} = \lambda^{-2}M(\lambda)F(x) \end{aligned}$$

and a similar calculation applies for G. Also, if $x \in C$ then $x_2 \geq 0$ and $x_2 \leq x_1^3$. These conditions imply $\lambda^3 x_2 \geq 0$ and $\lambda^3 x_2 \leq (\lambda x_1)^3$ for all $\lambda > 0$, i.e., $M(\lambda)x \in C$. Thus $M(\lambda)C \subset C$. while a reverse inclusion follows from this one by considering λ^{-1}. Similar arguments apply to D. □

The significance of homogeneity is that the behavior of solutions from different initial conditions that are related through the dilation matrix can be related through the dilation and a scaling of ordinary time.

Given a $\lambda > 0$ and $d \in \mathbb{R}$, consider a hybrid system obtained from (1) by scaling the flow map F:

$$\mathcal{H}_{\lambda,d}: \quad \left\{ \begin{array}{ll} \dot{x} \in \lambda^{-d}F(x) & x \in C, \\ x^+ \in G(x), & x \in D. \end{array} \right. \tag{3}$$

Lemma 3.4: *Let x be a hybrid arc. For each $\lambda > 0$, $d \in \mathbb{R}$, the function ψ defined at each $(t, j) \in \operatorname{dom}\phi$ by*

$$\psi(t, j) = M(\lambda)\phi(t, j)$$

is a hybrid arc, with $\operatorname{dom}\psi = \operatorname{dom}\phi$. *Moreover, if the hybrid system* (1) *is homogeneous with respect to the dilation* (2) *and degree* $d \in \mathbb{R}$, *then for each* $\lambda > 0$, ϕ *is a solution to* (1) *if and only if* ψ *is a solution to* (3).

The next result states that, in the context of homogeneity, a negative degree is necessary for USOT pre-AS, unless there are purely discrete solutions that converge to the origin.

Theorem 3.5: *Under Assumption 2.1, if the origin of* $\mathcal{H}$ *is pre-asymptotically stable,* $\mathcal{H}$ *is homogeneous of degree* $d \geq 0$, *and* $T_{\mathbb{R}^n\setminus\{0\}}(\phi) > 0$ *for all complete solutions* ϕ *with* $\omega(\phi(0,0)) = 1$ *then* $T_{\mathbb{R}^n\setminus\{0\}}(\phi) = \infty$ *for all complete solutions* ϕ *with* $\omega(\phi(0,0)) > 0$.

Corollary 3.6: *Under Assumption 2.1, if the origin of* $\mathcal{H}$ *is Zeno asymptotically stable and* $\mathcal{H}$ *is homogeneous of degree* $d \in \mathbb{R}$ *then* $d < 0$.

If $\mathcal{H}$ is homogeneous of degree $d \geq 0$ and yet $T_{\mathbb{R}^n\setminus\{0\}}(\phi) = 0$ for some ϕ with $\omega(\phi(0,0)) = 1$ then it is possible that $T_{\mathbb{R}^n\setminus\{0\}}(\phi) < \infty$ for all solutions ϕ with $\omega(\phi(0,0)) > 0$. This is illustrated by the following example.

Example 3.7: Consider the hybrid system in $\mathbb{R}^2$ with data

$$\begin{aligned} F(x) &= \rho(x)\begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}, \quad & C &= \{x : x_1 \geq 0\ ,\ x_2 \geq 0\}\,, \\ G(x) &= x/2, & D &= \{x : x_2 = 0\ ,\ x_1 \geq 0\}\,, \end{aligned}$$

and ρ is a continuous function that is positive on $\mathbb{R}^2\setminus\{0\}$ and homogeneous of degree $d \geq 0$ with respect to the standard dilation, i.e., $\rho(\lambda x) = \lambda^d \rho(x)$ for all $x \in \mathbb{R}^2$ and $\lambda > 0$. It follows that $F(\lambda x) = \lambda^d \lambda F(x)$, $G(\lambda x) = \lambda G(x)$, $\lambda C = C$ and $\lambda D = D$, i.e, the hybrid system is homogeneous of degree $d \geq 0$ with respect to the standard dilation. It is not difficult to see that, for each initial condition, solutions evolve on a circle and flow to the set D from which flowing is not possible and solutions jump toward the origin. Thus, for each solution ϕ, $T_{\mathbb{R}^2\setminus\{0\}}(\phi)$ is finite and given by the time required to flow from the initial condition to D. Even though all solutions have time domains that are bounded in the ordinary time direction, the origin is not Zeno asymptotically stable in the sense of Definition 2.6 since there are solutions starting near the origin that are not Zeno in the sense of Definition 2.4. In particular, all solution ϕ have an positive integer j such that $(\sup_t \operatorname{dom}\phi, j) \in \operatorname{dom}\phi$. □

The next example shows that "truly" Zeno behavior is possible when $d = 0$ but there are purely discrete solutions.

Example 3.8: Consider the hybrid system with data $F(x) = \begin{pmatrix} x_2 & -x_1 \end{pmatrix}^T$ and G, C and D are formed as follows. For each positive integer i, let

$$\begin{aligned} C_i &:= \left\{|x| = 1, x_1 \geq 0, x_2 \in [(2i+1)^{-1}, 2i^{-1}]\right\} \\ D_i &:= \left\{|x| = 1, x_1 \geq 0, x_2 \in [(2i+2)^{-1}, (2i+1)^{-1}]\right\} \\ C &:= \overline{\{x : x = \lambda z\ ,\ \lambda > 0\ ,\ z \in C_i \text{ for some } i\}} \\ D &:= \overline{\{x : x = \lambda z\ ,\ \lambda > 0\ ,\ z \in D_i \text{ for some } i\}} \end{aligned}$$

Let $g\begin{pmatrix}1\\0\end{pmatrix} = 0.5\begin{pmatrix}1\\0\end{pmatrix}$ and $g(z) = 0.5\begin{pmatrix}\sqrt{1-(2i+4)^{-2}} \\ (2i+4)^{-1}\end{pmatrix}$ for each point $z \in D_i$. Then, for each $x \in D$, define $G(x) := |x| g(x/|x|)$. All solutions of this system have $T_{\mathbb{R}^2\setminus\{0\}}(\phi) \leq \pi/2$ and those not starting on the x_1-axis are Zeno in the sense defined earlier. Because of the behavior of the solutions starting on the x_1 axis, the origin is not Zeno asymptotically stable in the sense of Definition 2.6. □

For homogenous hybrid systems, the next theorem relates asymptotic stability, which is a global property, to the behavior of those solutions starting a fixed distance from the origin as measure using a homogeneous norm.

Theorem 3.9: *Consider a hybrid system* $\mathcal{H}$ *such that*

(a) $\mathcal{H}$ *is homogeneous with respect to dilation* M *in* (2) *with degree* $d \in \mathbb{R}$;
(b) *there exist* $R > r > 0$ *and* $m > 0$ *such that for any solution* ϕ *to* $\mathcal{H}$ *with* $\omega(\phi(0,0)) = r$ *either*
    (i) $\operatorname{dom}\phi$ *is compact, with* $t + j \leq m$ *for all* $(t,j) \in \operatorname{dom}\phi$, *and for all such* (t,j), $\omega(\phi(t,j)) \leq R$, *or*
    (ii) *there exists* $(T,J) \in \operatorname{dom}\phi$ *with* $T + J \leq m$, $\omega(\phi(T,J)) \leq r/2$, *and such that* $\omega(\phi(t,j)) \leq R$ *for all* $(t,j) \in \operatorname{dom}\phi$, $t \leq T$, $j \leq J$.

Then, $0 \in \mathbb{R}^n$ *is pre-AS for* $\mathcal{H}$. *Moreover, if* $d < 0$ *then* $0 \in \mathbb{R}^n$ *is USOT pre-AS for* $\mathcal{H}$.

Example 3.10: Recall the system in Example 2.5. It was shown to be homogeneous of degree $d = -2$ in Example 3.3. Here we note that conditions (i) and (ii) in Lemma 3.9 are simple to verify, and hence the system has the origin USOT pre-AS. □

The next result is a consequence of Theorem 3.9.

Corollary 3.11: *Under Assumption 2.1, if the origin of* $\mathcal{H}$ *is pre-asymptotically stable and* $\mathcal{H}$ *is homogeneous of degree* $d < 0$ *with respect to some dilation then the origin of* $\mathcal{H}$ *is USOT pre-AS.*

IV. Homogeneous perturbations

The stability properties for homogeneous perturbations of homogeneous systems will play a fundamental role in subsequent results on stability from homogeneous approximations.

Definition 4.1: *Given a hybrid system* $\mathcal{H}$, *a dilation* M, *and a homogeneous norm* ω, *a* homogeneous perturbation of $\mathcal{H}$ *of size* $\rho > 0$ *is the hybrid system*

$$\mathcal{H}_\rho: \quad \left\{ \begin{array}{ll} \dot{x} \in F_\rho(x) & x \in C_\rho, \\ x^+ \in G_\rho(x), & x \in D_\rho, \end{array} \right. \tag{4}$$

with the data given by

$$\begin{aligned} F_\rho(x) &= conF\left(x + \rho M(\omega(x))\mathcal{B}\right) + \rho\omega^d(x) M(\omega(x))\mathcal{B} \\ G_\rho(x) &= \{v \,|\, v \in u + \rho M(\omega(u))\mathcal{B}, u \in G\left(x + \rho M(\omega(x))\mathcal{B}\right)\}\,, \\ C_\rho &= \{x \,|\, (x + \rho M(\omega(x))\mathcal{B}) \cap C \neq \emptyset\}\,, \\ D_\rho &= \{x \,|\, (x + \rho M(\omega(x))\mathcal{B}) \cap D \neq \emptyset\}\,. \end{aligned}$$

Using "homogeneous" in the name of the perturbation above is justified by the following result.

Proposition 4.2: *If the hybrid system* $\mathcal{H}$ *in* (1) *is homogeneous with respect to the dilation* $M(\cdot)$ *in* (2) *and degree* $d \in \mathbb{R}$ *then, for each* $\rho > 0$, *the hybrid system* $\mathcal{H}_\rho$ *in* (4) *is homogeneous with respect to the dilation* $M(\cdot)$ *in* (2) *and degree* d.

It needs to be noted that homogeneous perturbations of a hybrid system, i.e., those defining the data in (4), may lead to F_ρ that is not locally bounded in a neighborhood of 0. Indeed, when $d < 0$ it may happen that $\omega^d(x)M(\omega(x))$ is unbounded on a neighborhood of 0. For example:

Example 4.3: For $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$, consider $F(x) = \begin{pmatrix} 0 \\ x_1 \end{pmatrix}$, $C = \{x \in \mathbb{R}^2 \,|\, x_2 = 0\}$, D empty, and $G(x) = 0$ for all $x \in \mathbb{R}^2$. It is easy to verify that the hybrid system with such data is homogeneous with respect to the dilation $M(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^3 \end{pmatrix}$ and the degree is $d = -2$. We can take $\omega(x) = \left(x_1^6 + x_2^2\right)^{1/6}$. Then $\omega^d(x)M(\omega(x)) = \begin{pmatrix} \omega^{-1}(x) & 0 \\ 0 & \omega(x) \end{pmatrix}$ and this blows up as $x \to 0$. □

Theorem 4.4: *Suppose that the hybrid system $\mathcal{H}$ in* (1)

- *satisfies Assumption 2.1,*
- *is homogeneous with respect to dilation $M(\cdot)$ in* (2) *with degree $d \in \mathbb{R}$,*
- *has $0 \in \mathbb{R}^n$ pre-asymptotically stable.*

Then there exists $\rho > 0$ such that, for the system $\mathcal{H}_\rho$, when $d \geq 0$ the point $0 \in \mathbb{R}$ is pre-asymptotically stable and when $d < 0$ the point 0 is USOT pre-AS.

V. Local approximate homogeneity and linearizations

Frequently systems are not homogeneous but, locally, their stability properties are determined by a homogeneous approximation of the data. This section addresses such situations. The results parallel results for differential equations like in [18, Theorem 3.3]. They also extend results in [2] when applied in the context of Zeno behavior.

Definition 5.1: *A hybrid system $\mathcal{H} := (F, C, G, D)$ is said to be locally, approximately contained in the homogeneous hybrid system $\mathcal{H}^* = (F^*, C^*, G^*, D^*)$ if, for each $\rho > 0$, there exists a compact set $K \subset \mathbb{R}^n$ containing the origin in its interior such that that the hybrid system $\mathcal{H}_K := (F, C \cap K, G, D \cap K)$ is contained in the homogeneous perturbation of $\mathcal{H}^*$ of size ρ, denoted $\mathcal{H}^*_\rho$, i.e., $F(x) \subset F^*_\rho(x)$ for all $x \in C \cap K$, $C \cap K \subset C^*_\rho$, $G(x) \subset G^*_\rho(x)$ for all $x \in D \cap K$, and $D \cap K \subset D^*_\rho$.*

The following corollary is derived from Theorem 4.4.

Corollary 5.2: *If the hybrid system $\mathcal{H}$ is locally, approximately contained in a homogeneous system $\mathcal{H}^*$ for which the origin is pre-asymptotically stable then there exists a compact set $K \subset \mathbb{R}^n$ containing the origin in its interior such that, for the system $\mathcal{H}_K := (F, C \cap K, G, D \cap K)$,*

1) *the origin pre-asymptotically stable;*
2) *if the degree of $\mathcal{H}^*$ is negative then the origin of $\mathcal{H}_K$ is USOT pre-AS;*
3) *if the degree of $\mathcal{H}^*$ is not negative and $T_{\mathbb{R}^n\setminus\{0\}}(\phi^*) > 0$ for each solution ϕ^* of $\mathcal{H}^*$ then $T_{\mathbb{R}^n\setminus\{0\}}(\phi) = \infty$ for each complete solution ϕ of $\mathcal{H}_K$.*

This result can be applied to establish linear/conical approximation results for hybrid systems, as follows. Let the hybrid system $\mathcal{H} = (f, C, g, D)$ be such that f and g are continuously differentiable and $g(0) = 0$. Define $f^0(x) = f(0)$ and, if $f(0) = 0$, define $f^1(x) = \left(\left.\frac{\partial f(x)}{\partial x}\right|_{x=0} \right) x$. Also define $g^1(x) := \left(\left.\frac{\partial g(x)}{\partial x}\right|_{x=0} \right) x$. Finally, let $T_C(0)$, respectively, $T_D(0)$, denote the tangent cone to C, respectively, D, at the origin. In other words $T_C(0) = \limsup_{\tau \searrow 0} C/\tau$ and similarly for D. For more details, see [27, Section 6A]. When the sets C and D are given in terms of inequalities involving smooth constraint functions, the tangent cones can often be described using inequalities involving linear mappings and gradients of the constraint functions [27, Theorem 6.31].

Lemma 5.3: *The system $\mathcal{H}^i := (f^i, T_C(0), g^1, T_D(0))$ is homogeneous with respect to the standard dilation for $i = 0$ and, if $f(0) = 0$, for $i = 1$. The degree of homogeneity of $\mathcal{H}^i$ is $i - 1$. Moreover, $\mathcal{H}$ is locally, approximately contained in $\mathcal{H}^i$.*

In Lemma 5.3, the hybrid system $\mathcal{H}^0$, together with the extensions of the next section, constitutes of a type of generalization of the constant approximations considered in [2]. The hybrid system $\mathcal{H}^1$ is a generalization of linearization for continuous-time and discrete-time systems.

Corollary 5.4: *Consider a hybrid system $\mathcal{H} = (f, C, g, D)$ and suppose f and g are continuously differentiable and $g(0) = 0$. Consider $i = 0$ and, if $f(0) = 0$, $i = 1$. Define $\mathcal{H}^i := (f^i, T_C(0), g^1, T_D(0))$. If the hybrid system $\mathcal{H}^i$ has the origin pre-asymptotically stable then there exists a compact set K containing the origin in its interior such that $\mathcal{H}_K = (f, C \cap K, g, D \cap K)$ has the origin pre-asymptotically stable. When $i = 0$, $\mathcal{H}_K$ has the origin USOT pre-AS. When $i = 1$ and $T_{\mathbb{R}^n\setminus\{0\}}(\phi^i) > 0$ for each solutions ϕ^i of $\mathcal{H}^i$, $T_{\mathbb{R}^n\setminus\{0\}}(\phi) = \infty$ for each complete solution ϕ of $\mathcal{H}_K$.*

Example 5.5: Recall the system $\mathcal{H}$ in Example 2.5. We have $T_C(0) = T_D(0) = \{x : x_1 \geq 0, x_2 = 0\}$. The approximation of $\mathcal{H}$ to consider is $\mathcal{H}^1 = (F, T_C(0), G, T_D(0))$ since both F and G are linear. Flow is impossible for $\mathcal{H}^1$ and all maximal solutions are discrete (sometimes called instantaneously Zeno): we have $\operatorname{dom} x = \{0\} \times \mathbb{N}$ and $x(0, j) = x(0, 0)/2^j$ for $j \in \mathbb{N}$ for each maximal solution x to $\mathcal{H}^1$. Obviously, the origin is pre-AS for $\mathcal{H}^1$. Corollary 5.4 implies that the origin is (locally) pre-AS for $\mathcal{H}$. Since it was previously shown in Example 3.3 that $\mathcal{H}$ is homogeneous with a negative degree, this implies further that $\mathcal{H}$ is (globally) pre-AS and in fact USOT pre-AS. □

It may of course happen that a hybrid system has 0 pre-AS but $\mathcal{H}^i$ does not. A simple example is provided by the bouncing ball system of Example 3.2. For that system, the flow map for $\mathcal{H}^0$ is $\begin{pmatrix} 0 \\ -g \end{pmatrix}$ while the other data does not change. Solutions of the form $x(t, 0) = \begin{pmatrix} x_1(0) \\ x_2(0) - gt \end{pmatrix}$ show that 0 is not pre-AS for $\mathcal{H}^0$.

Other implications may fail too. In Example 5.5, $\mathcal{H}$ was uniformly Zeno stable (i.e., USOT pre-AS and Zeno AS) while the approximation $\mathcal{H}^1$ was USOT pre-AS but not Zeno AS: solutions to $\mathcal{H}^1$ did not flow at all. Finally, it can happen that an approximation has 0 uniformly Zeno stable but the original system does not have any Zeno solutions. This is shown in the example below and underlines that delicate conditions on existence on solutions are needed to guarantee Zeno behavior.

Example 5.6: Consider a hybrid system in $\mathbb{R}^2$ with data

$$F(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad C = \{x : 0 \le x_2 \le x_1\},$$
$$G(x) = \begin{pmatrix} 0.5x_1 \\ 0 \end{pmatrix}, \quad D = \{x : x_1 \ge 0\ ,\ x_2 = x_1 + x_1^2\}.$$

The origin is USOT pre-AS. Pre-asymptotic stability can be seen directly. (For this system, the only complete maximal solution is the solution $x(0,j) = 0$, $j \in \mathbb{N}$.) Furthermore, $T_{\mathbb{R}^2\setminus\{0\}}(x) \le x_1(0)$. Since $C \cap D = \{0\}$ and flowing while remaining in $\{0\}$ is impossible, there are no Zeno solutions.

Considering $\mathcal{H}^0$ leads to a system given by F, C, G, and $T_D(0) = \{x : x_1 = x_2 \ge 0\}$. The key difference from $\mathcal{H}$ is that now $C \cap T_D(0) = T_D(0)$. Every maximal solution x to $\mathcal{H}^0$ with $x_1(0) > 0$ is Zeno and satisfies $T_{\mathbb{R}^2\setminus\{0\}}(x) \le 2x_1(0)$. In fact, $\mathcal{H}^0$ is uniformly Zeno stable. However, only USOT pre-AS is guaranteed to carry over to $\mathcal{H}$. □

VI. Systems with logic variables

Let $Q = \{1, 2, \ldots, q_{max}\}$ and for each $q \in Q$, let $F^q : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, $G^q : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, $g^q : \mathbb{R}^n \rightrightarrows Q$ be set-valued mappings and $C^q, D^q \subset \mathbb{R}^n$ be sets. Consider a hybrid system

$$\mathcal{H}^Q : \begin{cases} \dot{x} \in F^q(x) & x \in C^q, \\ (q^+, x^+) \in G^q(x) & x \in D^q, \end{cases} \qquad (5)$$

with the state $\xi = (q, x) \in Q \times \mathbb{R}^n \subset \mathbb{R}^{n+1}$. Note that during flow, q remains constant. Such a system can be written in the form of $\mathcal{H}$ as in (1), as explained for example in [10, Section IV] Results of this paper, developed for (1), can be then translated to systems like (5). This is done by considering pre-AS not of 0 but of $\{0\} \times Q$, and systems (5) for which the associated system (1) is homogeneous with respect to $\widetilde{M} = \text{diag}\,\{1, \lambda^{r_1}, \lambda^{r_2}, \ldots, \lambda^{r_n}\}$.

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