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# Feedforward Control Design for the Wave Equation with Nonlinear Boundary Conditions Modelling a Torsional Rod

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***Abstract*— This paper presents a flatness–based approach for the feedforward control design of the wave equation with nonlinear boundary conditions modelling a torsional rod with a tip load. It uses a subdivision of the relevant time–space region based on the characteristic curves of the system as well as the property of formal power series parameterizability of the underlying partial differential equation on the subregions to derive an expression for the nominal input to achieve perfect tracking performance of a prescribed desired output trajectory. Therefore, this contribution proves the applicability of the formal power series approach to second order hyperbolic partial differential equations.**

***Index Terms*— Wave equation, nonlinear boundary conditions, feedforward control, trajectory planning, infinite–dimensional systems**

## I. Introduction

The wave equation modelling a torsional elastic rod can be used as a benchmark problem for evaluating the applicability of feedforward control design methods to hyperbolic distributed–parameter systems (see [10], [14]). While the exact solution is known for the linear case by d'Alembert's solution (see, e.g., [3]), there is no general control design method, if nonlinear variants of the wave equation are considered. Nonlinearities, however, are of great practical relevance, since there are many applications like propellers or driveshafts for which the tip load experiences a nonlinear characteristics. This results in nonlinear boundary conditions which have to be taken into account.

For the feedforward control design, flatness–based methods are particularly appealing due to their simplicity in use and their broad applicability. They are based on a parameterization of the system state and input in terms of a flat output as first described in [1] for nonlinear finite–dimensional systems. For infinite–dimensional systems, two major approaches have been proposed: For linear hyperbolic systems, the partial differential equation (PDE) is transformed into an ordinary differential equation using Mikusiński's operational calculus (see [10]). This equation is explicitly solved and the solution is transformed back into the original coordinates. This leads to solutions in terms of concentrated and/or distributed shift operators acting on the flat output, i.e., pre– and post–actuations. Using this approach, the feedforward control design has already been applied to the linear wave equation in [2], the linear torsional rod in [10], [14], the linear heat exchanger in [9], the linear heavy chain in [7], [8], and a linear gantry crane in [11]. This approach, however, essentially relies on the linearity of the system and hence cannot be extended to the nonlinear case.

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For parabolic systems, the state is represented as a formal power series in the spatial variable whose coefficients are functions of time, which produces a differential recurrence for the solution of the coefficients and does not require the considered system to be linear. Thus, the approach has so far been applied to the heat equation and diffusion–convection–reaction systems as given in [4], [5], [6], [13].

In [12] the formal power series approach has recently been modified and extended to quasi–linear first order hyperbolic systems. Motivated by these results, in the following a trajectory planning problem for the wave equation modelling a torsional rod is considered with the objective to determine the trajectory for the angular position at one end of the rod, which is required to ensure that an attached tip load on the opposite end of the rod follows a prescribed trajectory.

The paper is organized as follows. In Section II the problem under consideration is formulated. Section III states the solution approach and explains the modifications being made compared to previous flatness–based approaches. In Section IV, a feedforward tracking control is determined for the wave equation with nonlinear boundary conditions by using the formal power series approach. Section V provides some simulation results and Section VI concludes the paper.

## II. Problem Formulation

Consider the angular position $x = x(z,t)$ of the cross–section of a torsional elastic rod of scaled length $L = 1$ with a tip load at the one end $z = 0$, where $z$ describes the position along the rod and $t$ the time. The angular position of the cross–section is given by the PDE

$$\rho x_{tt}(z,t) = G x_{zz}(z,t), \tag{1}$$

where $\rho$ is the mass density and $G$ is the shear modulus of the elastic rod. Let the rod be initially at rest and assume that the angular position of the rod at $z = 1$ can be imposed by an actuator. The initial and boundary conditions are then given by

$$x(z,t^-) = 0, \tag{2}$$

$$x_t(z,t^-) = 0, \tag{3}$$

$$x(1,t) = u(t), \tag{4}$$

$$\theta x_{tt}(0,t) = GI x_z(0,t) - \alpha x_t(0,t)|x_t(0,t)|, \tag{5}$$

where $\theta$ is the mass moment of inertia of the tip load, $I$ is the geometrical moment of inertia of the rod, $\alpha$ is the coefficient of a nonlinear resistance torque, and $t^-$ is the time at which a change in the input is necessary to produce a change in the output at $t = 0$. Hence, $t^-$ constitutes the delay in the system, which depends on the values of $\rho$ and $G$. Furthermore, $u(t)$ is the system input, and the angular position of the tip load

$$y(t) = x(0,t) \tag{6}$$

serves as the system output, which can be shown to be a flat or basic output as demonstrated in [10], [14]. The nonlinear term represents a frictional torque whose magnitude is proportional to the rotational speed of the tip load. For $\alpha = 0$, (1) is a linear, space– and time–invariant, second order hyperbolic PDE for all

$$(z,t) \in \mathcal{R} := \{(z,t) | z \in [0,1], t \in \mathbb{R}\}. \tag{7}$$

For $\alpha \neq 0$, the system is nonlinear. The control objective considered in this contribution is to steer the output $y(t)$, i.e., the angular position of the tip load, along a prescribed trajectory $y_d(t)$. For this, we focus on trajectories that produce a set–point change from an initial to a final steady state. Since the only nonlinearity is with respect to $x_t(0,t)$, the set of initial and final states can be chosen as $x(z,t) = 0$ and $x(z,t) = 1$, respectively, without any loss of generality.

## III. Solution Approach

The solution approach considered in the following consists of subdividing the transition region $\mathcal{R}$ into suitable subregions $\mathcal{R}_{[j]}$ and representing the solution as a formal power series

$$\hat{x}^{[j]}(z_{[j]}, t_{[j]}) = \sum_{n,i=0}^{\infty} \hat{x}^{[j]}_{n,i} \frac{t^i_{[j]}}{i!} \frac{z^n_{[j]}}{n!} \tag{8}$$

on each subregion in the space and time coordinates $z_{[j]}$ and $t_{[j]}$. Here, the double index in the sum is used as a short form notation for a double sum. The bracketed index $j$ defines the subregion of the transition region on which the power series is used. The series is called formal since a–priori no restriction on its convergence or summability is imposed.

### A. Trajectory Planning

For hyperbolic systems, the formal power series approach requires the solution to be piecewise analytical in both $z$ and $t$ as explained in [12]. Therefore, both the PDE and the desired output trajectory are required to be piecewise analytical. In addition, the desired output trajectory needs to be continuously differentiable with respect to $t$ to fulfill the initial conditions. Hence, the easiest choice for a desired trajectory that produces a set–point change from the steady state solution $x(z,t) = 0$ to the final steady state $x(z,t) = 1$ in the scaled transition time $T = 1$ is given by

$$y_d(t) = \begin{cases} 0, & t < 0 \\ 3t^2 - 2t^3, & 0 \le t \le 1 \\ 1, & 1 < t \end{cases} \tag{9}$$

i.e., a piecewise polynomial function of the lowest possible degree while still assuring continuous differentiability of the solution. Note that equivalently to (8), the series

$$\tilde{x}^{[j]}(z_{[j]}, t_{[j]}) = \sum_{n=0}^{\infty} \tilde{x}^{[j]}_n(t_{[j]}) \frac{z^n_{[j]}}{n!} \tag{10}$$

can be used. However, the choice (8) is motivated by the fact that the desired trajectory (9) allows the explicite computation of the coefficients $\hat{x}^{[j]}_{n,i}$ as shown in the following.

### B. Modification of the PDE

Power series can only provide the solution, if the solution is an analytical function on the region where the approach is used. Therefore, $|x_t(0,t)|$ in (5) is replaced by $x_t(0,t)$ to eliminate the non–analytical character of the PDE. The modified boundary condition then reads

$$\theta x_{tt}(0,t) = GI x_z(0,t) - \alpha \left[x_t(0,t)\right]^2 . \tag{11}$$

This modification is admissible if $x_t(0,t) \ge 0$ for all $t \in \mathbb{R}$, which can be guaranteed by adequately choosing the desired trajectory as done in (9). Obviously, the case $x_t(0,t) \le 0$ can be treated in an analogous way. In case of a change of sign in $x_t(0,t)$, however, a further subdivision of the transition region is needed in addition to the structural subdivision that is presented in the following section. Due to the rising complexity of the presentation, this case is not considered in this contribution.

### C. Subdivision of the Transition Region

For parabolic and certain first order hyperbolic PDEs the solution is analytical in $z$ on the entire transition region or can at least be continued to an analytical function in $z$ if the desired trajectory is adequately chosen. For second order hyperbolic systems this is not the case. The only analyticity property that can be achieved by an adequately chosen trajectory is piecewise analyticity in both $z$ and $t$. Since a power series approach can only provide the solution if the solution is analytical, $\mathcal{R}$ must be subdivided into regions $\mathcal{R}_{[j]}$ on which the solution is analytical. The suitable subdivision is given by the characteristic curves passing through the non–analyticity points $t = 0$ and $t = 1$ of the desired trajectory. The ordinary differential equation describing the characteristic curves reads as

$$\det \begin{pmatrix} \rho & 0 & -G \\ \mathrm{d}t & \mathrm{d}z & 0 \\ 0 & \mathrm{d}t & \mathrm{d}z \end{pmatrix} = 0, \tag{12}$$

and, using the abbreviation $\nu = \sqrt{G/\rho}$, yields two families of characteristic curves given by

$$\gamma_{1,t_0} : \ t = -\frac{1}{\nu} z + t_0, \tag{13}$$

$$\gamma_{2,t_0} : \ t = +\frac{1}{\nu} z + t_0, \tag{14}$$

where $(0, t_0)$ is the point in the $(z,t)$–plane at which the characteristic curves intersect the $t$-axis. The characteristic curves passing through the non–analyticity points $(0,0)$ and

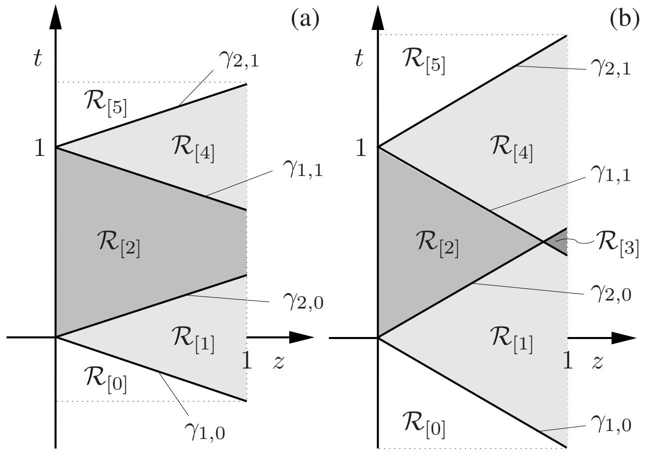


Fig. 1. Illustration of the characteristic curves of the wave equation modelling a torsional rod and subdivision of $\mathcal{R}$. (a) Subdivision for the case that $\gamma_{2,0}(z)$ and $\gamma_{1,1}(z)$ do not intersect on $\mathcal{R}$. (b) Subdivision for the case that $\gamma_{2,0}(z)$ and $\gamma_{1,1}(z)$ intersect on $\mathcal{R}$.

$(0,1)$ in the $(z,t)$–plane define the regions $\mathcal{R}_{[j]}$ as depicted in Fig. 1. Which of the depicted schemes applies depends on whether the two characteristic curves $\gamma_{2,0}(z)$ and $\gamma_{1,1}(z)$ do or do not intersect on $\mathcal{R}$. The regions $\mathcal{R}_{[j]}$ are defined by

$$\begin{aligned}
&\mathcal{R}_{[0]}:=\{(z,t)\in\mathcal{R} \mid t<\gamma_{1,0}(z)\},\\
&\mathcal{R}_{[1]}:=\{(z,t)\in\mathcal{R} \mid \gamma_{1,0}(z)<t<\min(\gamma_{2,0}(z),\gamma_{1,1}(z))\},\\
&\mathcal{R}_{[2]}:=\{(z,t)\in\mathcal{R} \mid \gamma_{2,0}(z)<t<\gamma_{1,1}(z)\},\\
&\mathcal{R}_{[3]}:=\{(z,t)\in\mathcal{R} \mid \gamma_{1,1}(z)<t<\gamma_{2,0}(z)\},\\
&\mathcal{R}_{[4]}:=\{(z,t)\in\mathcal{R} \mid \max(\gamma_{1,1}(z),\gamma_{2,0}(z))<t<\gamma_{2,1}(z)\},\\
&\mathcal{R}_{[5]}:=\{(z,t)\in\mathcal{R} \mid \gamma_{2,1}(z)<t\}.
\end{aligned} \tag{15}$$

To find the solution by applying a formal power series approach, the formulation of individual power series $\hat{x}^{[j]}(z_{[j]},t_{[j]})$ as defined in (8) is required on each of the regions $\mathcal{R}_{[j]}$, $j=0,\dots,5$.

## IV. Formal Power Series Solution

The solution process requires the individual subregions $\mathcal{R}_{[j]}$ to be considered in a certain order: First, the steady state solutions on $\mathcal{R}_{[0]}$ and $\mathcal{R}_{[5]}$, respectively, are determined. Then, the solution is calculated on $\mathcal{R}_{[2]}$, which includes the region on which the desired trajectory is defined, followed by the regions $\mathcal{R}_{[1]}$ and $\mathcal{R}_{[4]}$, which are adjacent to $\mathcal{R}_{[2]}$. Finally, if the characteristic curves $\gamma_{2,0}(z)$ and $\gamma_{1,1}(z)$ intersect on $\mathcal{R}$, the solution is determined on $\mathcal{R}_{[3]}$.

### A. *Solution on $\mathcal{R}_{[0]}$ and $\mathcal{R}_{[5]}$*

On $\mathcal{R}_{[0]}$ and $\mathcal{R}_{[5]}$, the solution is the trivial steady state solution $x^{[0]}(z_{[0]},t_{[0]})=0$ and $x^{[5]}(z_{[5]},t_{[5]})=1$, respectively, with $z_{[0]}:=z$, $t_{[0]}:=t$, $z_{[5]}:=z$, and $t_{[5]}:=t$. This can be verified by inserting $x^{[j]}(z_{[j]},t_{[j]})$, $j=0,5$ with $u_{[0]}(t)=0$ and $u_{[5]}(t)=1$, respectively, into (1)–(6).

### B. *Solution on $\mathcal{R}_{[2]}$*

On $\mathcal{R}_{[2]}$, the adequate coordinates are given by $z_{[2]}:=z$, $t_{[2]}:=t$. Inserting (8) with $j=2$ into the PDE (1) and using formal differentiation results in

$$\rho\sum_{n,i=0}^{\infty}\hat{x}^{[2]}_{n,i+2}\frac{t^i_{[2]}}{i!}\frac{z^n_{[2]}}{n!}=G\sum_{n,i=0}^{\infty}\hat{x}^{[2]}_{n+2,i}\frac{t^i_{[2]}}{i!}\frac{z^n_{[2]}}{n!}. \tag{16}$$

Using the output relation (6) and setting $y(t)=y_d(t)$ as well as the modified boundary condition (11) and equating terms of equal power in $z_{[2]}$ and $t_{[2]}$ yields the recurrence

$$\hat{x}^{[2]}_{0,i}=\left.\frac{\mathrm{d}^i}{\mathrm{d}t^i_{[2]}}y_d(t)\right|_{t_{[2]}=0}=\left.\frac{\mathrm{d}^i}{\mathrm{d}t^i}y_d(t)\right|_{t=0}, \tag{17}$$

$$\hat{x}^{[2]}_{1,i}=\frac{\theta}{GI}\hat{x}^{[2]}_{0,i+2}+\frac{\alpha}{GI}\sum_{k=0}^{i}\binom{i}{k}\hat{x}^{[2]}_{0,k+1}\hat{x}^{[2]}_{0,i-k+1}, \tag{18}$$

$$\hat{x}^{[2]}_{n+2,i}=\frac{\rho}{G}\hat{x}^{[2]}_{n,i+2}, \tag{19}$$

for $n,i\geq 0$. Since $\mathcal{R}_{[2]}$ and $\mathcal{R}_{[0]}$ coincide in $(z,t)=(0,0)$, the desired trajectory $y_d(t)$ must fulfill the initial conditions (2) and (3). The solution to the feedforward control problem is then given by

$$u^{[2]}(t)=\hat{x}^{[2]}(1,t_{[2]})=\sum_{n,i}^{\infty}\hat{x}^{[2]}_{n,i}\frac{t^i}{i!n!}. \tag{20}$$

For piecewise polynomial trajectories, the series is finite in $n$ and $i$. The solution is therefore exact and no approximation is needed. The coefficients for the trajectory defined in (9) are given in Table I.

### C. *Solution on $\mathcal{R}_{[1]}$*

Having determined the solution on $\mathcal{R}_{[0]}$ and $\mathcal{R}_{[2]}$, the next step consists of finding the solution on the region in between, i.e., on $\mathcal{R}_{[1]}$. This region is bounded by the curves $\gamma_{1,0}(z)$, $\gamma_{2,0}(z)$, and the line $z=1$, along which the input must be calculated. A diffeomorphic transformation into characteristic coordinates is given by

$$z_{[1]}=\frac{1}{\nu}z_{[2]}-t_{[2]},\qquad z_{[2]}=\frac{\nu}{2}\left(z_{[1]}+t_{[1]}\right), \tag{21}$$

$$t_{[1]}=\frac{1}{\nu}z_{[2]}+t_{[2]},\qquad t_{[2]}=\frac{1}{2}\left(-z_{[1]}+t_{[1]}\right). \tag{22}$$

Since the solution is know on $\mathcal{R}_{[0]}$ and $\mathcal{R}_{[2]}$, it is known along $\gamma_{1,0}(z)$ and $\gamma_{2,0}(z)$, respectively. In order to apply the formal power series approach on $\mathcal{R}_{[1]}$, (1) needs to be transformed into the characteristic coordinates $z_{[1]}$ and $t_{[1]}$. By setting $x^{[1]}(z_{[1]},t_{[1]}):=x(z,t)$, the PDE (1) can be transformed into

$$x^{[1]}_{z_{[1]}t_{[1]}}=0. \tag{23}$$

TABLE I

Coefficients $\hat{x}^{[2]}_{n,i}$ on $\mathcal{R}_{[2]}$ for the trajectory defined in (9). All remaining coefficients are 0.

| $\hat{x}_{n,i}$ | $n=0$ | $n=1$ | $n=2$ | $n=3$ | $n=4$ | $n=5$ |
|---|---|---|---|---|---|---|
| $i=0$ | 0 | $6\frac{\theta}{GI}$ | $6\frac{\rho}{G}$ | $72\frac{\alpha\rho}{G^2I}$ | 0 | $864\frac{\alpha\rho^2}{G^3I}$ |
| $i=1$ | 0 | $-12\frac{\theta}{GI}$ | $-12\frac{\rho}{G}$ | $-432\frac{\alpha\rho}{G^2I}$ | 0 | 0 |
| $i=2$ | 6 | $72\frac{\alpha}{GI}$ | 0 | $864\frac{\alpha\rho}{G^2I}$ | 0 | 0 |
| $i=3$ | $-12$ | $-432\frac{\alpha}{GI}$ | 0 | 0 | 0 | 0 |
| $i=4$ | 0 | $864\frac{\alpha}{GI}$ | 0 | 0 | 0 | 0 |

Initial and boundary conditions follow from the known solutions along $\gamma_{1,0}(z)$, which determines $x^{[1]}(z_{[1]},0)$, and $\gamma_{2,0}(z)$, which defines $x^{[1]}(0,t_{[1]})$. Finally, (4) also needs to be transformed into the characteristic coordinates, thus yielding the complete set of conditions

$$x^{[1]}(z_{[1]},0) = \hat{x}^{[0]}(z_{[0]}(z_{[1]},0), t_{[0]}(z_{[1]},0)), \tag{24}$$

$$x^{[1]}(0,t_{[1]}) = \hat{x}^{[2]}(z_{[2]}(0,t_{[1]}), t_{[2]}(0,t_{[1]})), \tag{25}$$

$$u^{[1]}(t) = x^{[1]}(z_{[1]}(1,t), t_{[1]}(1,t)). \tag{26}$$

The solution on $\mathcal{R}_{[1]}$ is represented as a formal power series as given in (8) with $j=1$ in the transformed coordinates $z_{[1]}$ and $t_{[1]}$. The initial and boundary conditions (24) and (25) can be used to determine the coefficients $\hat{x}^{[1]}_{n,i}$ for $n=0$ and $i=0$, respectively. In order to do so, consider the solution along the characteristic curve $\gamma_{1,0}(z)$ expressed as a formal power series on $\mathcal{R}_{[0]}$ and $\mathcal{R}_{[1]}$ as well as along $\gamma_{2,0}(z)$ expressed as a formal power series on $\mathcal{R}_{[2]}$ and $\mathcal{R}_{[1]}$. Since the solution must be continuous, the expressions need to be equal, which yields

$$\sum_{n,i=0}^{\infty} \hat{x}^{[1]}_{n,i} \frac{t_{[1]}^i}{i!}\frac{z_{[1]}^n}{n!}\Bigg|_{\substack{t_{[1]}=0\\ z_{[1]}=-2t_{[0]}}} = \sum_{n=0}^{\infty} \hat{x}^{[1]}_{n,0}\frac{(-2t_{[0]})^n}{n!} = 0 \tag{27}$$

for $\gamma_{1,0}(z)$ and

$$\begin{aligned}\sum_{n,i=0}^{\infty} \hat{x}^{[1]}_{n,i} \frac{t_{[1]}^i}{i!}\frac{z_{[1]}^n}{n!}\Bigg|_{\substack{z_{[1]}=0\\ t_{[1]}=2t_{[2]}}} &= \sum_{i=0}^{\infty} \hat{x}^{[1]}_{0,i}\frac{(2t_{[2]})^i}{i!}\\ = \sum_{n,i=0}^{\infty} \hat{x}^{[2]}_{n,i} \frac{t_{[2]}^i}{i!}\frac{z_{[2]}^n}{n!}\Bigg|_{z_{[2]}=\nu t_{[2]}} &= \sum_{n,i=0}^{\infty} \hat{x}^{[2]}_{n,i}\frac{\nu^n t_{[2]}^{i+n}}{i!n!}\end{aligned} \tag{28}$$

for $\gamma_{2,0}(z)$. Equating coefficients of equal power in $t_{[0]}$ and $t_{[2]}$ in (27) and (28), respectively, determines the coefficients

$$\hat{x}^{[1]}_{n,0} = 0, \qquad n \geq 0, \tag{29}$$

$$\hat{x}^{[1]}_{0,i} = \frac{1}{2^i}\sum_{k=0}^{i}\binom{i}{k}\hat{x}^{[2]}_{k,i-k}\nu^k, \quad i \geq 0, \tag{30}$$

in terms of the coefficients $\hat{x}^{[0]}_{n,i}$ and $\hat{x}^{[2]}_{n,i}$, and therefore constitutes the transition conditions. Formally differentiating (8) with $j=1$ with respect to $z_{[1]}$ and $t_{[1]}$, inserting the derivatives into (23) and equating coefficients of equal power in $z_{[1]}$ and $t_{[1]}$ provides the remaining coefficients via the recurrence

$$\hat{x}^{[1]}_{n+1,i+1} = 0. \tag{31}$$

Since by the choice of the trajectory defined in (9), $\hat{x}^{[2]}(z_{[2]},t_{[2]})$ is a finite series, it directly results from (29)–(31) that $\hat{x}^{[1]}(z_{[1]},t_{[1]})$ is also finite. Evaluating (26) yields the solution

$$\hat{u}^{[1]}(t) = \sum_{n,i=0}^{\infty}\hat{x}^{[1]}_{n,i}\frac{\left(\frac{1}{\nu}+t\right)^i}{i!}\frac{\left(\frac{1}{\nu}-t\right)^n}{n!} \tag{32}$$

for the system input. The coefficients $\hat{x}^{[1]}_{n,i}$ for the trajectory defined in (9) are given in Table II.

TABLE II
COEFFICIENTS $\hat{x}^{[1]}_{n,i}$ ON $\mathcal{R}_{[1]}$ FOR THE TRAJECTORY DEFINED IN (9). ALL REMAINING COEFFICIENTS ARE 0.

| $\hat{x}^{[1]}_{n,i}$ | $n=0$ |
|---|---|
| $i=0$ | $0$ |
| $i=1$ | $3\frac{\theta\nu}{GI}$ |
| $i=2$ | $3-6\frac{\theta\nu}{GI}$ |
| $i=3$ | $-6+36\frac{\alpha\nu}{GI}$ |
| $i=4$ | $-216\frac{\alpha\nu}{GI}$ |
| $i=5$ | $432\frac{\alpha\nu}{GI}$ |

## D. Solution on $\mathcal{R}_{[4]}$

The solution on $\mathcal{R}_{[4]}$ is calculated in an analogous way as the solution on $\mathcal{R}_{[1]}$. Since the transformed coordinate axes must coincide with the characteristic curves $\gamma_{1,1}(z)$ and $\gamma_{2,1}(z)$, the coordinates are defined as

$$z_{[4]} = \frac{1}{\nu}z_{[2]}-(t_{[2]}-1), \quad z_{[2]} = \frac{\nu}{2}\left(z_{[4]}+t_{[4]}\right), \tag{33}$$

$$t_{[4]} = \frac{1}{\nu}z_{[2]}+(t_{[2]}-1), \quad t_{[2]} = \frac{1}{2}\left(-z_{[4]}+t_{[4]}\right)+1. \tag{34}$$

Applying the formal power series approach (8) with $j=4$ along $\gamma_{2,1}(z)$ and equating the series to the corresponding solution on $\mathcal{R}_{[5]}$ yields

$$\sum_{n,i=0}^{\infty} \hat{x}^{[4]}_{n,i} \frac{t_{[4]}^i}{i!}\frac{z_{[4]}^n}{n!}\Bigg|_{\substack{z_{[4]}=0\\ t_{[4]}=2(t_{[5]}-1)}} = \sum_{i=0}^{\infty} \hat{x}^{[4]}_{0,i}\frac{(2(t_{[5]}-1))^i}{i!} = 1 \tag{35}$$

and therefore determines the coefficients

$$\hat{x}^{[4]}_{0,0} = 1, \qquad \hat{x}^{[4]}_{0,i} = 0, \quad i \geq 1. \tag{36}$$

Following the same approach along $\gamma_{1,1}(z)$, where $t_{[4]}=0$ produces the relation $z_{[2]}=\nu(1-t_{[2]})$ and $z_{[4]}=2(1-t_{[2]})$, we find

$$\begin{aligned}\sum_{n,i=0}^{\infty} \hat{x}^{[4]}_{n,i} \frac{t_{[4]}^i}{i!}\frac{z_{[4]}^n}{n!}\Bigg|_{\substack{z_{[4]}=2(1-t_{[2]})\\ t_{[4]}=0}} &= \sum_{n=0}^{\infty} \hat{x}^{[4]}_{n,0}\frac{(2(1-t_{[2]}))^n}{n!}\\ = \sum_{n,i=0}^{\infty} \hat{x}^{[2]}_{n,i} \frac{t_{[2]}^i}{i!}\frac{z_{[2]}^n}{n!}\Bigg|_{z_{[2]}=\nu(1-t_{[2]})} &= \sum_{n,i=0}^{\infty} \hat{x}^{[2]}_{n,i}\frac{t_{[2]}^i}{i!}\frac{(\nu(1-t_{[2]})^n}{n!}.\end{aligned} \tag{37}$$

Here, a straightforward comparison of powers in $t_{[2]}$ is not possible, but requires the application of a binomial formula on $(1-t_{[2]})^n$. Instead of following this option, the solution on $\mathcal{R}_{[2b]} := \mathcal{R}_{[2]}$ can also be recalculated using a series expansion in $t_{[2b]} := (t_{[2]}-1)$ and $z_{[2b]} := z_{[2]}$, thus permitting a direct comparison of powers in $t_{[2b]}$. Although both approaches are basically equivalent, the latter is preferred due to the fact that it offers advantages with respect to convergence in the case when the series involved is infinite. The formal power series is defined by (8) with $j=2b$ and $z_{[2b]} := z$. Using the output relation (6) as well as the boundary condition (11) yields the recurrence

$$\hat{x}_{0,i}^{[2b]} = \left. \frac{\mathrm{d}^i}{\mathrm{d}t_{[2b]}^i} y(t) \right|_{t_{[2b]}=0} = \left. \frac{\mathrm{d}^i}{\mathrm{d}t^i} y(t) \right|_{t=1}, \tag{38}$$

$$\hat{x}_{1,i}^{[2b]} = \frac{\theta}{GI} \hat{x}_{0,i+2}^{[2b]} + \frac{\alpha}{GI} \sum_{j=0}^{i} \binom{i}{j} \hat{x}_{0,j+1}^{[2b]} \hat{x}_{0,i-j+1}^{[2b]}, \tag{39}$$

$$\hat{x}_{n+2,i}^{[2b]} = \frac{\rho}{G} \hat{x}_{n,i+2}^{[2b]}, \tag{40}$$

for $n, i \geq 0$. The coefficients for the trajectory defined in (9) are given in Table III. The required coordinate transformation for $\mathcal{R}_{[4]}$ based on the coordinates $z_{[2b]}$ and $t_{[2b]}$ are given by

$$z_{[4]} = \frac{1}{\nu} z_{[2b]} - t_{[2b]}, \qquad z_{[2b]} = \frac{\nu}{2} \left( z_{[4]} + t_{[4]} \right), \tag{41}$$

$$t_{[4]} = \frac{1}{\nu} z_{[2b]} + t_{[2b]}, \qquad t_{[2b]} = \frac{1}{2} \left( -z_{[4]} + t_{[4]} \right). \tag{42}$$

The transformed PDE reads as

$$x_{z_{[4]} t_{[4]}}^{[4]} = 0 \tag{43}$$

with initial and boundary conditions analogous to (24)–(26) With this approach, a direct comparison of powers in $t_{[2b]}$ is possible. Writing

$$\begin{aligned} \sum_{n,i=0}^{\infty} \hat{x}_{n,i}^{[4]} \frac{t_{[4]}^i}{i!} \frac{z_{[4]}^n}{n!} \Bigg|_{\substack{z_{[4]}=-2t_{[2b]} \\ t_{[4]}=0}} &= \sum_{n=0}^{\infty} \hat{x}_{n,0}^{[4]} \frac{(-2t_{[2b]})^n}{n!} \\ = \sum_{n,i=0}^{\infty} \hat{x}_{n,i}^{[2b]} \frac{t_{[2b]}^i}{i!} \frac{z_{[2b]}^n}{n!} \Bigg|_{\substack{z_{[2b]}= \\ -\nu t_{[2b]}}} &= \sum_{n,i=0}^{\infty} \hat{x}_{n,i}^{[2b]} \frac{(-\nu)^n t_{[2b]}^{i+n}}{i! n!}, \end{aligned} \tag{44}$$

we obtain

$$\hat{x}_{n,0}^{[4]} = \frac{1}{(-2)^n} \sum_{k=0}^{n} \binom{n}{k} \hat{x}_{n-k,k}^{[2b]} (-\nu)^{n-k}, \tag{45}$$

$$\hat{x}_{n+1,i+1}^{[4]} = 0 \tag{46}$$

for $n \geq 0$ and $i \geq 0$. Since $\hat{x}^{[2b]}(z_{[2b]}, t_{[2b]})$ is by the choice of a trajectory as defined in (9) a finite series, it directly results from (36), (45)–(46) that $\hat{x}^{[4]}(z_{[4]}, t_{[4]})$ is also finite. The solution for the input is given by

$$\hat{u}^{[4]}(t) = \sum_{n,i=0}^{\infty} \hat{x}_{n,i}^{[4]} \frac{\left(\frac{1}{\nu} + t - 1\right)^i}{i!} \frac{\left(\frac{1}{\nu} - t + 1\right)^n}{n!}. \tag{47}$$

The coefficients $\hat{x}_{n,i}^{[4]}$ for the trajectory defined in (9) are given in Table IV.

TABLE III

COEFFICIENTS $\hat{x}_{n,i}^{[2b]}$ ON $\mathcal{R}_{[2b]}$ FOR THE TRAJECTORY DEFINED IN (9). ALL REMAINING COEFFICIENTS ARE 0.

| $\hat{x}_{n,i}$ | $n=0$ | $n=1$ | $n=2$ | $n=3$ | $n=4$ | $n=5$ |
|---|---|---|---|---|---|---|
| $i=0$ | $1$ | $-6\frac{\theta}{GI}$ | $-6\frac{\rho}{G}$ | $72\frac{\alpha\rho}{G^2 I}$ | $0$ | $864\frac{\alpha\rho^2}{G^3 I}$ |
| $i=1$ | $0$ | $-12\frac{\theta}{GI}$ | $-12\frac{\rho}{G}$ | $432\frac{\alpha\rho}{G^2 I}$ | $0$ | $0$ |
| $i=2$ | $-6$ | $72\frac{\alpha}{GI}$ | $0$ | $864\frac{\alpha\rho}{G^2 I}$ | $0$ | $0$ |
| $i=3$ | $-12$ | $432\frac{\alpha}{GI}$ | $0$ | $0$ | $0$ | $0$ |
| $i=4$ | $0$ | $864\frac{\alpha}{GI}$ | $0$ | $0$ | $0$ | $0$ |

TABLE IV

COEFFICIENTS $\hat{x}_{n,i}^{[4]}$ ON $\mathcal{R}_{[4]}$ FOR THE TRAJECTORY DEFINED IN (9). ALL REMAINING COEFFICIENTS ARE 0.

| $\hat{x}_{n,i}^{[4]}$ | $n=0$ | $n=1$ | $n=2$ | $n=3$ | $n=4$ | $n=5$ |
|---|---|---|---|---|---|---|
| $i=0$ | $1$ | $-3\frac{\theta\nu}{GI}$ | $6\frac{\theta\nu}{GI} - 3$ | $6 + 36\frac{\alpha\nu}{GI}$ | $-216\frac{\alpha\nu}{GI}$ | $432\frac{\alpha\nu}{GI}$ |

### E. Solution on $\mathcal{R}_{[3]}$

The solution on $\mathcal{R}_{[3]}$, if the region $\mathcal{R}_{[3]}$ exists, is particularly simple. The adequate coordinates are $z_{[3]} := z_{[1]}$ and $t_{[3]} := t_{[1]} - 1$. Since the solution on $\mathcal{R}_{[1]}$ depends only on $t_{[1]}$, and since $t_{[1]}$ is constant along the part of $\gamma_{1,1}(z)$ that belongs to $\mathcal{R}_{[1]}$ and $\mathcal{R}_{[3]}$, the solution on $\mathcal{R}_{[3]}$ is constant along $\gamma_{1,1}(z)$. Evaluating the formal power series $\hat{x}^{[1]}(z_{[1]}, t_{[1]})$ along $\gamma_{1,1}(z)$ yields

$$\hat{x}^{[1]}(z_{[1]}, t_{[1]}) \Big|_{t_{[1]}=1} = \hat{x}^{[3]}(z_{[3]}, t_{[3]}) \Big|_{t_{[3]}=0} = \frac{1}{2} + \frac{3}{5} \frac{\alpha\nu}{GI}. \tag{48}$$

An analogous statement holds true on $\mathcal{R}_{[4]}$. Here, the solution along $\gamma_{2,0}(z)$ only depends on $z_{[4]}$, and $z_{[4]}$ is constant along the part of $\gamma_{2,0}(z)$ that belongs to $\mathcal{R}_{[3]}$ and $\mathcal{R}_{[4]}$. Therefore, the solution is also constant on the part of $\gamma_{2,0}(z)$ that belongs to $\mathcal{R}_{[3]}$. It evaluates to

$$\hat{x}^{[4]}(z_{[4]}, t_{[4]}) \Big|_{z_{[4]}=1} = \hat{x}^{[3]}(z_{[3]}, t_{[3]}) \Big|_{z_{[3]}=0} = \frac{1}{2} + \frac{3}{5} \frac{\alpha\nu}{GI}, \tag{49}$$

and is therefore equal to the solution along the part of $\gamma_{1,1}(z)$ that belongs to $\mathcal{R}_{[3]}$. Since, in addition to (48) and (49), the PDE reads

$$x_{z_{[3]} t_{[3]}}^{[3]} = 0 \tag{50}$$

the solution neither depends on $z_{[3]}$ nor on $t_{[3]}$ and is therefore constant on all of $\mathcal{R}_{[3]}$, i.e.,

$$\hat{x}^{[3]}(z_{[3]}, t_{[3]}) = \frac{1}{2} + \frac{3}{5} \frac{\alpha\nu}{GI}. \tag{51}$$

The input on $\mathcal{R}_{[3]}$ follows directly as

$$\hat{u}^{[3]}(t) = \frac{1}{2} + \frac{3}{5} \frac{\alpha\nu}{GI}. \tag{52}$$

### F. Solution for the input $u(t)$

Summarizing, the complete formal power series solution to the feedforward control problem is given by

$$\hat{u}(t) = \begin{cases} \hat{u}^{[0]}(t), & t \leq -\frac{1}{\nu} \\ \hat{u}^{[1]}(t), & -\frac{1}{\nu} < t \leq \frac{1}{\nu} \\ \hat{u}^{[2]}(t), & \frac{1}{\nu} < t \leq 1 - \frac{1}{\nu} \\ \hat{u}^{[4]}(t), & 1 - \frac{1}{\nu} < t \leq 1 + \frac{1}{\nu} \\ \hat{u}^{[5]}(t), & 1 + \frac{1}{\nu} < t \end{cases} \tag{53}$$

for the case of non–intersecting characteristic curves $\gamma_{2,0}(z)$ and $\gamma_{1,1}(z)$, and

$$\hat{u}(t) = \begin{cases} \hat{u}^{[0]}(t), & t \leq -\frac{1}{\nu} \\ \hat{u}^{[1]}(t), & -\frac{1}{\nu} < t \leq 1 - \frac{1}{\nu} \\ \hat{u}^{[3]}(t), & 1 - \frac{1}{\nu} < t \leq \frac{1}{\nu} \\ \hat{u}^{[4]}(t), & \frac{1}{\nu} < t \leq 1 + \frac{1}{\nu} \\ \hat{u}^{[5]}(t), & 1 + \frac{1}{\nu} < t \end{cases} \tag{54}$$

for the case of intersecting characteristic curves.

## V. Simulation Results

For the simulation, a finite difference scheme with spatial and time discretization $\Delta z = 0.005$ and $\Delta t = 0.001$ is used for the case of non–intersecting trajectories $\gamma_{2,0}(z)$ and $\gamma_{1,1}(z)$, whereas $\Delta z = 0.002$ and $\Delta t = 0.002$ are used for the case of intersecting ones. The results in Fig. 2 and Fig. 3 confirm the perfect tracking by plotting the desired trajectory $y_d(t)$, the calculated input $\hat{u}(t)$ using the formal power series approach as well as the output $y(t)$ determined by numerical simulation of the system (1)–(5) with the input $\hat{u}(t)$.

Fig. 3 illustrates the case when the delay in the system is equal to the chosen transition time. In this case, all the energy necessary to move the tip load along the desired trajectory must be put into the system before the tip load starts moving. During the motion of the tip load, the input remains constant. When the tip load reaches its final position, the excess energy needs to be taken out of the system by an appropriate action on the input to prevent the tip load from swinging.

## VI. Summary

This paper demonstrates the applicability of the formal power series approach to the trajectory planning and feedforward control design task for a second order hyperbolic PDE illustrated for the wave equation with nonlinear boundary conditions modelling a torsional rod with a tip load. We proved that the solution to a piecewise polynomial desired trajectory is analytical on each of the subregions of the transition region separated by the characteristic curves passing through the non–analyticity points of the desired trajectory.

Future research will be directed towards the feedforward control of a nonlinear wave equation that results in infinite series, as well as various other second order hyperbolic systems like the telegraph equation, the heavy chain, and heat exchangers, both in their linear and nonlinear variants.

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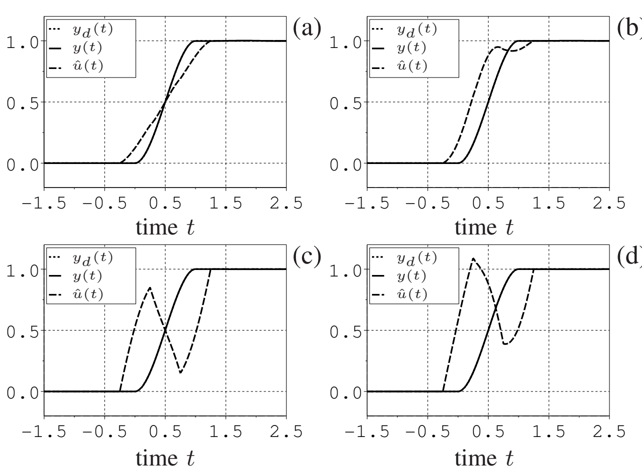


Fig. 2. Simulation results for the case that $\gamma_{2,0}(z)$ and $\gamma_{1,1}(z)$ do not intersect on $\mathcal{R}$ with the fixed parameters $L = 1$, $\rho = 1$, $G = 16$, and $I = \frac{1}{16}$. (a) Solution for the linear case $\alpha = 0$ and a tip load with $\theta = 0.02$. (b) Solution for the nonlinear case $\alpha = 0.2$ and a tip load with $\theta = 0.02$. (c) Solution for the linear case $\alpha = 0$ and a heavy tip load with $\theta = 0.2$. (d) Solution for the nonlinear case $\alpha = 0.2$ and a heavy tip load with $\theta = 0.2$.

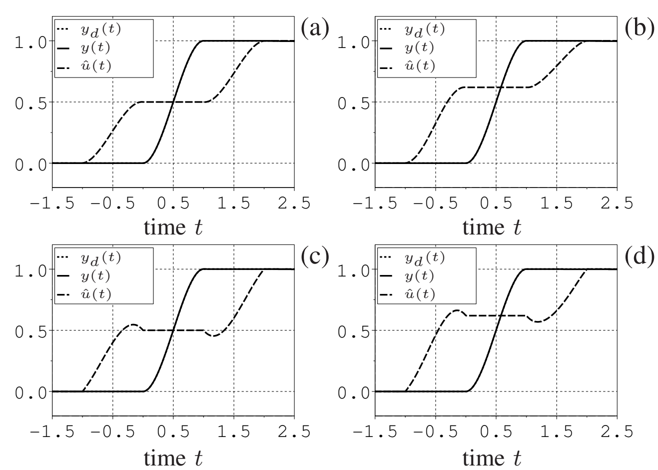


Fig. 3. Simulation results for the case that $\gamma_{2,0}(z)$ and $\gamma_{1,1}(z)$ intersect on $\mathcal{R}$ with the fixed parameters $L = 1$, $\rho = 1$, $G = 1$, and $I = 1$. (a) Solution for the linear case $\alpha = 0$ and a tip load with $\theta = 0.02$. (b) Solution for the nonlinear case $\alpha = 0.2$ and a tip load with $\theta = 0.02$. (c) Solution for the linear case $\alpha = 0$ and a heavy tip load with $\theta = 0.2$. (d) Solution for the nonlinear case $\alpha = 0.2$ and a heavy tip load with $\theta = 0.2$.