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On the Choice of an Optimal Interpolation Point in Krylov-Based Order Reduction

Behnam Salimbahrami, *Member, IEEE*, Rudy Eid, *Student Member, IEEE* and Boris Lohmann, *Member, IEEE*

Abstract— **In this paper, the well-known problem of finding a suitable interpolation point in order reduction via moment matching by Krylov subspaces is investigated. By using the equivalence property of moment matching and Laguerre-based order reduction, the problem is reformulated as finding the best choice for the free parameter α in the Laguerre basis. Minimizing appropriate cost functions with very few iterations is the key point toward finding this best interpolation point.**

I. INTRODUCTION

Due to the increasing complexity of physical systems in engineering and rising demand on higher modeling accuracy, the need for model order reduction is continuously growing. In order reduction of large-scale systems, moment matching by Krylov subspaces are among the best choices [1], [2].

Even though the reduced-order model is calculated, via a projection, in a relatively short time with a good numerical accuracy, the interpretation of these methods is restricted to being a local approximation of the frequency response of the original system. The frequency range of interest is determined by the so-called interpolation point about which the moments in the frequency-domain are matched. Consequently, this family of methods can not directly guarantee a good approximation of the impulse response, as it is quite hard in most practical cases, to predict the accuracy of the time-domain response of the reduced-order model from its frequency-domain one. It is then more natural to do order reduction directly in the time-domain, for instance, through the approximation of the impulse response.

Furthermore, the appropriate choice of the interpolation point in moment matching is not straightforward and is still an active field of research. In the literature, different choices for the single or multiple interpolation points have been presented, targeting different aims. In [3], the problem of passivity preserving order reduction has been addressed and a rational Krylov algorithm with interpolation points selected as spectral zeros of the original transfer function has been presented. In [4], an iteratively corrected rational Krylov algorithm for H_2 model reduction has been suggested.

Lately, several successful methods for approximating the impulse response using orthogonal polynomials have been proposed [5], [6], [7], [8], [9]. Among these approaches, the Laguerre-based reduction has shown to be very suitable for the reduction of large-scale systems as it can be reformulated (both in time and frequency domain) to benefit from the numerical and computational advantages of the Krylov subspace-based methods.

In order to optimize the approximation using the Laguerre basis functions, the choice of the Laguerre pole, also known as time-scale factor, is crucial. Numerous works treated this problem in system identification [10], approximation [11], [12], [13], [14], [15], [16], and signal processing [17].

In [18], [8] the equivalence between the Laguerre-based order reduction and moment matching, both in time- and frequency-domain, has been shown. Based on these results, the open problem of choosing an optimal expansion point in the rational Krylov subspace reduction methods (moment matching about $s_0 \neq 0$) can be reformulated to the problem of finding the optimal parameter α in the Laguerre-based reduction methods.

In this paper, it is first shown that the key parameter for the impulse response approximation of the original system can be calculated optimally in a closed-form by solving appropriate Lyapunov equations. Then, two methods for the choice of the optimal Laguerre parameter and consequently the single expansion point in rational interpolation order reduction are presented. Accordingly, different model reduction algorithms are suggested and their advantages and disadvantages are pointed out. The importance of these approaches lies in the fact that they try to minimize the effect of the higher order terms in the infinite Laguerre series expansions of the impulse response, and that they offer a time-domain interpretation of moment matching which is originally developed in frequency domain. In addition, the methods have a simple structure and are numerically efficient and thus suitable for the reduction of large-scale systems.

II. KRYLOV-BASED ORDER REDUCTION

Consider the stable Linear Time Invariant (LTI) system,

$$\Sigma : \begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \\ y = \mathbf{c}^T\mathbf{x}, \end{cases} \tag{1}$$

having the transfer function

$$H(s) = \mathbf{c}^T(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b},$$

with its moments about zero calculated as follows [2]:

$$\mathbf{m}_i = \mathbf{c}^T\mathbf{A}^{-i-1}\mathbf{b}, \qquad i = 0, 1, \cdots . \tag{2}$$

The aim of order reduction by Krylov-subspace methods is to find a reduced order model of order $q \ll n$, whose moments match some of those of the original one [1]. This family of methods is also known as moment matching.

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The authors are with Institute of Automatic Control, Technische Universität München, Boltzmann str. 15, D-85748 Garching, Germany, {salimbahrami, eid, lohmann}@mytum.de

A numerically robust and efficient way to calculate this reduced order model is based om applying a projection to the original model,

$$\Sigma_r : \begin{cases} \dot{\mathbf{x}}_r(t) = \mathbf{W}^T\mathbf{A}\mathbf{V}\mathbf{x}_r(t) + \mathbf{W}^T\mathbf{b}\mathbf{u}(t), \\ y(t) = \mathbf{c}^T\mathbf{V}\mathbf{x}_r(t), \end{cases} \tag{3}$$

by means of the so-called projection matrices, $\mathbf{V}$ and $\mathbf{W}$ with $\mathbf{W}^T\mathbf{V} = \mathbf{I}$. For the choice of the projection matrices, the Krylov subspace, defined in e.g., [2] is used,

$$\mathcal{K}_q(\mathbf{A}_1, \mathbf{b}_1) = span\{\mathbf{b}_1, \mathbf{A}_1\mathbf{b}_1, \cdots, \mathbf{A}_1^{q-1}\mathbf{b}_1\}$$

where $\mathbf{A}_1 \in \mathbb{R}^{n\times n}$, and $\mathbf{b}_1 \in \mathbb{R}^n$ is called the starting vector. Now, when the projection matrices are chosen such that,

$$colspan(\mathbf{V}) \subset \mathcal{K}_q\left((\mathbf{A} - s_0\mathbf{I})^{-1}, (\mathbf{A} - s_0\mathbf{I})^{-1}\mathbf{b}\right),$$
$$colspan(\mathbf{W}) \subset \mathcal{K}_q\left((\mathbf{A} - s_0\mathbf{I})^{-T}, (\mathbf{A} - s_0\mathbf{I})^{-T}\mathbf{c}^T\right),$$

$2q$ moments around s_0 match and the method is known as rational interpolation. As it is not simple to find an appropriate choice of s_0, in many applications $s_0 = 0$ is chosen which leads to a good approximation of the low-frequency behavior but does not necessarily lead to good results if the high or middle frequency behavior is of interest.

Note that in the so-called one-sided method, only one Krylov subspace is used with a common choice $\mathbf{W} = \mathbf{V}$ and only q moments match. For the numerical computation of the matrices $\mathbf{V}$ and $\mathbf{W}$, the known Lanczos or Arnoldi or one of their modified versions are employed. For more details, see e.g. [2], [19] and the references therein.

III. LAGUERRE-BASED ORDER REDUCTION AND THE EQUIVALENCE PROPERTY

The ith Laguerre polynomial is defined as,

$$l_i(t) = \frac{e^t}{i!}\frac{d^i}{dt^i}(e^{-t}t^i),\ i = 0, 1, \cdots$$

and the scaled Laguerre functions are,

$$\phi_i^\alpha(t) = \sqrt{2\alpha}e^{-\alpha t}l_i(2\alpha t),\ i = 0, 1, \cdots$$

where α is a positive scaling parameter called time-scale factor. As these functions form a uniformly bounded orthonormal basis for the Hilbert space $\mathcal{L}_2(R_+)$ [20], the impulse response of (1) can be written as

$$h(t) = \sum_{i=0}^{\infty} F_i\phi_i^\alpha(t).$$

The key idea of the time-domain Laguerre-based order reduction consists of projecting the state vector $\mathbf{x}$ in (1) onto the q-th order subspace spanned by the first q Laguerre functions [8]. This results in a reduced system with impulse response $h_r(t)$, whose Laguerre coefficients match some of the first coefficients of the original impulse response $h(t)$.

The reduced order system is obtained by applying the projection as shown in (3), however with the matrices $\mathbf{V}$ and $\mathbf{W}$ forming a basis for the subspace spanned by the columns of the Laguerre coefficients of the expansion of the states $\mathbf{x}$ of the system. In fact, it can be shown that the subspace in question is equal to the Krylov subspace,

$$colspan(\mathbf{V}) \subset \mathcal{K}_q\left((\mathbf{A} - \alpha\mathbf{I})^{-1}\mathbf{A}, (\mathbf{A} - \alpha\mathbf{I})^{-1}\mathbf{b}\right), \tag{4}$$
$$colspan(\mathbf{W}) \subset \mathcal{K}_q\left((\mathbf{A} - \alpha\mathbf{I})^{-T}\mathbf{A}^T, (\mathbf{A} - \alpha\mathbf{I})^{-T}\mathbf{c}^T\right). \tag{5}$$

Hence, the projection matrices needed in the Laguerre-based order reduction can be efficiently calculated using the Krylov-subspace machinery used for moment matching. This formulation shows the dependency of the reduction approach, and consequently the reduced system on the parameter α. Based on this fact, by changing this parameter, new basis functions ϕ_i^α are generated and different approximations of the impulse response of the original model are obtained.

Thus, there is a need of a method to calculate an α leading to good impulse response's approximations while satisfying some optimality conditions.

A. The Equivalence

Based on the work in [18], the Laguerre-based order reduction and the moment matching about a single interpolation point can be shown equivalent. Using the fact that the transfer function of the reduced-order model depends only on the choice of the Krylov subspaces and not on the bases of these subspaces, it is enough for the equivalence of both approaches to show that the Krylov subspaces involved in both methods are equal. This is in fact true as

$$\mathcal{K}_q\left((\mathbf{A} - \alpha\mathbf{I})^{-1}, \mathbf{v}\right) = \mathcal{K}_q\left((\mathbf{A} - \alpha\mathbf{I})^{-1}\mathbf{A}, \mathbf{v}\right),$$

for a starting vector $\mathbf{v}$; see [18] for a detailed discussion.

Theorem 1: Reducing a state space model in time-domain by matching the Laguerre coefficients of the impulse responses of the original and reduced models is exactly equivalent to matching the moments of their transfer functions around $s = \alpha$ in the frequency-domain.

In other words, if order reduction is carried out completely in time-domain to match some of the first Laguerre coefficients with a certain parameter α as proposed at the beginning of this section, the same number of moments around $s_0 = \alpha$ in the frequency-domain automatically match. Similarly, if order reduction is carried out completely in frequency-domain to match some of the first moments around s_0, the same number of the first Laguerre coefficients of the Laguerre series expansion of the impulse response with $\alpha = s_0$ automatically match.

B. Property of the Laguerre Function

The key point to investigate the Laguerre parameter is the differential equation that the Laguerre functions satisfy. It is well-known that the Laguerre polynomial $l_i(t)$ satisfies the following differential equation [20],

$$t\ddot{l}_i(t) + (1 - t)\dot{l}_i(t) + il_i(t) = 0.$$

Considering the Laguerre function and the variable $\tilde{t} = 2\alpha t$, the following relations hold,

$$l_i(\tilde{t}) = \frac{1}{\sqrt{2\alpha}}e^{\alpha t}\phi_i^\alpha(t)$$
$$\frac{d}{d\tilde{t}}l_i(\tilde{t}) = \frac{1}{2\alpha\sqrt{2\alpha}}e^{\alpha t}\left(\dot{\phi}_i^\alpha(t) + \alpha\phi_i^\alpha(t)\right)$$

$$\frac{d^2}{d\tilde{t}^2}l_i(\tilde{t}) = \frac{1}{4\alpha^2\sqrt{2\alpha}}e^{\alpha t}\left(\ddot{\phi}_i^\alpha(t) + 2\alpha\dot{\phi}_i^\alpha(t) + \alpha^2\phi_i^\alpha(t)\right)$$

Combining these equations with the following equation,

$$2\alpha t\ddot{l}_i(2\alpha t) + (1-2\alpha t)\dot{l}_i(2\alpha t) + il_i(2\alpha t) = 0,$$

leads to the differential equation that is satisfied by the Laguerre function,

$$\begin{aligned} &t\left(\ddot{\phi}_i^\alpha(t) + 2\alpha\dot{\phi}_i^\alpha(t) + \alpha^2\phi_i^\alpha(t)\right) + \\ &\qquad (1-2\alpha t)\left(\dot{\phi}_i^\alpha(t) + \alpha\phi_i^\alpha(t)\right) + 2\alpha i\phi_i^\alpha(t) = 0 \Rightarrow \\ &t\ddot{\phi}_i^\alpha(t) + \dot{\phi}_i^\alpha(t) - \alpha^2 t\phi_i^\alpha(t) + \alpha\phi_i^\alpha(t) + 2\alpha i\phi_i^\alpha(t) = 0 \Rightarrow \\ &-t\ddot{\phi}_i^\alpha(t) - \dot{\phi}_i^\alpha(t) + \alpha^2 t\phi_i^\alpha(t) = 2\alpha(i+\frac{1}{2})\phi_i^\alpha(t). \qquad (6) \end{aligned}$$

The differential equation (6) which is found by a direct calculation in time-domain is the same as in [13] where it was derived in the s-domain using the Laplace transform of the Laguerre function. This property will be involved in finding the optimal α in the following sections.

IV. RATIONAL KRYLOV WITH AN OPTIMAL INTERPOLATION POINT

Following the results in [13] and assuming that $h(t)$ is an impulse response of a stable system, one can write

$$\begin{aligned} &-\int_0^\infty th(t)\ddot{h}(t)dt - \int_0^\infty h(t)\dot{h}(t)dt + \alpha^2\int_0^\infty th^2(t)dt \\ &= \int_0^\infty \left[\sum_{i=0}^\infty F_i\left(-t\ddot{\phi}_i^\alpha(t) - \dot{\phi}_i^\alpha(t) + \alpha^2 t\phi_i^\alpha(t)\right)\right]h(t)dt \\ &= \sum_{i=0}^\infty\left[F_i 2\alpha(i+\frac{1}{2})\int_0^\infty \phi_i^\alpha(t)h(t)\right]dt = \sum_{i=0}^\infty 2F_i^2\alpha(i+\frac{1}{2}). \end{aligned}$$

Now define the cost function,

$$J(\alpha) = \sum_{i=0}^\infty iF_i^2(\alpha). \qquad (7)$$

When the first coefficients of the system's impulse response are used in order reduction, minimizing J with the weighting i for every coefficient F_i is very effective. In other words, applying the optimal α found by minimizing the cost function J leads to a meaningful result as it puts more weight on the coefficients with higher index. This property accelerates the convergence of the infinite sum of Laguerre functions by making the higher order terms less significant.

To calculate J, assume that $\lim_{t\to\infty} h(t) = 0$, $\lim_{t\to 0} h(t) < \infty$, $\lim_{t\to\infty} \dot{h}(t) < \infty$ and $\sum_{i=0}^\infty F_i^2 = \|h(t)\|_2^2 = \int_0^\infty h^2(t)dt$ and,

$$\int_0^\infty \dot{h}^2(t)tdt + \alpha^2\int_0^\infty h^2(t)tdt = 2\alpha J + \alpha\|h(t)\|_2^2.$$

Define,

$$M_1 = \frac{\int_0^\infty h^2(t)tdt}{\int_0^\infty h^2(t)dt}, M_2 = \frac{\int_0^\infty \dot{h}^2(t)tdt}{\int_0^\infty h^2(t)dt} \Rightarrow \qquad (8)$$

$$J = \|h(t)\|_2^2\frac{\alpha^2 M_1 + M_2}{2\alpha} - \frac{1}{2}\|h(t)\|_2^2. \qquad (9)$$

The optimal value of α can be found as follows,

$$\frac{dJ}{d\alpha} = \|h(t)\|_2^2\frac{2\alpha^2 M_1 - 2M_2}{4\alpha^2} = 0 \Rightarrow$$

$$\alpha^* = \sqrt{\frac{M_2}{M_1}}\ ,\ J^* = \|h(t)\|_2^2\left(\sqrt{M_2M_1} - \frac{1}{2}\right). \qquad (10)$$

The main question arising in this context is the calculation of the optimal parameter α in practice. Although the problem looks complicated in the general case, it may be easily solved for special classes of systems including LTI systems.

Let $h(t) = \mathbf{c}^T e^{\mathbf{A}t}\mathbf{b}$ be the impulse response (with zero initial condition) of the system (1). The square of the two-norm of this system is,

$$\|h(t)\|_2^2 = \int_0^\infty h^2(t)dt = \mathbf{c}^T\mathbf{X}\mathbf{c}.$$

where $\mathbf{X}$ is called the controllability gramian and satisfies the following Lyapunov equation,

$$\mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{A}^T + \mathbf{b}\mathbf{b}^T = \mathbf{0},\ \mathbf{X} = \int_0^\infty e^{\mathbf{A}t}\mathbf{b}\mathbf{b}^T e^{\mathbf{A}^T t}dt. \qquad (11)$$

Lemma 1: For system (1), the optimal parameter that minimizes the cost function (9) can be calculated as follows,

$$\alpha^* = \sqrt{\frac{\mathbf{c}^T\mathbf{A}\mathbf{Y}\mathbf{A}^T\mathbf{c}}{\mathbf{c}^T\mathbf{Y}\mathbf{c}}}, \qquad (12)$$

where $\mathbf{Y}$ is the solution of the Lyapunov equation,

$$\mathbf{A}\mathbf{Y} + \mathbf{Y}\mathbf{A}^T + \mathbf{X} = \mathbf{0}, \qquad (13)$$

and $\mathbf{X}$ is the controllability gramian.

Proof: Let us calculate M_1 for system (1),

$$\int_0^\infty h^2(t)tdt = \mathbf{c}^T\underbrace{\int_0^\infty e^{\mathbf{A}t}\mathbf{b}\mathbf{b}^T e^{\mathbf{A}^T t}tdt}_{\mathbf{Y}}\,\mathbf{c}.$$

Assuming that the system is stable and considering the Lyapunov equation (11), we have,

$$\begin{aligned} \mathbf{A}\mathbf{Y} &= \int_0^\infty \mathbf{A}e^{\mathbf{A}t}\mathbf{b}\mathbf{b}^T e^{\mathbf{A}^T t}tdt \\ &= e^{\mathbf{A}t}\mathbf{b}\mathbf{b}^T e^{\mathbf{A}^T t}t\Big]_0^\infty - \int_0^\infty e^{\mathbf{A}t}\mathbf{b}\mathbf{b}^T e^{\mathbf{A}^T t}dt \\ &\qquad - \int_0^\infty e^{\mathbf{A}t}\mathbf{b}\mathbf{b}^T e^{\mathbf{A}^T t}tdt\mathbf{A}^T = \mathbf{0} - \mathbf{X} - \mathbf{Y}\mathbf{A}^T. \end{aligned}$$

Therefore,

$$\mathbf{A}\mathbf{Y} + \mathbf{Y}\mathbf{A}^T + \mathbf{X} = \mathbf{0},\ M_1 = \frac{\mathbf{c}^T\mathbf{Y}\mathbf{c}}{\mathbf{c}^T\mathbf{X}\mathbf{c}}. \qquad (14)$$

To calculate M_2, we have,

$$\begin{aligned} \int_0^\infty \dot{h}^2(t)tdt &= \mathbf{c}^T\mathbf{A}\int_0^\infty e^{\mathbf{A}t}\mathbf{b}\mathbf{b}^T e^{\mathbf{A}^T t}tdt\mathbf{A}^T\mathbf{c} \\ &= \mathbf{c}^T\mathbf{A}\mathbf{Y}\mathbf{A}^T\mathbf{c} \Rightarrow M_2 = \frac{\mathbf{c}^T\mathbf{A}\mathbf{Y}\mathbf{A}^T\mathbf{c}}{\mathbf{c}^T\mathbf{X}\mathbf{c}}. \qquad (15) \end{aligned}$$

Applying equation (10) completes the proof. ■

Remark 1: If the original system is stable then $\mathbf{X}$ and $\mathbf{Y}$ are positive definite. Therefore, $M_1, M_2 > 0$ and α^* is real.

According to Lemma 1, the cost function J can be easily minimized and the optimal parameter α is found by (12). Such a calculation is straightforward as J depends only on the original system and the parameter α.

Algorithm 1: Rational Krylov with an Optimal Point (RK-OP)

1) Solve the Lyapunov equations (11), (13) to calculate $\mathbf{X}$ and $\mathbf{Y}$.
2) Calculate α^* using (12) to minimize the function (7).
3) Find the reduced system (3) using α^* and (4), (5) with a given order q.

However, the calculation for α is costly as two Lyapunov equations in the size of the original system are to be solved. By finding an approximate solution of the Lyapunov equations involved in the RK-OP algorithm, the cost of calculation can be reduced dramatically. In the following, $\mathbf{X}$ and $\mathbf{Y}$ are calculated using a reduced system by means of the so-called Galerkin conditions [21], [22].

Consider the Lyapunov equation associated with the reduced system Σ_r in (3),

$$\mathbf{W}^T\mathbf{A}\mathbf{V}\mathbf{X}_r + \mathbf{X}_r\mathbf{V}^T\mathbf{A}^T\mathbf{W} + \mathbf{W}^T\mathbf{b}\mathbf{b}^T\mathbf{W} = \mathbf{0}. \quad (16)$$

If we approximate the original controllability gramian as,

$$\mathbf{X} \approx \hat{\mathbf{X}} = \mathbf{V}\mathbf{X}_r\mathbf{V}^T, \quad (17)$$

the following Galerkin condition is satisfied,

$$\mathbf{W}^T\left(\mathbf{A}\hat{\mathbf{X}} + \hat{\mathbf{X}}\mathbf{A}^T + \mathbf{b}\mathbf{b}^T\right)\mathbf{W} = \mathbf{0}. \quad (18)$$

Using the approximate gramian to calculate $\mathbf{Y}$ results in

$$\mathbf{W}^T\mathbf{A}\mathbf{V}\mathbf{Y}_r + \mathbf{Y}_r\mathbf{V}^T\mathbf{A}^T\mathbf{W} + \mathbf{X}_r = \mathbf{0}, \quad (19)$$

$$\mathbf{Y} \approx \hat{\mathbf{Y}} = \mathbf{V}\mathbf{Y}_r\mathbf{V}^T, \quad (20)$$

$$\mathbf{W}^T\left(\mathbf{A}\hat{\mathbf{Y}} + \hat{\mathbf{Y}}\mathbf{A}^T + \mathbf{X}_r\right)\mathbf{W} = \mathbf{0}. \quad (21)$$

Accordingly, the optimal parameter is approximated as,

$$\alpha^* \approx \sqrt{\frac{\mathbf{c}^T\mathbf{A}\mathbf{V}\mathbf{Y}_r\mathbf{V}^T\mathbf{A}^T\mathbf{c}}{\mathbf{c}^T\mathbf{V}\mathbf{Y}_r\mathbf{V}^T\mathbf{c}}}. \quad (22)$$

Such an approximation depends on the reduced system itself and the optimization is hence not so simple. To converge to an optimal solution as in the RK-OP algorithm, it is proposed to iterate between the optimal parameter and the reduced system starting from an initial parameter.

Algorithm 2: Rational Krylov with an Iteratively Calculated Optimal Point (RK-ICOP)

1) Reduce the original system using the initial value of α_0 and set $i = 1$.
2) Solve the corresponding Lyapunov equations for the reduced system to calculate $\mathbf{X}_r$ and $\mathbf{Y}_r$.
3) Calculate the approximation of the optimal parameter α_i using (22).
4) Reduce the system using α_i with a given order q.
5) Increase i and go back to step 2.

The algorithm may be terminated if $\alpha_i - \alpha_{i-1} \leq \epsilon$ for a given tolerance ϵ. The convergence of this algorithm will be discussed in the following section.

Remark 2: A common method to approximate or identify complex systems is based on *truncating* the Laguerre series expansion. Assume that the system h is approximated by the sum of the first N terms as $\hat{h} = \sum_{i=0}^{N-1} F_i$. Then,

$$\|h(t) - \hat{h}(t)\|_2^2 = \sum_{i=N}^{\infty} F_i^2 \leq \frac{1}{N}\sum_{i=0}^{\infty} iF_i^2 = \frac{1}{N}J \Rightarrow$$

$$\frac{\|h(t) - \hat{h}(t)\|_2^2}{\|h(t)\|_2^2} \leq \frac{1}{N}\frac{\alpha^2 M_1 + M_2}{2\alpha} - \frac{1}{2N}. \quad (23)$$

This suggests to minimize J to find an optimal α that minimizes the upper bound of the relative error norm of the approximation. Such a reduced system usually does not lead to satisfactory result mainly because all its poles are located in a single point $-\alpha$. Furthermore, although the cost function J appears as the upper bound of the error, it does not really reflect the magnitude of the error system. In most applications, the bound given above is far from the real two-norm of the error as the weighting i increases to infinity.

V. RATIONAL KRYLOV WITH AN OPTIMAL ERROR MINIMIZING INTERPOLATION POINT

Consider the order reduction problem by matching the first N Laguerre coefficients. A natural cost function would be to minimize the difference between the rest of the coefficients. This suggests a new cost function,

$$J_d = \sum_{i=N}^{\infty} i(F_i - F_{ri})^2 = \sum_{i=0}^{\infty} iF_i^2 + \sum_{i=0}^{\infty} iF_{ri}^2 - 2\sum_{i=0}^{\infty} iF_iF_{ri}. \quad (24)$$

The value of α suggested by minimizing the cost function J_d should lead to good reduced systems when the first coefficients match. In the following, the optimal parameter for this cost function is calculated.

The first two terms in (24) are calculated using equation (9) and the result of section IV. For the last term, the method in section III-B is followed and the differential equation (6) is used,

$$-\int_0^\infty th_r(t)\ddot{h}(t)dt - \int_0^\infty h_r(t)\dot{h}(t)dt + \alpha^2\int_0^\infty th(t)h_r(t)dt$$
$$= \int_0^\infty \left[\sum_{i=0}^{\infty} F_i 2\alpha\left(i + \frac{1}{2}\right)\phi_i^\alpha(t)\right] h_r(t)dt$$
$$= 2\alpha\sum_{i=0}^{\infty} iF_iF_{ri} + \alpha\sum_{i=0}^{\infty} F_iF_{ri} = 2\alpha\sum_{i=0}^{\infty} iF_iF_{ri} + \alpha\int_0^\infty h(t)h_r(t)dt.$$

Finally by simplifying the integral terms we have,

$$\sum_{i=0}^{\infty} iF_iF_{ri} = \frac{1}{2\alpha}\left(\int_0^\infty \dot{h}_r(t)\dot{h}(t)tdt + \alpha^2\int_0^\infty h(t)h_r(t)tdt\right) - \frac{1}{2}\int_0^\infty h(t)h_r(t)dt. \quad (25)$$

Consider the original and its projected reduced system,

$$\int_0^\infty h(t)h_r(t)dt = \mathbf{c}^T\int_0^\infty e^{\mathbf{A}t}\mathbf{b}\mathbf{b}^T\mathbf{W}e^{\mathbf{V}^T\mathbf{A}^T\mathbf{W}t}dt\mathbf{V}^T\mathbf{c}$$
$$= \mathbf{c}^T\tilde{\mathbf{X}}\mathbf{V}^T\mathbf{c}. \quad (26)$$

where $\tilde{\mathbf{X}}$ is the solution of the following Sylvester equation,

$$\mathbf{A}\tilde{\mathbf{X}} + \tilde{\mathbf{X}}\mathbf{V}^T\mathbf{A}^T\mathbf{W} + \mathbf{b}\mathbf{b}^T\mathbf{W} = \mathbf{0}. \quad (27)$$

Assuming that the original and reduced systems are stable and considering the solution of the Sylvester equation (27), a new variable $\tilde{\mathbf{Y}}$ satisfies the following,

$$\mathbf{A}\tilde{\mathbf{Y}} + \tilde{\mathbf{Y}}\mathbf{V}^T\mathbf{A}^T\mathbf{W} + \tilde{\mathbf{X}} = \mathbf{0},$$

$$\tilde{\mathbf{Y}} = \int_0^\infty \mathbf{A}e^{\mathbf{A}t}\mathbf{b}\mathbf{b}^T\mathbf{W}e^{\mathbf{V}^T\mathbf{A}^T\mathbf{W}t}tdt$$

$$\int_0^\infty h_r(t)h(t)tdt = \mathbf{c}^T\tilde{\mathbf{Y}}\mathbf{V}^T\mathbf{c}. \quad (28)$$

Finally, it is simple to show that,

$$\int_0^\infty \dot{h}_r(t)\dot{h}(t)tdt = \mathbf{c}^T\mathbf{A}\tilde{\mathbf{Y}}\mathbf{V}^T\mathbf{A}^T\mathbf{W}\mathbf{V}^T\mathbf{c}. \quad (29)$$

Lemma 2: Consider an LTI system that has been reduced by projection. The cost function J_d in equation (24) satisfies,

$$J_d = \frac{1}{N}\left(\frac{\alpha^2\tilde{M}_1 + \tilde{M}_2}{2\alpha} - \frac{\|h(t)\|_2}{2} - \frac{\|h_r(t)\|_2}{2} + \mathbf{c}^T\tilde{\mathbf{X}}\mathbf{V}^T\mathbf{c}\right),$$

where,

$$\tilde{M}_1 = \mathbf{c}^T\left(\mathbf{Y} + \mathbf{V}\mathbf{Y}_r\mathbf{V}^T - 2\tilde{\mathbf{Y}}\mathbf{V}^T\right)\mathbf{c}$$

$$\tilde{M}_2 = \mathbf{c}^T\left(\mathbf{A}\mathbf{Y}\mathbf{A}^T + \mathbf{V}\mathbf{W}^T\mathbf{A}\mathbf{V}\mathbf{Y}_r\mathbf{V}^T\mathbf{A}^T\mathbf{W}\mathbf{V}^T - 2\mathbf{A}\tilde{\mathbf{Y}}\mathbf{V}^T\mathbf{A}^T\mathbf{W}\mathbf{V}^T\right)\mathbf{c}.$$

What restricts the application of the cost function given in lemma 2 is the dependency of the right hand side on the reduced system. However, if the reduced system is assumed to be given, the upper bound can be minimized leading to the optimal parameter $\alpha^* = \sqrt{\frac{\tilde{M}_2}{\tilde{M}_1}}$.

Algorithm 3: Rational Krylov with an Optimal Error Minimizing Point (RK-OEMP)

1) Solve the corresponding Lyapunov equations for the original system to calculate $\mathbf{X}$, $\mathbf{Y}$.
2) Reduce the original system using the initial value of α_0 and set $i = 1$.
3) Solve the corresponding Lyapunov and Sylvester equations for the reduced system to calculate $\mathbf{X}_r$, $\mathbf{Y}_r$, $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$.
4) Calculate the parameter $\alpha_i = \sqrt{\frac{\tilde{M}_2}{\tilde{M}_1}}$ where $\tilde{M}_2$ and $\tilde{M}_1$ are defined in theorem 2.
5) Reduce the system using α_i with a given order q.
6) Increase i and go back to step 3.

It should be noted that the value of $\mathbf{X}$ and $\mathbf{Y}$ should be calculated only once and the best choice for the starting parameter is $\alpha_0 = \sqrt{\frac{M_2}{M_1}}$. To reduce the computational cost and avoid solving Lyapunov equations in the size of original system, similar to algorithm 2, the method of Galerkin is employed leading to the next algorithm. Note that calculating $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$ is not numerically very expensive as they are $n \times q$ matrices with $q << n$.

Algorithm 4: Rational Krylov with a Near Optimal Error Minimizing Point (RK-NOEMP)

1) Reduce the original system using the initial value of α_0 and set $i = 1$.
2) Solve the corresponding Lyapunov and Sylvester equations for the reduced system to calculate $\mathbf{X}_r$, $\mathbf{Y}_r$, $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$ and set $\mathbf{Y} = \mathbf{V}\mathbf{Y}_r\mathbf{V}^T$.
3) Calculate the parameter $\alpha_i = \sqrt{\frac{\tilde{M}_2}{\tilde{M}_1}}$ where $\tilde{M}_2$ and $\tilde{M}_1$ are defined in theorem 2.
4) Reduce the system using α_i with a given order q.
5) Increase i and go back to step 2.

In order to analyze the convergence of the algorithms 2 and 4, consider the original system Σ that has been reduced to Σ_0 by matching the first N Laguerre coefficients associated with α_0. This system approximates some of the major dynamics of Σ from a Laguerre approximation point of view. In the next step, the parameter α_1 calculated from Σ_0 is employed to calculate the reduced system Σ_1. This is the first reduced system found based on the results of this section. Since α_2 is extracted from Σ_1, it is expected that the difference $\alpha_2 - \alpha^*$ tends to zero. Applying the proposed algorithms to several technical systems confirms this theoretical interpretation and shows that no significant changes occurs in the α_i after the third iteration.

VI. TECHNICAL EXAMPLE

In order to demonstrate the effectiveness of the presented algorithms, the model of a CD player is considered [23]. The most important part of this system is the optical unit (lenses, laser diode, and photo detectors) and its actuators. The main task in this system is to control the arm holding the optical unit to read the required track on the disk and to adjust the position of the focusing lens to adjust the depth of the laser beam penetrating the disc. In order to achieve this task, the system has been modeled by finite element method (FEM) leading to a differential equation of order 120.

The system has been reduced to order 8 by applying all four algorithms presented here. Minimizing the cost function (9) in RK-OP leads to $\alpha^* = 292.8794$. By running the algorithm 2, RK-ICOP, the parameter converges in three steps to $\alpha^* = 291.8036$ which has less than 0.4% error.

The two algorithms RK-OEMP and the approximated version RK-NOEMP lead to the optimized parameter $\alpha^* = 207.0667$ and 206.8629, respectively. The difference is less than 0.1% and the reduced systems are almost equal.

Figure 1 illustrates the parameter α in terms of iterations for all algorithms that show a fast convergence to the desired value of parameter. The impulse response of the reduced systems shows very good approximations, as in figure 2.

It is remarked that the optimal parameter for RK-ICOP and RK-NOEMP are different as they minimize different cost functions. Finally, it is clear from the presented results that, for this technical example, all algorithms lead to very similar results. However, the simulations with many other examples have shown that this fact is generally not true. All examples investigated so far have confirmed that algorithms 2 and 4 converge quickly (typically, within 3 steps) towards the results of algorithms 1 and 3, and that the approximation of the system response in time-domain is excellent.

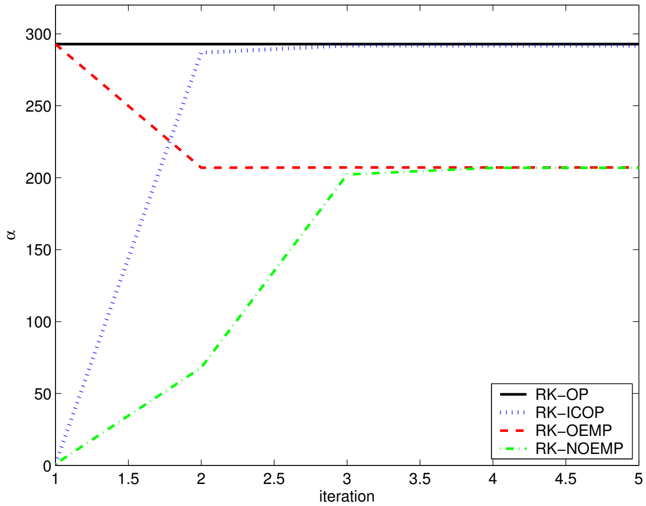

Fig. 1. The interpolation parameter.

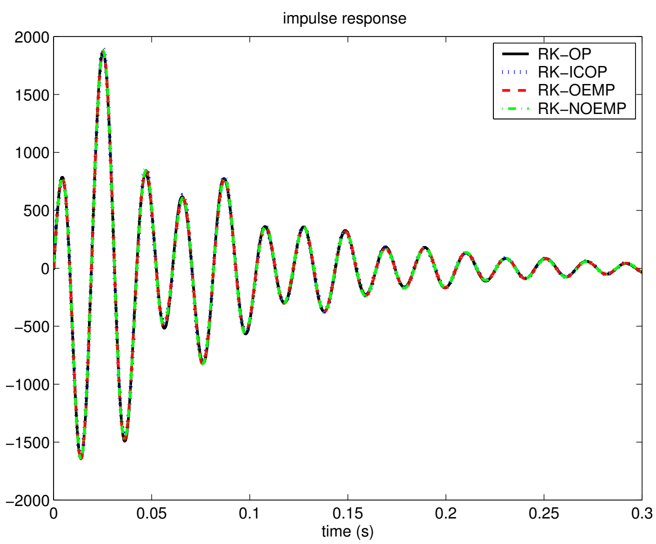

Fig. 2. Impulse response of the original and reduced systems.

VII. CONCLUSIONS

A new method, with various algorithms, for the choice of the interpolation point in Krylov-subspace method based on a Laguerre series representation of the system has been presented. To this end, the fact that the reduced order model, obtained by matching the Laguerre coefficients of the impulse response, is equal to the one obtained by moment matching about $s_0 = \alpha$, has been used. The optimal choice of α in the Laguerre domain has been then adjusted in several different ways to serve as an *optimal expansion point* for the rational Krylov order reduction while minimizing a certain objective function. Applying the proposed algorithms to several technical systems confirms the very fast convergence of all their variants and showed that negligible improvements in the accuracy of the reduced order model occur after the third iteration step. In addition, it is observed that the approximation using the Galerkin conditions showed to be particularly useful if the order of the reduced system is not very low, depending on complexity of the system. The extension of the results to the MIMO case and multi-point interpolation is being investigated within the authors' group.

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