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Max–Min Control Problems for Constrained Discrete Time Systems

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***Abstract*— We consider control synthesis problems for constrained discrete time nonlinear systems subject to uncertainty. The uncertainty affects the system in a form of a bounded, but known, persistent disturbance and leads, consequently, to the max–min control synthesis problems. A computational characterization of the max–min controllable sets is derived for a general nonlinear case. The max–min time optimal control of constrained piecewise affine discrete time systems is also discussed. Corresponding computational details are outlined and some illustrative examples are provided.**

I. Introduction

Control of constrained discrete time systems in the presence of uncertainty appears mainly in two different contexts in the control literature, depending on standing assumptions on the uncertainty. The first one is the case when the uncertainty (including disturbances, measurement noise and uncertainties in system dynamics) is bounded and unknown to the controller or the decision maker. The corresponding class of control problems leads to the *min–max* optimal control problems, see, for instance, pioneering contributions [1]–[3]. The second class of problems reflects the case when the set–bounded uncertainty *is revealed* to the controller at the current time instance (the moment of the decision making), while its future realization remains unknown but bounded. The corresponding control synthesis results in the *max–min* optimal control problems that are also encountered in the classical *theory of dynamic games* (cf. [4] and references therein). The treatment of the constrained *max–min* problems in the literature is somehow scant. Notable references include discussions on max–min controllability of unconstrained continuous time systems [5] and in the presence of control constraints [6]. Recent publication by the authors [7] discusses the characterization of the solution to max–min optimal control for several classes of linear discrete time systems (time invariant, parameter and time varying) subject to both state and input constraints. This note extends initial results of [7] by discussing the max–min control problems in a more general framework. We base our approach on set–theoretic methods by placing the reachability analysis (cf. [8], [9]) in the center of the discussion. As an application of the developed theory we consider max–min time optimal control of piecewise affine (PWA) systems.

PWA systems have received significant attention of control researchers in last two decades as a modeling paradigm for nonlinear systems [10] and equivalent representation for several classes of discrete time *hybrid systems* [11]. A number of authors considered various types of optimal control problems for PWA systems [12]–[16]. In this paper, we complement above listed contributions by addressing max–min controllability and *max–min time optimal control* of PWA systems. Our main motivation comes from the fact that PWA systems are popularly used for approximation of nonlinear systems, primarily in the hope of easier control design. If strict constraints need to be satisfied, one may wish to utilize robust control design by taking into account induced approximation errors. If the dynamics of the original nonlinear system is known (as often is the case), the approximation error can be modeled as a bounded but *known* uncertainty, leading naturally to max–min (rather than the min–max) robust control problems.

Outline of the paper: The problem statement and preliminaries are given in Section II. The main result that provides the characterization of max–min controllable sets for a general case of discrete time systems is stated in Section III. Particular aspects of constrained max–min control for PWA system are discussed in Section IV. An illustrative example and concluding remarks are provided, respectively, in Sections V and VI.

Nomenclature and Basic Definitions: Let $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{N}_{[q_1,q_2]} := \{q_1, q_1 + 1, \ldots, q_2 - 1, q_2\}$ for a given $q_1 \in \mathbb{N}$ and $q_2 \in \mathbb{N}$ such that $q_1 < q_2$ and $\mathbb{N}_q$ denotes $\mathbb{N}_{[0,q]}$ for $q \in \mathbb{N}$. Given two sets $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^n$, the complement of A is $A^c := \{x \in \mathbb{R}^n : x \notin A\}$ and the set difference between A and B is $A \backslash B := \{x \in A : x \notin B\} = A \cap B^c$. A power set (set of all subsets) of a set A is denoted as 2^A. The *orthogonal projection* of a set $A \subseteq X \times Y \subseteq \mathbb{R}^n \times \mathbb{R}^m$ is defined as $\mathrm{Proj}_X(A) := \{x \in X : \exists y \in Y \text{ such that } (x, y) \in A\}$. The Minkowski set addition of two (nonempty) sets X and Y, such that $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^n$, is denoted by $X \oplus Y := \{x + y \ : \ x \in X, \ y \in Y\}$. The Minkowski (Pontryagin) subtraction of sets $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{Y} \subset \mathbb{R}^n$ is denoted by $\mathcal{X} \ominus Y := \{z \in \mathbb{R}^n \ : \ z \oplus \mathcal{Y} \subseteq \mathcal{X}\}$. A *polyhedron* is an intersection of a finite number of open and/or closed half-spaces. A *polytope* is a closed and bounded polyhedron. A union of (polytopes) polyhedra is referred to as a (compact) polygon. A restriction of a function $f : \mathbb{R}^n \to \mathbb{R}$ to a set $X \subseteq \mathbb{R}^n$ is denoted as $f \mid X$. A function f is *continuous relative to a set* X if $f \mid X$ is continuous. The interior of a set $X \subset \mathbb{R}^n$ is denoted as $\mathrm{int}\, X$. If $f(\cdot)$ is a set-valued function from X into U, namely, its values are subsets of U, then its *graph* is the set $\{(x, y) : x \in X, \ y \in f(x)\}$.

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II. Problem statement and preliminaries

We start our discussion by considering a discrete time system given by the state update equation:

$$x^+ = f(x, u, w), \tag{1}$$

where $x \in \mathbb{R}^n$ is the current state, $u \in \mathbb{R}^m$ is the current control, $w \in \mathbb{R}^p$ is the current disturbance, $x^+ \in \mathbb{R}^n$ is the successor state and $f(\cdot,\cdot,\cdot) \; : \; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n$ is the state transition map. The system variable x, u and w are subject to constraints:

$$(x, u) \in \Omega_{xu} \subseteq \mathbb{R}^n \times \mathbb{R}^m \text{ and } w \in \mathcal{W} \subseteq \mathbb{R}^p. \tag{2}$$

For the time being, no special properties of the sets $\mathcal{W}$ and Ω_{xu} are assumed. For the orthogonal projection of the set Ω_{xu} to x– and u–subspaces, we use, respectively, Ω_x and Ω_u. Similarly, we denote: $\Omega_{xw} := \Omega_x \times \mathcal{W}$. Also, we denote:

$$\Omega := \Omega_{xu} \times \mathcal{W}.$$

The crucial interpretation of robust control problems considered in this note is the following:

Interpretation 1: At any time $k \in \mathbb{N}_0$ when the decision concerning the control input u_k is taken, both the state x_k and the disturbance $w_k \in \mathcal{W}$ are known while future disturbances $w_{k+i}, \; i \in \mathbb{N}$ are not known and can take arbitrary values $w_{k+i} \in \mathcal{W}, \; i \in \mathbb{N}$.

In the light of Interpretation 1, the information available to the controller for control synthesis, at any time instance $k \in \mathbb{N}_0$, is the pair (x, w). Using the notation and terminology introduced in [7], we invoke the *information set* $\mathcal{Z}$ given by:

$$\mathcal{Z} := \{(x, w) \colon (x, w) \in \Omega_x \times \mathcal{W}\}. \tag{3}$$

We use the term *control policy* for a sequence of *control laws* $\pi_i \colon \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m$ over the finite horizon $N \in \mathbb{N}$ and denote it as $\Pi_N := \{\pi_i(\cdot) \; : \; i \in \mathbb{N}_{N-1}\}$. The set of all control policies over the horizon N is denoted as $\mathbf{\Pi}_N$. A sequence of admissible disturbances is denoted as $\mathbf{w}_N := \{w_0, w_1, \ldots, w_{N-1}\}$ (here $w_i \in \mathcal{W}$ for all $i \in \mathbb{N}_{N-1}$) and the set of all admissible disturbance sequences over the horizon N is denoted as $\mathbf{W}_N$. Accordingly, we use $\phi(i; x_0, \Pi_N, \mathbf{w}_N)$ to denote *the solution* to the state update equation (1) at the time instance i, given the initial condition x_0, the control policy Π_N and the disturbance sequence $\mathbf{w}_N$.

We recall the concept of *N-step max-min controllability*.

Definition 2.1 (N–step max–min controllability): A state $x \in \Omega_x$ is *N–step max–min controllable*, $N \in \mathbb{N}$, to a target set $\mathcal{X}_f \subseteq \Omega_x$ if and only if for all admissible disturbance sequences $\mathbf{w}_N \in \mathbf{W}_N$ there exists a control policy $\Pi_N \in \mathbf{\Pi}_N$ such that

$$\forall i \in \mathbb{N}_{N-1}, \; (x_i, \pi_i(z_i)) \in \Omega_{xu} \text{ and } x_N \in \mathcal{X}_f, \tag{4}$$

where $x_i := \phi(i; x, \Pi_N, \mathbf{w}_N)$ and $z_i := (x_i, w_i)$.

A central role in the max–min controllability is played by the mapping $\mathcal{B}(\cdot) : 2^{\mathbb{R}^n} \to 2^{\mathbb{R}^n}$ given by:

$$\begin{aligned} \mathcal{B}(X) := \{x \; : \; & \forall w \in \mathcal{W} \; \exists u \text{ such that} \\ & (x, u) \in \Omega_{xu} \text{ and } f(x, u, w) \in X\}. \end{aligned} \tag{5}$$

An equally important role is played by the set–valued control map $\mathcal{U}(\cdot,\cdot)$:

$$\begin{aligned} \mathcal{U}(x, w) := \{u \; : \; & (x, u, w) \in \Omega_{xu} \times \mathcal{W} \\ & \text{and } f(x, u, w) \in X\}, \end{aligned} \tag{6}$$

defined for all $(x, w) \in \mathcal{B}(X) \times \mathcal{W}$ and for any given set $X \subseteq \mathbb{R}^n$. Clearly, given a set X, the set $\mathcal{B}(X)$ is the set of all states x such that for all disturbances $w \in \mathcal{W}$ there exists a control input u (where obviously $u = u(x, w)$) for which the state and control satisfy constraint $(x, u) \in \Omega_{xu}$ and the successor state $f(x, u, w)$ is in the set X. Similarly, given any $(x, w) \in \mathcal{B}(X) \times \mathcal{W}$ the set of all controls u ensuring that state and control satisfy constraint $(x, u) \in \Omega_{xu}$ and that the successor state $f(x, u, w)$ lies in the set X is precisely given by the set–valued control map $\mathcal{U}(x, w)$. Thus, in terms of the max–min controllability, given a target set, say $\mathcal{X}_f$, the set $\mathcal{B}(\mathcal{X}_f)$ is the 1–step max–min controllable set to $\mathcal{X}_f$. The set of j–step max–min controllable states is hereafter referred to as the j–step controllable set (we omit the max–min attribute for typographical reasons) and is denoted as $\mathcal{X}_j$. By a direct inspection of definition 2.1 and (5), given a target set $\mathcal{X}_f$, j–step controllable sets, $j \in \mathbb{N}_N$, are given by the direct iteration of the mapping $\mathcal{B}(\cdot)$:

$$\mathcal{X}_j = \mathcal{B}(\mathcal{X}_{j-1}), \; j \in \mathbb{N}_{[1,N]} \text{ and } \mathcal{X}_0 = \mathcal{X}_f. \tag{7}$$

The set–valued control maps $\mathcal{U}_j(\cdot,\cdot)$ given, for all $(x, w) \in \mathcal{X}_j \times \mathcal{W}$, by:

$$\begin{aligned} \mathcal{U}_j(x, w) := \{u \; : \; & (x, u, w) \in \Omega_{xu} \times \mathcal{W} \text{ and} \\ & f(x, u, w) \in \mathcal{X}_{j-1}\}, \end{aligned} \tag{8}$$

are, as is customary, associated with the j–step controllable sets $\mathcal{X}_j$ and the disturbance set $\mathcal{W}$.

The max–min control problems considered in this note are described by the following generic formulation.

Problem 1 (N-horizon max-min optimal robust control): Given an integer $N \in \mathbb{N}$ and a target set $\mathcal{X}_f \subseteq \Omega_x$:

1) characterize the j-step controllable sets $\mathcal{X}_j$, $j \in \mathbb{N}_{[1,N]}$, and
2) for a given *cost function* $V_N(x, \Pi_N, \mathbf{w}_N)$, select a control policy Π_N^0 from the set of control policies $\tilde{\mathbf{\Pi}}_N$ satisfying (4) which yields the optimal cost:

$$V_N^0(x) := \sup_{\mathbf{w}_N \in \mathbf{W}_N} \inf_{\Pi_N \in \mathbf{\Pi}_N} V_N(x, \Pi_N, \mathbf{w}_N). \tag{9}$$

Apart from ensuring that state trajectories of the controlled uncertain system reach the target set, a desirable feature of the control algorithm is to ensure that the state trajectories of the controlled uncertain system remain within the target set if the initial state is in the target set, i.e. that the target set $\mathcal{X}_f$ is a *robust control invariant set* [8], [17]–[19].

Definition 2.2 (Max–min robust control invariance): A set $\mathcal{S} \subseteq \Omega_x$ is max–min robust control invariant if and only if $\mathcal{S} \subseteq \mathcal{B}(\mathcal{S})$.

In the sequel, we simply use the term "robust control invariance" rather than the "max–min robust control invariance" - no confusion should arise.

III. MAX–MIN REACHABILITY ANALYSIS

Our first step is to provide an alternative form of the mapping $\mathcal{B}(\cdot)$ that permits the utilization of the standard computational geometry tools for some classes of the underlying system and involved constraint sets. We provide our first two main results in a more general setting and then discuss their utilization for the class of PWA systems in Section IV. We define for a given set $X \subseteq \mathbb{R}^n$:

$$\Phi(X) := \{(x, u, w) \in \Omega_{xu} \times \mathcal{W} \colon\ f(x, u, w) \in X\}, \quad (10)$$

$$\Psi(X) := \mathrm{Proj}_{\Omega_{xw}}(\Phi(X)), \quad (11)$$

$$\Delta(X) := \Omega_{xw} \backslash \Psi(X). \quad (12)$$

We can now state our first main result.

Theorem 3.1 (The alternative form of the map $\mathcal{B}(\cdot)$): The 1-step controllable set $\mathcal{B}(X)$ to a set $X \subseteq \mathbb{R}^n$ is given by:

$$\mathcal{B}(X) = \Omega_x \backslash \mathrm{Proj}_{\Omega_x}(\Delta(X)). \quad (13)$$

Proof: See [20]. ■

The result of Theorem 3.1 does not depend on the particular nature neither of the sets X, Ω_{xu} and $\mathcal{W}$ nor of the state transition mapping $f(\cdot,\cdot,\cdot)$. Although the statement of Theorem 3.1 is very compact, the computation of the set $\mathcal{B}(X)$ according to (13) could be difficult. The set operations defined in (10)–(12) are, however, potentially tractable, from the computational point of view, for some classes of the state transition mapping $f(\cdot,\cdot,\cdot)$ and involved sets X, Ω_{xu} and $\mathcal{W}$ as indicated in Section IV.

Our second main result is concerned with topological properties of the mapping $\mathcal{B}(\cdot)$.

Theorem 3.2 (The topological property of the map $\mathcal{B}(\cdot)$): Assume that sets $X \subseteq \Omega_x$, $\mathcal{W}$ and Ω_{xu} are compact and that the function $f(\cdot,\cdot,\cdot) \colon \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n$ is continuous relative to the set Ω, where $\Omega = \Omega_{xu} \times \mathcal{W}$. Then, the set $\mathcal{B}(X)$ is a compact (possibly empty) set.

Proof: See [20]. ■

Remark 3.1: We now recall a few well known facts [2], [9], [17], [19] relevant to the max–min controllability and invariance. These standard observations are concerned with the j–step controllable sets $\mathcal{X}_j$ obtained via (7). A j–step robust controllable set $\mathcal{X}_j$ is robust control invariant if and only if $\mathcal{X}_j \subseteq \mathcal{X}_{j+1} = \mathcal{B}(\mathcal{X}_j)$. The j–step controllable sets $\mathcal{X}_j$ are robust control invariant for all $j \in \mathbb{N}_0$ if and only if $\mathcal{X}_f$ is robust control invariant (i.e. $\mathcal{X}_f \subseteq \mathcal{B}(\mathcal{X}_f)$). The maximal robust control invariant set contained in Ω_x, say $\tilde{\mathcal{X}}_\infty$, is unique (possibly empty). If $\mathcal{X}_f = \Omega_x$ the maximal robust control invariant set $\tilde{\mathcal{X}}_\infty$ contained in Ω_x satisfies $\tilde{\mathcal{X}}_\infty \subseteq \bigcap_{j=0}^{\infty} \mathcal{X}_j$ and, in addition, $\tilde{\mathcal{X}}_\infty = \mathcal{X}_j$ for some $j \in \mathbb{N}_0$ if and only if $\mathcal{X}_j = \mathcal{X}_{j+1} = \mathcal{B}(\mathcal{X}_j)$. If, however, sets $\mathcal{X}_f$, $\mathcal{W}$ and Ω_{xu} are compact and the function $f(\cdot,\cdot,\cdot) \colon \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n$ is continuous relative to the set Ω, where $\Omega = \Omega_{xu} \times \mathcal{W}$, then the j–step controllable set $\mathcal{X}_j$ are compact sets. Furthermore the maximal robust control invariant set $\tilde{\mathcal{X}}_\infty$ contained in Ω_x is compact (possibly empty) set and $\tilde{\mathcal{X}}_\infty = \bigcap_{j=0}^{\infty} \mathcal{X}_j$ if, in addition, $\mathcal{X}_f = \Omega_x$. The relevance of Theorems 3.1 and 3.2 is paramount from the computational point of view, since it makes the explicit characterization (computation) of the controllable sets $\mathcal{X}_j$ considerably easier for certain classes of problems, as elaborated next in Section IV.

IV. MAX–MIN CONTROL OF PWA SYSTEMS

We consider in this section max–min control of discrete time PWA systems given by the state update equation:

$$x^+ = f(x, u, w), \quad (14)$$

where, as before, the current and successor states are, respectively, x and x^+, the current control is u, the current disturbance is w and the state transition mapping $f(\cdot,\cdot,\cdot) \colon \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n$. Standing assumptions in this section are:

A1 $(x, u) \in \Omega_{xu} \subset \mathbb{R}^n \times \mathbb{R}^m$ where Ω_{xu} is a compact polygon defining constraints on the states and the control inputs,

A2 $w \in \mathcal{W}$, where $\mathcal{W}$ is a polytope in $\mathbb{R}^p$,

A3 A subdivision of the set $\Omega := \Omega_{xu} \times \mathcal{W}$ is defined:

$$\Omega = \bigcup_{i=1}^{d} \mathcal{P}_i, \quad (15)$$

where $\mathcal{P}_i \subset \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$, $i \in \mathbb{N}_{[1,d]}$ are full dimensional polytopes in $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$ such that

$$\mathrm{int}\, \mathcal{P}_i \cap \mathrm{int}\, \mathcal{P}_j = \emptyset \text{ for } i \neq j,\ i, j \in \mathbb{N}_{[1,d]},$$

A4 $f_i := f \mid \mathcal{P}_i$, $i \in \mathbb{N}_{[1,d]}$ is an affine function in (x, u, w), i.e.

$$f_i(x, u, w) := A_i x + B_i u + G_i w + c_i$$

where A_i, B_i, c_i and G_i are system matrices of appropriate dimensions,

A5 the function $f(\cdot,\cdot,\cdot)$ is a continuous function relative to the set Ω.

Additional structure of the state transition map $f(\cdot,\cdot,\cdot)$ and involved constraint sets Ω_{xu} and $\mathcal{W}$ allows us to establish that the polygonal structure of j–step controllable sets $\mathcal{X}_j$ is invariant under the mapping $\mathcal{B}(\cdot)$, i.e. that the set $\mathcal{B}(X)$ is a (compact) polygon if the set X is a (compact) polygon. We utilize Theorems 3.1 and 3.2 to verify this property. Hence, suppose that the j–step controllable set $\mathcal{X}_j$ is a compact polygon for some $j \in \mathbb{N}_0$ so that it admits the following representation:

$$\mathcal{X}_j = \bigcup_{k=1}^{q} \mathcal{X}_{(j,k)}, \quad (16)$$

where $\mathcal{X}_{(j,k)}$, $k \in \mathbb{N}_{[1,q]}$ (here q is a finite integer), are polytopes in $\mathbb{R}^n$. The set $\Phi(\mathcal{X}_j)$ defined by (10) is, in this case, given by:

$$\begin{aligned}\Phi(\mathcal{X}_j) &= \{(x, u, w) \in \Omega \colon f(x, u, w) \in \mathcal{X}_j\} \\ &= \bigcup_{i=1}^{d} \bigcup_{k=1}^{q} \Phi_{(j,i,k)} \text{ where} \\ \Phi_{(j,i,k)} &= \{(x, u, w) \in \mathcal{P}_i \ :\ f_i(x, u, w) \in \mathcal{X}_{(j,k)}\}. \quad (17)\end{aligned}$$

Since functions $f_i(\cdot,\cdot,\cdot)$ are affine and sets $\mathcal{X}_{(j,k)}$ and $\mathcal{P}_i$ are polytopes it follows that sets $\Phi_{(j,i,k)}$ are polytopes (if nonempty) and, consequently, the set $\Phi(\mathcal{X}_j)$ is, clearly, a compact polygon (if nonempty). Hence, the set $\Psi(\mathcal{X}_j)$ defined by (11), is, in this case, given by:

$$\begin{aligned}\Psi(\mathcal{X}_j) &= \operatorname{Proj}_{\Omega_{xw}}(\Phi(\mathcal{X}_j)) = \operatorname{Proj}_{\Omega_{xw}}\left(\bigcup_{i=1}^{d}\bigcup_{k=1}^{q}\Phi_{(j,i,k)}\right)\\ &= \bigcup_{i=1}^{d}\bigcup_{k=1}^{q}\Psi_{(j,i,k)}, \text{ where} \qquad (18)\\ \Psi_{(j,i,k)} &= \operatorname{Proj}_{\Omega_{xw}}\left(\Phi_{(j,i,k)}\right). \qquad (19)\end{aligned}$$

Sets $\Psi_{(j,i,k)}$ are polytopes (if nonempty) because they are orthogonal projections of polytopes $\Phi_{(j,i,k)}$. Hence, it follows that the set $\Psi(\mathcal{X}_j)$ is a compact polygon (if nonempty). The set $\Delta(\mathcal{X}_j)$ defined by (12), is, in this case, given by the set difference of a compact polygon Ω_{xw} and a compact polygon $\Psi(\mathcal{X}_j)$ and hence it is a polygon (if nonempty) since the set difference between two polygons is a polygon (not necessarily compact). Consequently, the set $\operatorname{Proj}_{\Omega_x}(\Delta(\mathcal{X}_j))$ is a polygon (not necessarily compact) since it is a projection onto the set Ω_x of the polygon $\Delta(\mathcal{X}_j)$. It follows that, since the set Ω_x is a compact polygon and the set $\operatorname{Proj}_{\Omega_x}(\Delta(\mathcal{X}_j))$ is a polygon , the set $\Omega_x \setminus \operatorname{Proj}_{\Omega_x}(\Delta(\mathcal{X}_j))$ is also a polygon. Consequently, since by Theorem 3.1 $\mathcal{B}(\mathcal{X}_j) = \Omega_x \setminus \operatorname{Proj}_{\Omega_x}(\Delta(\mathcal{X}_j))$, the set $\mathcal{B}(\mathcal{X}_j)$ is a polygon (if nonempty). Since, under Assumptions **A1–A5**, the set $\mathcal{B}(\mathcal{X}_j)$ is also compact by Theorem 3.2, it follows that the $j+1$–step controllable set $\mathcal{X}_{j+1}$ is a compact polygon (if nonempty) if the j–step controllable set $\mathcal{X}_j$ is a compact polygon.

A similar set algebra leads to a relevant conclusion related to the polygonal structure of the set–valued control maps $\mathcal{U}_j(\cdot,\cdot)$. Necessary, but only brief, details are as follows. When the j–step controllable set is a polygon, we have by (8), for any $(x,w) \in \mathcal{X}_{j+1} \times \mathcal{W}$:

$$\begin{aligned}\mathcal{U}_{j+1}(x,w) &= \bigcup_{i=1}^{d}\bigcup_{k=1}^{q}\mathcal{U}_{(j+1,i,k)}(x,w) \text{ where}\\ \mathcal{U}_{(j+1,i,k)}(x,w) &= \{u \;:\; (x,u,w) \in \mathcal{P}_i \text{ and}\\ &\qquad f_i(x,u,w) \in \mathcal{X}_{(j,k)}\},\end{aligned} \qquad (20)$$

and where some of sets $\mathcal{U}_{(j+1,i,k)}(x,w)$ might be empty. However, when the $j+1$–step controllable set $\mathcal{X}_{j+1}$ is nonempty it is clear that there exist integers i and k (depending on x and w) such that the set $\mathcal{U}_{(j+1,i,k)}(x,w)$ is nonempty. Since functions $f_i(\cdot,\cdot,\cdot)$ are affine in polytopes $\mathcal{P}_i$ and sets $\mathcal{W}$ and $\mathcal{X}_{(j,k)}$ are polytopes, it follows directly from the expression (20) that the set–valued control map $\mathcal{U}_{j+1}(\cdot,\cdot)$ is a compact– and polygonal–valued whenever the set $\mathcal{X}_j$ is nonempty and a compact polygon; In fact, it is easy to see that the graph of the set–valued control map $\mathcal{U}_{j+1}(\cdot,\cdot)$ is, in this case, a compact polygon. Therefore, as a computationally relevant refinement of results established in Section III we have:

Proposition 4.1: Suppose that Assumptions **A1–A5** hold and that the target set $\mathcal{X}_f = \mathcal{X}_0$ is a compact polygon. Then for all $j \in \mathbb{N}$: (i) the j–step controllable sets $\mathcal{X}_j$, are compact polygons (if nonempty), (ii) the graphs of the corresponding set–valued control maps $\mathcal{U}_j(\cdot,\cdot)$ are compact polygons (if nonempty), (iii) the j–step controllable sets $\mathcal{X}_j$ and the graphs of the corresponding set–valued control maps $\mathcal{U}_j(\cdot,\cdot)$ are nonempty compact polygons if, in addition, $\mathcal{X}_f$ is a robust control invariant set and, furthermore, the j–step controllable sets $\mathcal{X}_j$ are nested in this case (i.e. $\forall j \in \mathbb{N}_0, \mathcal{X}_j \subseteq \mathcal{X}_{j+1}$).
We wish to point out that there is no reason to expect that continuous selections of the set–valued control maps $\mathcal{U}_j(\cdot,\cdot)$ exist, as even in the case of constrained linear systems, the nonconvexity of the target invariant set may result in the nonexistence of a continuous control law inducing the robust control invariance [21].

We now discuss the max–min time optimal control synthesis problem for the considered class of PWA systems. The max–min time optimal control problem is stated concisely as follows:

$$\begin{aligned}N^0(x) := \min_{N\in\mathbb{N}_0}\sup_{\mathbf{w}_N\in\mathbf{W}_N}\inf_{\Pi_N\in\mathbf{\Pi}_N}\{N \;:\;&\\ \forall \mathbf{w}_N \in \mathbf{W}_N,\ \forall i \in \mathbb{N}_{N-1},\ (x_i, \pi_i(x_i,w_i)) \in \Omega_{xu}&\\ \text{and } x_N \in \mathcal{X}_f\},& \qquad (21)\end{aligned}$$

where as before, for all $i \in \mathbb{N}_N$, $x_i := \phi(i; x, \mathbf{w}_N, \Pi_N)$ and $\phi(i; x, \mathbf{w}_N, \Pi_N)$ denotes the solution to (14) at time instance i when the initial state is x, the disturbance sequence is $\mathbf{w}_N$ and the control policy is Π_N. A direct utilization of the max–min controllable sets $\mathcal{X}_j$ and corresponding set–valued control laws $\mathcal{U}_j(\cdot,\cdot)$ yields the following fact ([2], [17], [19], [22]):

Proposition 4.2: Fix an $N \in \mathbb{N}$. Then there exist $x \in \mathbb{R}^n$ and $\Pi_N \in \mathbf{\Pi}_N$ such that:

$$\begin{aligned}&\forall \mathbf{w}_N \in \mathbf{W}_N,\ \forall i \in \mathbb{N}_{N-1},\\ &(\phi(i;x,\mathbf{w}_N,\Pi_N), \pi_i(\phi(i;x,\mathbf{w}_N,\Pi_N), w_i)) \in \Omega_{xu}\\ &\text{and } \phi(N;x,\mathbf{w}_N,\Pi_N) \in \mathcal{X}_f\end{aligned}$$

if and only if the N–step controllable set $\mathcal{X}_N \neq \emptyset$. Furthermore, the corresponding control policy $\Pi_N = \{\pi_0(\cdot,\cdot), \pi_1(\cdot,\cdot), \ldots, \pi_{N-1}(\cdot,\cdot)\}$ is such that:

$$\forall (y,w) \in \mathcal{X}_{N-i} \times \mathcal{W},\ \forall i \in \mathbb{N}_{N-1},\ \pi_i(y,w) \in \mathcal{U}_{N-i}(y,w).$$

Proposition 4.2 states that the max–min time optimal control problem is solvable if and only if the state x belongs to one of the j–step controllable sets $\mathcal{X}_j$ and this set is nonempty. Hence, the max–min time optimal control synthesis is direct providing that the j–step controllable sets $\mathcal{X}_j$ and corresponding set–valued control maps $\mathcal{U}_j(\cdot,\cdot)$ are precomputed and are available to the controller. Namely, for all nonempty j–step controllable sets $\mathcal{X}_j$ the max–min time optimal control problem is solvable and the following fact is, clearly, true for all $j \in \mathbb{N}_0$ such that $\mathcal{X}_j \neq \emptyset$:

$$N^0(x) = \min_{j\in\mathbb{N}_0}\{j \;:\; x \in \mathcal{X}_j\}.$$

We would like to point out that the max–min time optimal control guarantees the upper bound of the actual cost incurred for a particular disturbance sequence as illustrated by the following simple example.

Example 4.1: In case the system is linear, i.e. given by $x^+ = Ax + Bu + w$ and the system variables are subject to constraints $(x, u, w) \in \mathcal{X} \times \mathcal{U} \times \mathcal{W}$ the mapping $\mathcal{B}(\cdot)$, i.e. the 1–step controllable set is given explicitly by:

$$\mathcal{B}(X) = \{x \in \mathcal{X} \colon Ax \in [X \oplus (-B\mathcal{U})] \ominus \mathcal{W}\}. \qquad (22)$$

Consider the scalar linear system with the state-update equation: $x^+ = 2x + u + w$ and constraint sets $\mathcal{U} = [-5, 5]$, $\mathcal{W} = [-1, 1]$ and $\mathcal{X} = \mathbb{R}$. Let the target set $\mathcal{X}_f = \mathcal{X}_0 = [-1, 1]$. Note that the set $\mathcal{X}_0$ is robust control invariant. It is easy to verify that the maximal robust control invariant set is $\tilde{\mathcal{X}}_\infty = [-4, 4]$. The j–step controllable sets $\mathcal{X}_j$, $j \in \mathbb{N}_0$ are given by $\mathcal{X}_j = \left[-4 + 3/2^j, 4 - 3/2^j\right]$. Thus, the set $\tilde{\mathcal{X}}_\infty$ is never attained through the recursion (7) for a finite j. The sets $\Psi(\mathcal{X}_{j-1})$, $j \in \mathbb{N}_{[1,5]}$, are shown on Figure 1. Consider the initial state $x_0 = 4 - \varepsilon$, where $\varepsilon > 0$ is such that $x_0 \in \mathcal{X}_5$. For a constant disturbance $w = 1$, the target set $\mathcal{X}_f = \mathcal{X}_0$ can be reached in 5 time steps. On the other hand, for a constant disturbance $w = -1$, the pair $(x, w) \in \Psi(\mathcal{X}_1)$ and, consequently, the target set $\mathcal{X}_0$ can be reached in at most 2 time steps. It is not difficult to see that by choosing the $\varepsilon > 0$ arbitrarily close to 0 the number of steps needed to reach the target set $\mathcal{X}_0$ (for the constant disturbance sequence $\{1, 1, \ldots, 1 \ldots\}$) can be made arbitrarily large. Note also that states -4 and 4 can not be max–min controlled to the target set $\mathcal{X}_0$ since they are limit points and can only be kept in the maximal robust control invariant set $\tilde{\mathcal{X}}_\infty$.

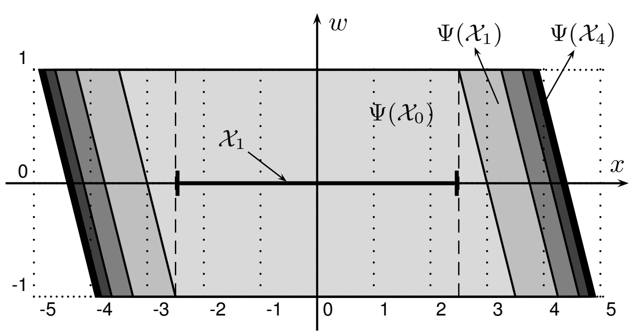

Fig. 1. The sets $\Psi(\mathcal{X}_0), \ldots, \Psi(\mathcal{X}_4)$ for the Example 4.1.

Remark 4.1: Utilizing Propositions 4.1 and 4.2 it is clear that in the case when the target set $\mathcal{X}_f$ is, in addition, robust control invariant set, the j–step controllable sets are nonempty and robust control invariant for all $j \in \mathbb{N}$ and, in addition, $\forall j \in \mathbb{N}$, $\mathcal{X}_{j-1} \subseteq \mathcal{X}_j$. However, in a practical application of the time–optimal controllers, it is a standard practice to fix an integer $N_{\max} \in \mathbb{N}$ (which can be arbitrary large) and pre–compute the j–step controllable sets $\mathcal{X}_j$, $j \in \mathbb{N}_{[1,N_{\max}]}$ and corresponding set–valued control maps $\mathcal{U}_j(\cdot, \cdot)$ [8], [23], [24]. In this case, the actual implementation of the max–min time–optimal control reduces to, for a given (measured) state $x \in \mathcal{X}_j$, $j \in \mathbb{N}_{[1,N_{\max}]}$, and disturbance realization $w \in \mathcal{W}$, finding the index

$$N^*(x, w) = \min_{j \in \mathbb{N}_{[1,N_{\max}]}} \{j \colon (x, w) \in \Psi(\mathcal{X}_{j-1})\}$$

where $\Psi(\mathcal{X}_{j-1})$ is given by (18) and applying any control lying in the corresponding control set $\mathcal{U}_{N^*(x,w)}(x, w)$:

$$u(x, w) \in \mathcal{U}_{N^*(x,w)}(x, w).$$

We observe that, for $x \in \mathcal{X}_j$, $j \in \mathbb{N}_{[1,N_{\max}]}$, and $w \in \mathcal{W}$, the following holds:

$$N^*(x, w) \leq N^0(x) \text{ and } N^0(x) = \sup_{w \in \mathcal{W}} N^*(x, w).$$

Under Assumptions **A1–A5** and when the target set $\mathcal{X}_f$ is a compact polygon and robust control invariant set, the j–step controllable sets $\mathcal{X}_j$ and the graphs of set–valued control maps $\mathcal{U}_j(\cdot, \cdot)$ are, nonempty, compact polygons and hence are, in principle, pre–computable by utilizing the standard computational geometry software [25], [26]. Furthermore, in this case, a suitable selections of the set–valued control maps $\mathcal{U}_j(\cdot, \cdot)$ can be obtained by utilizing (20) and employing parametric programming [27]. Using (20), each (nonempty) set–valued control map $\mathcal{U}_{(j,i,k)}(\cdot, \cdot)$ admits a selection $\pi_{(j,i,k)}(\cdot, \cdot)$ given by the following parametric programming problem:

$$\pi_{(j,i,k)}(x, w) \in \arg \inf_u \left\{V_{sel}(x, u, w) \colon u \in \mathcal{U}_{(j,i,k)}(x, w)\right\}, \qquad (23)$$

where $V_{sel}(\cdot, \cdot) \colon \Omega \to \mathbb{R}$ is a selection criteria and $(i, k) \in \mathbb{N}_{[1,q]} \times \mathbb{N}_{[1,d]}$ and $\mathcal{U}_{(j,i,k)}(\cdot, \cdot)$ is given by (20).

When the selection criteria $V_{sel}(\cdot, \cdot) \colon \Omega \to \mathbb{R}$ is a linear or a convex quadratic function, the optimization problem (23) can be written as a parametric (linear or quadratic) program and solved using various implementations of convex linear/quadratic parametric programming algorithms [27]. In this case, there exists a continuous, piecewise affine, selection $\pi_{(j,i,k)}(\cdot, \cdot) \in \mathcal{U}_{(j,i,k)}(\cdot, \cdot)$ defined for all $(x, w) \in \Psi_{(j,i,k)}$ where $\Psi_{(j,i,k)}$ is given by (19). Control laws $\pi_{(j,i,k)}(\cdot, \cdot)$, then yield a control law $\pi_{N_{\max}-j}(\cdot, \cdot) \in \mathcal{U}_j(\cdot, \cdot)$ by utilizing (19) in a transparent way. However, since the set–valued control map $\mathcal{U}_j(\cdot, \cdot)$ given by (20) is the union of set–valued maps of $\mathcal{U}_{(j,i,k)}(\cdot, \cdot)$, the resulting control law $\pi_{N_{\max}-j}(\cdot, \cdot) \in \mathcal{U}_j(\cdot, \cdot)$ defined for all $(x, w) \in \Psi(\mathcal{X}_{j-1})$, where $\Psi(\mathcal{X}_{j-1})$ is given by (18), may not be continuous (though a set–valued control law $\tilde{\pi}_{N_{\max}-j}(x, w) \in \mathcal{U}_j(x, w)$, $(x, w) \in \Psi(\mathcal{X}_{j-1})$ with a closed (compact) graph and which is set–valued only on, possibly, subsets (of $\Psi(\mathcal{X}_{j-1})$) of zero measure is easily obtained with a modest increase of the computational effort).

V. Numerical example

Consider a continuous PWA system given by:

$$x^+ = \begin{cases} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w & \text{if } x_2 + u \leq 0 \\ \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 0.5 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w & \text{if } x_2 + u > 0 \end{cases} \qquad (24)$$

The constraints set Ω is given as $\Omega = \Omega_x \times \Omega_u \times \mathcal{W}$, where $\Omega_x = \{x \colon |x|_\infty \leq 5\}$, $\Omega_u = [-1, 1]$ and $\mathcal{W} = [-1, -0.5]$. It is easy to verify that the PWA mapping (24) is continuous. A target set is $\mathcal{X}_0 = \{x \colon |x|_\infty \leq 1\}$. Polytopic computations are performed using the Multi–Parametric Toolbox [25] for Matlab, though the computations can be performed by any software tool implementing basic polytopic operations (e.g. [26]). The controllable sets $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{X}_3$ are shown in

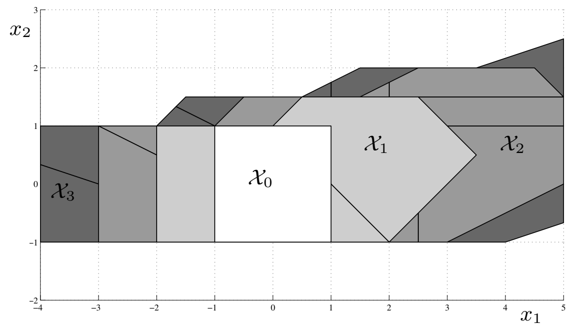

Fig. 2. Controllable sets for the continuous PWA system.

Figure 2. Figure 3 depicts the set $\Psi(\mathcal{X}_1)$ and the controllable set $\mathcal{X}_2$. One can see a geometric interpretation of the max-min controllability: the product $\mathcal{X}_2 \times \mathcal{W}$ is a subset of $\Psi(\mathcal{X}_1)$ - for each state in the set $\mathcal{X}_2$ and *all* $w \in \mathcal{W}$ there exists an admissible control that drives the state into the set $\mathcal{X}_1$. Computational tractability is not significantly impaired by

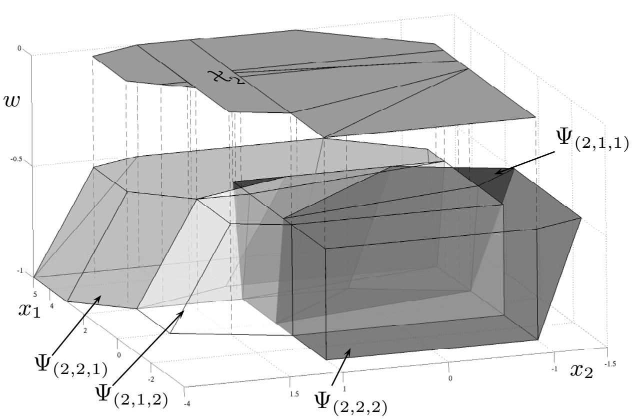

Fig. 3. The set $\Psi(\mathcal{X}_1)$ for the example of continuous PWA system.

the fact that the controllable sets in the example are non-convex since the required polytopic manipulations are easily extendable to polygons. Nevertheless, the required projection operation limits potential applications of the method to lower dimensional problems.

VI. Conclusion

We have presented a general framework for the design of max-min controllers of constrained discrete time systems. We have given computational details for a class of continuous PWA systems. Computational requirements are the key factor that may potentially limit the applicability of the method. The exact computations are, however, tractable for particular, but frequently encountered, classes of systems.

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