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Direct Adaptive Control for a Mutli-Compartmental Model of a Pressure-Limited Respirator and Lung Mechanics System

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Abstract— **In this paper, we develop an adaptive control framework for a multi-compartmental model of a pressure-limited respirator and lung mechanics system. Specifically, we develop a model reference direct adaptive controller framework where the plant and reference model involve switching and time-varying dynamics. We then apply the proposed adaptive feedback controller framework to stabilize a given limit cycle corresponding to a clinically plausible respiratory pattern.**

I. Introduction

Mechanical ventilation of a patient with respiratory failure is one of the most common life-saving procedures performed in the intensive care unit. However, mechanical ventilation is physically uncomfortable due to the noxious interface between the ventilator and patient, and mechanical ventilation evokes substantial anxiety on the part of the patient. This will often be manifested by the patient "fighting the ventilator." In this situation, there is dyssynchrony between the ventilatory effort of the patient and the ventilator. The patient will attempt to exhale at the time the ventilator is trying to expand the lungs or the patient will try to inhale when the ventilator is decreasing airway pressure to allow an exhalation. When patient-ventilator dyssynchrony occurs, at the very least there is excessive work of breathing with subsequent ventilatory muscle fatigue and in the worst case, elevated airway pressures that can actually rupture lung tissue. In this situation, it is a very common clinical practice to sedate patients to minimize "fighting the ventilator." Sedative-hypnotic agents act on the central nervous system to ameliorate the anxiety and discomfort associated with mechanical ventilation and facilitate patient-ventilator synchrony. In this paper, we develop an adaptive feedback controller for addressing this dyssynchrony for intensive care unit sedation.

In a recent paper [1], we extended the existing models for ventilation systems [2–9] to obtain a general mathematical model for the dynamic behavior of a multi-compartment respiratory system in response to an arbitrary applied inspiratory pressure. Specifically, we used compartmental dynamical system theory to model and analyze the dynamics of a pressure-limited respirator and lung mechanics system, and showed that the periodic orbit generated by this system is globally asymptotically stable. Furthermore, we showed that the individual compartmental volumes, and hence the total lung volume, converge to steady-state end-inspiratory and end-expiratory values. In this paper, we develop a model reference direct adaptive controller framework where the plant and reference model involve switching and time-varying dynamics. Then, we apply the proposed adaptive framework to the multi-compartmental model of a pressure-limited respirator and lung mechanics system. Specifically, we develop an adaptive feedback controller that stabilizes a given limit cycle corresponding to a clinically plausible breathing pattern. Finally, we apply the proposed adaptive control framework to a mechanical ventilation model to quantify patient-ventilator dyssynchrony for intensive care unit sedation.

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II. Notation and Mathematical Preliminaries

In this section, we introduce notation, several definitions, and some key results that are necessary for developing the main results of this paper. Specifically, for $x \in \mathbb{R}^n$ we write $x \geq\geq 0$ (resp., $x >> 0$) to indicate that every component of x is nonnegative (resp., positive). In this case, we say that x is *nonnegative* or *positive*, respectively. Likewise, $A \in \mathbb{R}^{n\times m}$ is *nonnegative* or *positive* if every entry of A is nonnegative or positive, respectively, which is written as $A \geq\geq 0$ or $A >> 0$, respectively. Furthermore, for $A \in \mathbb{R}^{n\times n}$ we write $A \geq 0$ (resp., $A > 0$) to indicate that A is a nonnegative-definite (resp., positive-definite) matrix. In addition, $(\cdot)^{\mathrm{T}}$ denotes transpose and $(\cdot)^{-1}$ denotes inverse. Let $\overline{\mathbb{R}}_+^n$ and $\mathbb{R}_+^n$ denote the nonnegative and positive orthants of $\mathbb{R}^n$, that is, if $x \in \mathbb{R}^n$, then $x \in \overline{\mathbb{R}}_+^n$ and $x \in \mathbb{R}_+^n$ are equivalent, respectively, to $x \geq\geq 0$ and $x >> 0$. Finally, $\mathbf{e}_n \in \mathbb{R}^n$ denotes the ones vector of order n, that is, $\mathbf{e}_n = [1, \cdots, 1]^{\mathrm{T}}$; if the order of $\mathbf{e}_n$ is clear from context we simply write $\mathbf{e}$ for $\mathbf{e}_n$.

The following definitions introduce the notions of essentially nonnegative, compartmental, and strictly ultrametric matrices.

Definition 2.1 ([10]): Let $A \in \mathbb{R}^{n\times n}$. A is *essentially nonnegative* if $A_{(i,j)} \geq 0$, $i,j = 1,\ldots,n$, $i \neq j$. A is *compartmental* if A is essentially nonnegative and $\mathbf{e}^{\mathrm{T}}A \leq\leq 0$.

Definition 2.2 ([11]): Let $A \in \mathbb{R}^{n\times n}$ be such that $A \geq\geq 0$. A is *strictly ultrametric* if A is symmetric, $A_{(i,i)} > \max\{A_{(i,k)} : k = 1,\ldots,n, k \neq i\}$, $i = 1,\ldots,n$, and $A_{(i,j)} \geq \min\{A_{(i,k)}, A_{(k,j)}\}$, $k = 1,\ldots,n$, $i,j = 1,\ldots,n, i \neq j$.

In this paper, we consider nonlinear periodic dynamical systems of the form

$$\dot{x}(t) = f(t, x(t)), \qquad x(0) = x_0, \qquad t \in \mathcal{I}_{x_0}, \quad (1)$$

where $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $t \in \mathcal{I}_{x_0}$, is the system state vector, $\mathcal{D}$ is an open set, $f : [0,\infty) \times \mathcal{D} \to \mathbb{R}^n$ satisfies $f(t,x) = f(t+T,x)$, $x \in \mathcal{D}$, $t \geq 0$, for some $T > 0$, and $\mathcal{I}_{x_0} = [0,\tau_{x_0})$, $0 < \tau_{x_0} \leq \infty$, is the maximal interval of existence for the solution $x(\cdot)$ of (1). A function $x : \mathcal{I}_{x_0} \to \mathcal{D}$ is said to be a *solution* to (1) on the interval $\mathcal{I}_{x_0} \subseteq [0,\infty)$ with initial condition $x(0) = x_0$ if $x(t)$ satisfies (1) for all $t \in \mathcal{I}_{x_0}$. It is assumed that $f(\cdot,\cdot)$ is such that the solution to (1) is unique for every initial condition in $\mathcal{D}$ and jointly continuous in t and x_0. A sufficient condition ensuring this is Lipschitz continuity of $f(t,\cdot) : \mathcal{D} \to \mathbb{R}^n$ for all $t \in [0,t_1]$

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and continuity of $f(\cdot, x) : [0, t_1] \to \mathbb{R}^n$ for all $x \in \mathcal{D}$. Here, we assume that all solutions to (1) are bounded over $\mathcal{I}_{x_0}$, and hence, by the Peano-Cauchy theorem can be extended to infinity.

Next, we introduce the notions of periodic solutions and periodic orbits for (1). For the next definition, we denote the solution $x(\cdot)$ to (1) with initial conditon $x_0 \in \mathcal{D}$ by $s(t, x_0)$.[1]

Definition 2.3: A solution $s(t, x_0)$ of (1) is *periodic* if there exists a finite time $T > 0$ such that $s(t + T, x_0) = s(t, x_0)$ for all $t \geq 0$. A set $\mathcal{O} \subset \mathcal{D}$ is a *periodic orbit* of (1) if $\mathcal{O} = \{x \in \mathcal{D} : x = s(t, x_0), 0 \leq t \leq T\}$ for some periodic solution $s(t, x_0)$ of (1).

Finally, we introduce the notions of Lyapunov and asymptotic stability of a periodic orbit of the nonlinear dynamical system (1). For this definition, $\mathrm{dist}(p, \mathcal{M})$ denotes the distance from a point p to any point in the set $\mathcal{M}$, that is, $\mathrm{dist}(p, \mathcal{M}) \triangleq \inf_{x \in \mathcal{M}} \|p - x\|$.

Definition 2.4: A periodic orbit $\mathcal{O}$ of (1) is *Lyapunov stable* if, for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $\mathrm{dist}(x_0, \mathcal{O}) < \delta$, then $\mathrm{dist}(s(t, x_0), \mathcal{O}) < \varepsilon, t \geq 0$. A periodic orbit $\mathcal{O}$ is *asymptotically stable* if $\mathcal{O}$ is Lyapunov stable and there exists $\varepsilon > 0$ such that if $\mathrm{dist}(x_0, \mathcal{O}) < \varepsilon$, then $\mathrm{dist}(s(t, x_0), \mathcal{O}) \to 0$ as $t \to \infty$.

III. Compartmental Modeling of Lung Dynamics

In this section, we present a general mathematical model for the dynamic behavior of a multi-compartment respiratory system in response to an arbitrary applied inspiratory pressure [1]. Here, we assume that the bronchial tree has a *dichotomy architecture* [1], [12], that is, in every generation each airway unit branches into two airway units of the subsequent generation. First, however, we start by considering a single-compartmental lung model as shown in Figure 1. At time $t = 0$, an arbitrary pressure $p_{\mathrm{in}}(t)$ is applied to

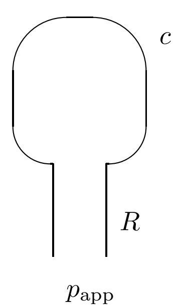

Fig. 1. Single-compartment lung model

the opening of the parent airways. At time $t = T_{\mathrm{in}}$, the applied airway pressure is released and expiration takes place passively, that is, the external pressure $p_{\mathrm{ex}}(t)$ is applied during the time interval $T_{\mathrm{in}} \leq t \leq T_{\mathrm{in}} + T_{\mathrm{ex}}$, where T_{ex} is the duration of expiration.

The state equation for inspiration (inflation of lung) is given by

$$R_{\mathrm{in}}\dot{x}(t) + \frac{1}{c}x(t) = p_{\mathrm{in}}(t),\ x(0) = x_0^{\mathrm{in}},\ 0 \leq t \leq T_{\mathrm{in}}, \quad (2)$$

where $x(t) \in \mathbb{R}$, $t \geq 0$, is the lung volume, $R_{\mathrm{in}} \in \mathbb{R}$ is the resistance to air flow during the inspiration period, and $x_0^{\mathrm{in}} \in \mathbb{R}$ is the lung volume at the start of the inspiration and serves as the system initial condition. We assume that expiration is passive (due to elastic stretch of lung unit). During the expiration process, the state equation is given by

$$R_{\mathrm{ex}}\dot{x}(t) + \frac{1}{c}x(t) = p_{\mathrm{ex}}(t),\ x(T_{\mathrm{in}}) = x_0^{\mathrm{ex}}, \quad T_{\mathrm{in}} \leq t \leq T_{\mathrm{in}} + T_{\mathrm{ex}}, \quad (3)$$

where $x(t) \in \mathbb{R}$, $t \geq 0$, is the lung volume, $R_{\mathrm{ex}} \in \mathbb{R}$ is the resistance to air flow during the expiration period, and $x_0^{\mathrm{ex}} \in \mathbb{R}$ is the lung volume at the start of expiration.

Next, we present the state equations for a multi-compartment model. In this model, the lungs are represented as 2^n lung units which are connected to the pressure source by n generations of airway units, where each airway is divided into two airways of the subsequent generation leading to 2^n compartments (see Figure 2 for a four-compartment model).

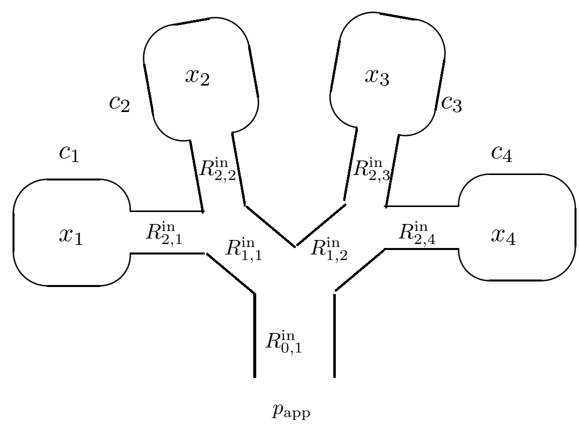

Fig. 2. Four-compartment lung model

Let $c_i,\ i = 1, 2, \ldots, 2^n$, denote the compliance of each compartment and let $R_{j,i}^{\mathrm{in}}$ (resp., $R_{j,i}^{\mathrm{ex}}$), $i = 1, 2, \ldots, 2^j,\ j = 0, \ldots, n$, denote the resistance (to air flow) of the i-th airway in the j-th generation during the inspiration (resp., expiration) period with R_{01}^{in} (resp., R_{01}^{ex}) denoting the inspiration (resp., expiration) of the *parent* (i.e., 0 generation) airway. As in the single-compartment model we assume that a pressure of $p_{\mathrm{app}}(t)$ is applied during inspiration. Next, let $x_i,\ i = 1, 2, \ldots, 2^n$, denote the lung volume in the i-th compartment. Now, the state equations for inspiration and expiration are given by

$$R_{\mathrm{in}}\dot{x}(t) + Cx(t) = p_{\mathrm{in}}\mathbf{e},\ x(0) = x_0^{\mathrm{in}},\ 0 \leq t \leq T_{\mathrm{in}}, \quad (4)$$

$$R_{\mathrm{ex}}\dot{x}(t) + Cx(t) = p_{\mathrm{ex}}\mathbf{e},\ x(T_{\mathrm{in}}) = x_0^{\mathrm{ex}}, \quad T_{\mathrm{in}} \leq t \leq T_{\mathrm{ex}} + T_{\mathrm{in}}, \quad (5)$$

where $x \triangleq [x_1, x_2, \cdots, x_{2^n}]^{\mathrm{T}}$, $C \triangleq \mathrm{diag}[\frac{1}{c_1}, \cdots, \frac{1}{c_{2^n}}]$, and

$$R_{\mathrm{in}} \triangleq \sum_{j=0}^{n}\sum_{k=1}^{2^j} R_{j,k}^{\mathrm{in}}\ Z_{j,k}\ Z_{j,k}^{\mathrm{T}}, \quad (6)$$

$$R_{\mathrm{ex}} \triangleq \sum_{j=0}^{n}\sum_{k=1}^{2^j} R_{j,k}^{\mathrm{ex}}\ Z_{j,k}\ Z_{j,k}^{\mathrm{T}}, \quad (7)$$

[1]Note that since (1) is a time-varying dynamical system it is typical to denote its solution as $\hat{s}(t, t_0, x_0)$ to indicate the dependence on both the initial time t_0 and the initial state x_0. In this paper, we assume that $t_0 = 0$ and define $s(t, x_0) \triangleq \hat{s}(t, 0, x_0)$.

where $Z_{j,k} \in \mathbb{R}^{2^n}$ is such that the l-th element of $Z_{j,k}$ is 1 for all $l = (k-1)2^{n-j}+1,\ (k-1)2^{n-j}+2, \ldots, k2^{n-j}, k = 1, \ldots, 2^j, j = 0, 1, \ldots, n$, and zero elsewhere.

Note that if $R_{\rm in}$ and $R_{\rm ex}$ are invertible, then (4) and (5) can be equivalently written as

$$\dot{x}(t) = A_{\rm in}x(t) + B_{\rm in}p_{\rm in}(t),\ x(0) = x_0^{\rm in}, \quad 0 \le t \le T_{\rm in}, \quad (8)$$

$$\dot{x}(t) = A_{\rm ex}x(t) + B_{\rm ex}p_{\rm ex}(t),\ x(T_{\rm in}) = x_0^{\rm ex}, \quad T_{\rm in} \le t \le T_{\rm ex} + T_{\rm in}, \quad (9)$$

where $A_{\rm in} \triangleq -R_{\rm in}^{-1}C$, $B_{\rm in} \triangleq R_{\rm in}^{-1}\mathbf{e}$, $A_{\rm ex} \triangleq -R_{\rm ex}^{-1}C$, and $B_{\rm ex} \triangleq R_{\rm ex}^{-1}\mathbf{e}$.

The following proposition states several important properties of $R_{\rm in}, R_{\rm ex}, A_{\rm in}$, and $A_{\rm ex}$ that are essential for the main results of this paper.

Proposition 3.1 ([1]): Consider the dynamical system (4) and (5) or, equivalently, (8) and (9). Then the following statements hold:

i) $R_{\rm in} > 0$ and $R_{\rm ex} > 0$.
ii) $A_{\rm in}^{\rm T}C + CA_{\rm in} < 0$.
iii) $A_{\rm ex}^{\rm T}C + CA_{\rm ex} < 0$.
iv) $R_{\rm in}$ and $R_{\rm ex}$ are strictly ultrametric.
v) $A_{\rm in}$ and $A_{\rm ex}$ are compartmental and Hurwitz, and $B_{\rm in} \geq\geq 0$ and $B_{\rm ex} \geq\geq 0$, where $B_{\rm in} \triangleq R_{\rm in}^{-1}\mathbf{e}$ and $B_{\rm ex} \triangleq R_{\rm ex}^{-1}\mathbf{e}$.

Remark 3.1: It follows from Proposition 3.1 that the dynamical system (4) and (5) is nonnegative, that is, the state of (4) and (5) remains in the nonnegative orthant for $t \geq 0$. See [1] for details.

Remark 3.2: It follows from Proposition 3.1 that $R_{\rm in}$ and $R_{\rm ex}$ are invertible. Hence, $A_{\rm in}$ and $A_{\rm ex}$ are well defined, which implies that the state equations for inspiration and expiration given by (8) and (9), respectively, are well defined.

In this paper, we assume that the inspiration process starts from a given initial state $x_0^{\rm in}$ followed by the expiration process where its initial state will be the final state of the inspiration. An inspiration followed by the expiration is called a *breathing cycle*. We assume that each breathing cycle is followed by another breathing cycle where the initial condition for the latter breathing cycle is the final state of the former breathing cycle. Furthermore, we assume that the duration of inspiration is $T_{\rm in}$ and that of expiration is $T_{\rm ex}$ so that the total duration of a breathing cycle is $T_{\rm in} + T_{\rm ex}$. It is clear that this process generates a periodic dynamical system with a period $T \triangleq T_{\rm in}+T_{\rm ex}$. Furthermore, the system dynamics switch from inspiration to expiration and back to inspiration. Hence, the dynamics for a breathing cycle can be characterized by the periodic switched dynamical system $\mathcal{G}$ given by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(0) = x_0^{\rm in}, \quad t \geq 0, \quad (10)$$

where

$$A(t) = A(t+T),\ u(t) = u(t+T),\ t \geq 0, \quad (11)$$

$$A(t) = \begin{cases} A_{\rm in}, & 0 \le t < T_{\rm in}, \\ A_{\rm ex}, & T_{\rm in} \le t < T, \end{cases} \quad (12)$$

$$B(t) = \begin{cases} B_{\rm in}, & 0 \le t < T_{\rm in}, \\ B_{\rm ex}, & T_{\rm in} \le t < T, \end{cases} \quad (13)$$

$$u(t) = \begin{cases} p_{\rm in}(t), & 0 \le t < T_{\rm in}, \\ p_{\rm ex}(t), & T_{\rm in} \le t < T. \end{cases} \quad (14)$$

Next, we characterize and analyze the stability of periodic orbits of the switched dynamical system $\mathcal{G}$ given by (10). First, note that (see [1] for details)

$$x_0^{\rm ex} = x(T_{\rm in}) = \Gamma_{\rm in}x_0^{\rm in} + \theta, \quad (15)$$

where

$$\Gamma_{\rm in} \triangleq e^{A_{\rm in}T_{\rm in}}, \quad (16)$$

$$\theta \triangleq e^{A_{\rm in}T_{\rm in}}\int_0^{T_{\rm in}} e^{-A_{\rm in}t}B_{\rm in}p_{\rm in}(t){\rm d}t. \quad (17)$$

Furthermore, note that

$$x(T) = \Gamma_{\rm ex}x_0^{\rm ex} + \delta, \quad (18)$$

where

$$\Gamma_{\rm ex} \triangleq e^{A_{\rm ex}T_{\rm ex}}, \quad (19)$$

$$\delta \triangleq e^{A_{\rm ex}T}\int_{T_{\rm in}}^{T} e^{-A_{\rm ex}t}B_{\rm ex}p_{\rm ex}(t){\rm d}t. \quad (20)$$

Next, let $x_{\rm m}^{\rm in}$ denote the initial condition for the m-th inspiration (and hence the m-th breathing cycle) and let $x_m^{\rm ex}$ denote the initial condition for the m-th expiration, that is, $x_{\rm m}^{\rm in} = x(mT)$ and $x_{\rm m}^{\rm ex} = x(mT+T_{\rm in})$, $m = 0, 1, \ldots$. Hence, it follows from (15) and (18) that

$$x_1^{\rm in} = \Gamma_{\rm ei}x_0^{\rm in} + \Gamma_{\rm ex}\theta + \delta, \quad (21)$$

where $\Gamma_{\rm ei} \triangleq \Gamma_{\rm ex}\Gamma_{\rm in}$. Similarly, it can be shown that

$$x_1^{\rm ex} = \Gamma_{\rm ie}x_0^{\rm ex} + \Gamma_{\rm in}\delta + \theta, \quad (22)$$

where $\Gamma_{\rm ie} \triangleq \Gamma_{\rm in}\Gamma_{\rm ex}$. More generally,

$$x_{m+1}^{\rm in} = \Gamma_{\rm ei}x_{\rm m}^{\rm in} + \Gamma_{\rm ex}\theta + \delta, \quad m = 0, 1, \ldots, \quad (23)$$

$$x_{m+1}^{\rm ex} = \Gamma_{\rm ie}x_{\rm m}^{\rm ex} + \Gamma_{\rm in}\delta + \theta, \quad m = 0, 1, \ldots. \quad (24)$$

The following proposition states two key properties for $\Gamma_{\rm ei}$ and $\Gamma_{\rm ie}$ which are useful in characterizing a periodic orbit for the switched dynamical system $\mathcal{G}$.

Proposition 3.2 ([1]): The following statements hold:

i) $\Gamma_{\rm ex}^{\rm T}C\Gamma_{\rm ex} < C$ and $\Gamma_{\rm in}^{\rm T}C\Gamma_{\rm in} < C$.
ii) $\Gamma_{\rm ei}^{\rm T}C\Gamma_{\rm ei} < C$ and $\Gamma_{\rm ie}^{\rm T}C\Gamma_{\rm ie} < C$.

For the next result, define $\hat{x}_{\rm in} \triangleq (I - \Gamma_{\rm ei})^{-1}(\Gamma_{\rm ex}\theta + \delta)$ and $\hat{x}_{\rm ex} \triangleq (I - \Gamma_{\rm ie})^{-1}(\Gamma_{\rm in}\delta + \theta)$.

Proposition 3.3 ([1]): Consider the switched dynamical system $\mathcal{G}$ given by (10). Then for every $x_0^{\rm in} \in \overline{\mathbb{R}}_+^n$, the following statements hold:

i) $\lim_{m\to\infty} x_{\rm m}^{\rm in} = \hat{x}_{\rm in}$ and $\lim_{m\to\infty} x_{\rm m}^{\rm ex} = \hat{x}_{\rm ex}$.
ii) For every $t \in [0, T_{\rm in}]$,

$$\lim_{m\to\infty} x(t+mT) = e^{A_{\rm in}t}\hat{x}_{\rm in} + \int_0^t e^{A_{\rm in}(t-\tau)}B_{\rm in}p_{\rm in}(\tau){\rm d}\tau,$$

and for every $t \in [T_{\rm in}, T]$,

$$\lim_{m\to\infty} x(t+mT+T_{\rm in}) = e^{A_{\rm ex}t}\hat{x}_{\rm ex} + \int_0^t e^{A_{\rm ex}(t-\tau)}B_{\rm ex}p_{\rm ex}(\tau+T_{\rm in}){\rm d}\tau.$$

Remark 3.3: It follows from Proposition 3.3 that the individual compartmental volumes, and hence the total volume, converge to the steady-state end-inspiratory and end-expiratory values of $(I - \Gamma_{\rm ei})^{-1}(\Gamma_{\rm ex}\theta + \delta)$ and $(I - \Gamma_{\rm ie})^{-1}(\Gamma_{\rm in}\delta + \theta)$, respectively.

Next, let $\hat{x} \triangleq (I-\Gamma_{\rm ei})^{-1}(\Gamma_{\rm ex}\theta+\delta)$ and define the orbit

$$\begin{aligned}\mathcal{O}_{\hat{x}} \;\triangleq\; & \{x \in \overline{\mathbb{R}}_{+}^{n} : x = s(t,\hat{x}), \\ & \text{where } s(t,\hat{x}) \text{ is the solution to (10)}\}. \end{aligned} \quad (25)$$

With $x_0^{\rm in} = \hat{x}$ note that $x_{\rm m}^{\rm in} = \hat{x}$, $m = 1,2,\ldots$, or, equivalently, $x(mT) = \hat{x}$, $m = 1,2,\ldots$, which implies that $\mathcal{O}_{\hat{x}}$ is a periodic orbit of (10).

Theorem 3.1 ([1]): Consider the switched dynamical system $\mathcal{G}$ given by (10). Then the periodic orbit $\mathcal{O}_{\hat{x}}$ of $\mathcal{G}$ generated by $x(0) = \hat{x} = (I-\Gamma_{\rm ei})^{-1}(\Gamma_{\rm ex}\theta+\delta)$ is globally asymptotically stable.

Remark 3.4: Note that Theorem 3.1 is valid for arbitrary nonnegative functions (possibly discontinuous) $p_{\rm in}(t)$ and $p_{\rm ex}(t)$ as long as

$$\int_0^{T_{\rm in}} e^{-A_{\rm in}t}B_{\rm in}p_{\rm in}(t){\rm d}t$$

and

$$\int_{T_{\rm in}}^{T} e^{-A_{\rm ex}t}B_{\rm ex}p_{\rm ex}(t){\rm d}t$$

are finite. In the case where $p_{\rm in}(t) = \alpha t + \beta$ and $p_{\rm ex}(t) = \gamma$ for some positive constants α, β, and γ, θ and δ are given by

$$\begin{aligned}\theta &= A_{\rm in}^{-2}[(\alpha I + \beta A_{\rm in})(e^{A_{\rm in}T_{\rm in}} - I) - \alpha A_{\rm in}T_{\rm in}]B_{\rm in}, \\ \delta &= \gamma A_{\rm ex}^{-1}(e^{A_{\rm ex}T_{\rm ex}} - I)B_{\rm ex}.\end{aligned}$$

Remark 3.5: Although Theorem 3.1 is presented for the simpler case of regular dichotomy architecture, it also holds for more general case of irregular dichotomy archietecture [1]. Hence, all the results presented in Section IV and V trivially apply to the case of irregular dichotomy architecture. Here, we consider the case of regular dichotomy architecture for simplicity of exposition.

IV. Direct Adaptive Control for Switched Linear Time-Varying Systems

In this section, we consider the problem of adaptive tracking of uncertain linear time-varying switching systems. Specifically, consider the following controlled uncertain switched linear time-varying system $\mathcal{G}$ given by

$$\dot{x}_{\rm p}(t) = A_{\rm p}(t)x_{\rm p}(t) + B_{\rm p}(t)u(t), \quad x_{\rm p}(0) = x_{\rm p0}, \quad t \geq 0, \quad (26)$$

where $x_{\rm p}(t) \in \mathbb{R}^n, t \geq 0$, is the state vector, $u(t) \in \mathbb{R}^p, t \geq 0$, is the control input, and $A_{\rm p}(t) \in \mathbb{R}^{n\times n}, t \geq 0$, and $B_{\rm p}(t) \in \mathbb{R}^{n\times p}, t \geq 0$, are *unknown* matrices. The control input $u(\cdot)$ in (26) is restricted to the class of *admissible controls* consisting of measurable functions such that $u(t) \in \mathbb{R}^p, t \geq 0$. Furthermore, for the uncertain linear time-varying system $\mathcal{G}$, we assume that $A_{\rm p}(\cdot)$ and $B_{\rm p}(\cdot)$ are piecewise continuous functions and we assume that the required properties for the existence and uniqueness of solutions are satisfied; that is, $A_{\rm p}(\cdot), B_{\rm p}(\cdot)$, and $u(\cdot)$ satisfy sufficient regularity conditions such that (26) has a unique solution forward in time.

Next, consider a reference model given by

$$\dot{x}_{\rm m}(t) = A_{\rm m}(t)x_{\rm m}(t) + B_{\rm m}(t)r(t), \quad x_{\rm m}(0) = x_{\rm m0}, \quad t \geq 0, \quad (27)$$

where $x_{\rm m}(t) \in \mathbb{R}^n, t \geq 0$, is the state vector, $r(t) \in \mathbb{R}^p, t \geq 0$, is the reference input, and $A_{\rm m}(t) \in \mathbb{R}^{n\times n}, t \geq 0$, and $B_{\rm m}(t) \in \mathbb{R}^{n\times p}, t \geq 0$, are known matrices. Moreover, let $A_{\rm m}(t)$ satisfy

$$A_{\rm m}^{\rm T}(t)C_{\rm m} + C_{\rm m}A_{\rm m}(t) \leq -\varepsilon_{\rm m}I, \quad (28)$$

where $\varepsilon_{\rm m} > 0$ and $C_{\rm m} \in \mathbb{R}^{n\times n}$ is positive definite. Furthermore, we assume that $A_{\rm m}(\cdot)$ and $B_{\rm m}(\cdot)$ are piecewise continuous and are such that (28) has a unique solution for all $t \geq 0$ and $x_{\rm m}(t)$ is uniformly bounded for all $t \geq 0$ and $x_{\rm m0} \in \mathbb{R}^n$.

For the next result, we assume that there exist positive definite matrices $Q^* \in \mathbb{R}^{p\times p}$ and $\Theta^* \in \mathbb{R}^{p\times n}$ such that the compatibility conditions

$$\begin{aligned}B_{\rm p}(t)Q^* &= B_{\rm m}(t), \quad t \geq 0, & (29)\\ A_{\rm p}(t) + B_{\rm p}(t)\Theta^* &= A_{\rm m}(t), \quad t \geq 0, & (30)\end{aligned}$$

hold.

Theorem 4.1: Consider the uncertain linear time-varying system $\mathcal{G}$ given by (26) and the reference model given by (27). Then the adaptive feedback control law

$$u(t) = \Theta(t)x_{\rm p}(t) + Q(t)r(t), \quad (31)$$

where $\Theta(t) \in \mathbb{R}^{p\times n}, t \geq 0$, and $Q(t) \in \mathbb{R}^{p\times p}, t \geq 0$, with updated laws

$$\dot{\Theta}(t) = -B_{\rm m}^{\rm T}(t)C_{\rm m}e(t)x_{\rm p}^{\rm T}(t), \; \Theta(0) = \Theta_0, \; t \geq 0, \quad (32)$$

$$\dot{Q}(t) = -B_{\rm m}^{\rm T}(t)C_{\rm m}e(t)r^{\rm T}(t), Q(0) = Q_0, \quad (33)$$

where $e(t) \triangleq x_{\rm p}(t) - x_{\rm m}(t)$, guarantees that the solution $(x_{\rm p}(t), \Theta(t), Q(t))$ of the closed-loop system given by (26), (27), (31), (32), and (33) is uniformly bounded for all $t \geq 0$ and $x_{\rm p}(t) \to x_{\rm m}(t)$ as $t \to \infty$.

Proof: Note that with $u(t), t \geq 0$, given by (31) it follows from (26) that

$$\begin{aligned}\dot{x}_{\rm p}(t) = A_{\rm p}(t)x_{\rm p}(t) + B_{\rm p}(t)\Theta(t)x_{\rm p}(t) + B_{\rm p}(t)Q(t)r(t),\\ x_{\rm p}(0) = x_{\rm p0}, \; t \geq 0, \quad (34)\end{aligned}$$

or, equivalently, using (29) and (30),

$$\begin{aligned}\dot{x}_{\rm p}(t) &= A_{\rm p}(t)x_{\rm p}(t) + B_{\rm p}(t)(\Theta^* + \Theta(t) - \Theta^*)x_{\rm p}(t)\\ &\quad + B_{\rm p}(t)(Q^* + Q(t) - Q^*)r(t)\\ &= (A_{\rm p}(t) + B_{\rm p}(t)\Theta^*)x_{\rm p}(t)\\ &\quad + B_{\rm p}(t)(\Theta(t) - \Theta^*)x_{\rm p}(t) + B_{\rm p}(t)Q^*r(t)\\ &\quad + B_{\rm p}(t)(Q(t) - Q^*)r(t)\\ &= A_{\rm m}(t)x_{\rm p}(t) + B_{\rm m}(t)r(t)\\ &\quad + B_{\rm p}(t)(\Theta(t) - \Theta^*)x_{\rm p}(t)\\ &\quad + B_{\rm p}(t)(Q(t) - Q^*)r(t)\\ &= A_{\rm m}(t)x_{\rm p}(t) + B_{\rm m}(t)r(t)\\ &\quad + B_{\rm p}(t)\Phi(t)x_{\rm p}(t) + B_{\rm p}(t)\Psi(t)r(t)\\ &\qquad x_{\rm p}(0) = x_0, \; t \geq 0, \quad (35)\end{aligned}$$

where $\Phi(t) \triangleq \Theta(t) - \Theta^*$ and $\Psi(t) \triangleq Q(t) - Q^*$. Then it follows from (27) and (35) that

$$\begin{aligned}\dot{e}(t) &= \dot{x}_{\rm p}(t) - \dot{x}_{\rm m}(t)\\ &= A_{\rm m}(t)e(t) + B_{\rm p}(t)\Phi(t)x_{\rm p}(t) + B_{\rm p}(t)\Psi(t)r(t)\\ &\qquad e(0) = x_{\rm p0} - x_{\rm m0}, \; t \geq 0. \quad (36)\end{aligned}$$

To show uniform boundedness of the closed-loop system (32), (33), and (36) consider the continuously differentiable function

$$V(e,\Phi,\Psi) = e^{\rm T}C_{\rm m}e + {\rm tr}\ \Psi^{\rm T}Q^{*-1}\Psi + {\rm tr}\ \Phi^{\rm T}Q^{*-1}\Phi, \quad (37)$$

and note that $V(0,0,0) = 0$. Since $C_{\rm m}$ and Q^* are positive definite, $V(e,\Psi,\Phi) > 0$ for all $(e,\Phi,\Psi) \neq (0,0,0)$. In addition, $V(e,\Phi,\Psi)$ is radially unbounded. Now, using (32) and (33), it follows that the derivative of $V(\cdot,\cdot,\cdot)$ along the closed-loop system trajectories is given by

$$\begin{aligned}\dot V(e(t),\Phi(t),\Psi(t)) &= e^{\mathrm T}(t)[A_{\mathrm m}^{\mathrm T}(t)C_{\mathrm m}+C_{\mathrm m}A_{\mathrm m}(t)]e(t)\\ &\quad +2e^{\mathrm T}(t)C_{\mathrm m}B_{\mathrm p}(t)\Phi(t)x_{\mathrm p}(t)\\ &\quad +2e^{\mathrm T}(t)C_{\mathrm m}B_{\mathrm p}(t)\Psi(t)r(t)\\ &\quad +2\mathrm{tr}\ \dot\Phi^{\mathrm T}(t)Q^{*-1}\Phi(t)\\ &\quad +2\mathrm{tr}\ \dot\Psi^{\mathrm T}(t)Q^{*-1}\Psi(t)\\ &= e^{\mathrm T}(t)[A_{\mathrm m}^{\mathrm T}(t)C_{\mathrm m}+C_{\mathrm m}A_{\mathrm m}(t)]e(t)\\ &\le -\varepsilon_{\mathrm m}e^{\mathrm T}(t)e(t),\ t\ge 0. \qquad (38)\end{aligned}$$

Hence, it follows from Corollary 2.4 of [13, pp. 68] that $(e(t),\Phi(t),\Psi(t))$ is uniformly bounded, and hence, $(x_{\mathrm p}(t),\Theta(t),Q(t))$ is uniformly bounded for all $t\ge 0$.

Next, with $W_1(e,\Phi,\Psi) = W_2(e,\Phi,\Psi) = V(e,\Phi,\Psi)$ and $W(e,\Phi,\Psi) = \varepsilon_{\mathrm m}e^{\mathrm T}e$, it follows from Theorem 2.5 of [13] that $(e(t),\Phi(t),\Psi(t)) \to \mathcal R$ as $t\to\infty$, where $\mathcal R \triangleq \{(e,\Phi,\Psi) : W(e,\Phi,\Psi)=0\} = \{(e,\Phi,\Psi): e=0\}$. In particular, note that

$$\begin{aligned}\dot W(e(t),\Phi(t),\Psi(t)) &= 2\varepsilon_{\mathrm m}e^{\mathrm T}\dot e\\ &= 2\varepsilon_{\mathrm m}e^{\mathrm T}(t)[A_{\mathrm m}(t)e(t)\\ &\quad +B_{\mathrm p}(t)\Phi(t)x_{\mathrm p}(t)\\ &\quad +B_{\mathrm p}(t)\Psi(t)r(t)] \qquad (39)\end{aligned}$$

is bounded for all $t\ge 0$, and hence, all conditions of Theorem 2.5 of [13, pp. 54] are satisfied proving that $e(t)\to 0$ or, equivalently, $x_{\mathrm p}(t)\to x_{\mathrm m}(t)$ as $t\to\infty$. □

Remark 4.1: Although the form of the adaptive control law given in Theorem 4.1 is identical to that of standard model reference adaptive controllers provided in the literature (see, for example, [14]), the dynamics of system considered in Theorem 4.1 are not Lipschitz continuous, and hence, standard proofs involving Barbalat's lemma do not hold. Consequently, Theorem 4.1 requires the more general result given by Theorem 2.5 of [13].

It is important to note that the adaptive laws (32) and (33) do *not* require explicit knowledge of Q^* or Θ^*. Furthermore, no specific structure on the uncertain dynamics $A_{\mathrm p}(\cdot)$ and $B_{\mathrm p}(\cdot)$ is required as long as (29) and (30) are satisfied.

V. Direct Adaptive Control for the Compartment Lung Model

In this section, we demonstrate the utility of the proposed direct adaptive control framework for a multi-compartmental lung model. First, we choose the reference model (27) to correspond to a respiratory system producing a plausible breathing pattern. Specifically, let $A_{\mathrm m}(t) = -R_{\mathrm m}^{-1}(t)C_{\mathrm m}$ and $B_{\mathrm m}(t) = R_{\mathrm m}^{-1}(t)\mathbf e$, where

$$R_{\mathrm m}(t) = \begin{cases} R_{\mathrm{in\_m}}, & 0\le t< T_{\mathrm{in}},\\ R_{\mathrm{ex\_m}}, & T_{\mathrm{in}}\le t<T,\end{cases} \qquad (40)$$

and where $R_{\mathrm m}(t) = R_{\mathrm m}(t+T), t>T$. Here, $R_{\mathrm{in\_m}}, R_{\mathrm{ex\_m}}, C_{\mathrm m}$, and $r(t)$ are chosen appropriately to obtain the desirable breathing pattern. It follows from Theorem 3.1 that $x_{\mathrm m}(t)$ converges to a stable limit cycle, and hence, $x_m(t)$, $t\ge 0$, is uniformly bounded.

Next, we assume that the switched linear time-varying system (26) is such that $A_{\mathrm p}(t) = -R_{\mathrm p}^{-1}(t)C_{\mathrm p}$ and $B_{\mathrm p}(t) = R_{\mathrm p}^{-1}(t)\mathbf e$, where

$$R_{\mathrm p}(t) = \begin{cases} R_{\mathrm{in\_p}}, & 0\le t< T_{\mathrm{in}},\\ R_{\mathrm{ex\_p}}, & T_{\mathrm{in}}\le t<T,\end{cases} \qquad (41)$$

and where $R_{\mathrm p}(t) = R_{\mathrm p}(t+T), t>T$, so that (26) has the form of a lung mechanics model. Here, we assume that $R_{\mathrm{in\_p}}, R_{\mathrm{ex\_p}}$, and $C_{\mathrm p}$ are *unknown* and we use Theorem 4.1 to design $u(t)$ so that $x_{\mathrm p}(t)\to x_{\mathrm m}(t)$ as $t\to\infty$.

Now, in order to apply Theorem 4.1, we need to show that the compatibility conditions (29) and (30) hold. The following proposition provides sufficient conditions under which (29) and (30) hold.

Proposition 5.1: Let $\Gamma \triangleq R_{\mathrm{in\_p}}R_{\mathrm{in\_m}}^{-1}$. Assume that the following conditions hold:

i) $\Gamma \triangleq R_{\mathrm{ex\_p}}R_{\mathrm{ex\_m}}^{-1}$.
ii) There exists a positive scalar Q^* such that $\Gamma\mathbf e = Q^*\mathbf e$.
iii) There exists $\Theta^*\in\mathbb R^{1\times n}$ such that $C_{\mathrm p} = \Gamma C_{\mathrm m} + \mathbf e\Theta^*$.

Then (29) and (30) hold.

Proof: The proof follows by noting that i) and ii) imply (29) holds, while i) and iii) imply (30) holds. □

Remark 5.1: In the absence of switching, conditions ii) and iii) are standard for model reference adaptive control [14]. Condition i) is an additional condition that ensures Theorem 4.1 holds for the switching periodic lung mechanics model.

Remark 5.2: Note that all three conditions in Proposition 5.1 are trivially satisfied if $\Gamma = kI_n - \mathbf e\mathbf e^{\mathrm T}$, where $k>n$.

To illustrative the adaptive controller framework on a numerical example, we consider a four-compartment lung mechanics model, that is, $n=4$. The reference model is assumed to correspond to a bronchial tree which has a regular dichotomy architecture (see Section III). Anatomically the human lung has around 24 generations of airway units. A typical value for lung compliance is $0.1\,\ell$/cm H_2O, that is, $\hat c_0 = 0.1\,\ell$/cm H_2O (see [1]). (Note that respiratory pressure is measured in terms of centimeters of water pressure.) The airway resistance varies with the branch generation and typical values can be found in [15]. Furthermore, the expiratory resistances will be higher than the inspiratory resistance by a factor of 2 to 3. For the reference model we assume that the factor is 2.5.

Next, for simulation purposes, we assume $\Gamma = 10I_4 - \mathbf e\mathbf e^{\mathrm T}$ so that all the conditions of Proposition 5.1, and hence, the compatibility conditions of Theorem 4.1 are satisfied. Furthermore, we let $\Theta_0 = [75, 75, 75, 75]$ and $Q_0 = 5$. However, note that no explicit knowledge of the plant model is needed to generate the control input $u(t)$ and the update laws. Figure 3 shows the error $x_{\mathrm p}(t) - x_{\mathrm m}(t)$ versus time t, verifying that $x_{\mathrm p}(t)\to x_{\mathrm m}(t)$ as $t\to\infty$. Here, we assumed that the applied pressure for the reference model is $p_{\mathrm{in}}(t) = \sin(20t)+5$ cm H_2O, $p_{\mathrm{ex}}(t) = 0$ cm H_2O, the inspiration time is $T_{\mathrm{in}} = 1$ sec, and the expiration time is $T_{\mathrm{ex}} = 2$ sec.

VI. Conclusion

In this paper, we developed an adaptive control framework for a multi-compartmental model of a pressure-limited respirator and lung mechanics system. Specifically, we developed a model reference direct adaptive controller framework where the plant and reference models involve switching and time-varying dynamics. Next, we applied the proposed adaptive feedback controller framework to stabilize a given limit cycle corresponding to a clinically plausible respiratory pattern.

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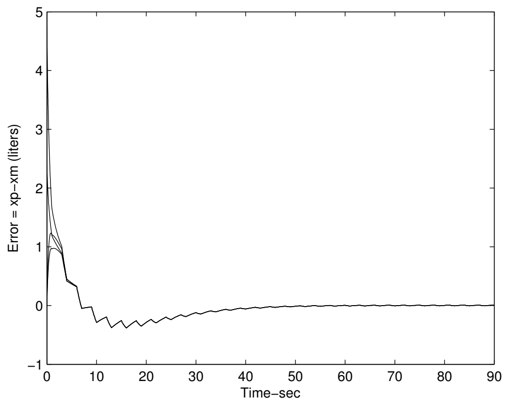

Fig. 3. Error versus time

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