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On the Number of Leaders Needed to Ensure Network Connectivity

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***Abstract*— In this paper we examine the leader-to-follower ratio needed to maintain connectivity in a leader-follower multi-agent network with proximity based communication topology. In the scenario we consider, only the leaders are aware of the global mission, which is to converge to a known destination point. Thus, the objective of the leaders is to drag the team to the desired goal. In the paper we obtain bounds on the number of leaders needed to complete the task while guaranteeing that connectedness of the communication graph is maintained. The results are first established for an initially complete communication graph and then extended to the incomplete case. The results are illustrated by computer simulations.**

I. Introduction

Being able to effectively control networked systems is a key capability in a number of applications, including multi-agent robotics [7], [9], [2] networked sensor and health maintenance [11],[15],[10] and formation control [5],[12],[6],[8] just to name a few. One way in which the user can interact with such systems is through so-called leader agents, whose dynamics need not conform to those of the non-leader agents. In this paper we study such systems, i.e. systems where a select subset of the agents are following a task-level controller encoding the transport of the network from one location to another. The rest of the agents have no notion of these objectives, and are instead executing a local interaction-based control strategy for keeping the team together.

The reasons for prescribing networked solutions to engineering systems range from cost considerations (many cheap systems for solving a problem rather than a single expensive system) to strength-in-numbers arguments. However, as of yet, few studies have addressed the question concerning how many agents one actually needs. In this paper, we pursue this question in the context of a leader-follower network. In particular, we ask the question "How many leaders do you really need?", in order to quantify the strength-in-numbers argument as it applies to leader-follower networks. While issues regarding controllability and stability of leader follower networks has been addressed recently in [14],[13], the issue of the number of leaders needed is a novel topic introduced in the current paper.

The particular scenario under consideration in this paper is one in which the leaders move towards a target location. At the same time, the followers try to maintain appropriate inter-agent cohesion using a standard, nearest neighbor control law. Adjacency in this network is supposed to be defined through inter-agent distances in a so-called disk proximity graph [1]. (Two agents are adjacent if they are within a given distance of each other.) If the leaders move too fast, or if there are not enough leaders to provide sufficient attraction to the followers, the network may become disconnected. As such, there is a critical number of leaders required to ensure that the sum of the attraction exerted on the followers is sufficient to ensure that connectivity is maintained. The establishment of this critical number is the main topic under consideration in this paper.

The outline of this paper is as follows: Section II describes the system and the problem treated in this paper. The number of leaders that ensure connectivity maintenance of the network topology is treated first for the case of complete interaction graphs in Section III. The results are extended to incomplete graphs in Section IV while Section V includes illustrating simulation examples. The results of the paper are summarized in Section VI, where we also include a discussion about further research directions.

II. System and Problem Statement

Consider N agents evolving in $\mathbb{R}$. Although this assumption seems restrictive, we argue here that the results of this paper can be extended to arbitrary dimensions in a relatively straightforward fashion. This however, is left for future publication endeavors. We use single integrator agents whose motions obey the model:

$$\dot{x}_i = u_i, i \in \mathcal{N} = [1, \ldots, N] \qquad (1)$$

We assume that agents belong either to the subset of leaders $\mathcal{N}^l$, or to the subset of followers, $\mathcal{N}^f$. We also have $\mathcal{N}^l \cup \mathcal{N}^f = \mathcal{N}$ and $\mathcal{N}^l \cap \mathcal{N}^f = \emptyset$. Agents have limited sensing capabilities, encoded by a limiting sensing zone of radius Δ around each agent within which it has knowledge of the relative positions of neighboring agents. For each agent $i \in \mathcal{N}$, we define by

$$\mathcal{N}_i = \{j \in \mathcal{N} : |x_i - x_j| \leq \Delta\} \qquad (2)$$

the set of agents of which agent i has knowledge of the relative positions at each time instant. The set $\mathcal{N}_i$ is called agent i's neighboring set and it is time-varying. In particular, $\mathcal{N}_i$ is updated every time an agent enters/leaves the sensing

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zone of another. We denote $|\mathcal{N}_i| = N_i$. The *communication graph* $\mathcal{G} = \{V, E\}$ of the group topology is a graph that consists of a set of vertices $V = \{1, ..., N\}$ indexed by the team members, and a set of edges, $E = \{(i,j) \in V \times V | i \in \mathcal{N}_j\}$ containing pairs of vertices that represent inter-agent communication links. Since the set E is time varying, the graph $\mathcal{G} = \mathcal{G}(t)$ itself, is time-varying.

The dynamics of each follower obey the following agreement equation:

$$\dot{x}_i = -\sum_{j \in \mathcal{N}_i} (x_i - x_j), \qquad \forall i \in \mathcal{N}^f \tag{3}$$

The leaders have the additional goal of dragging the team to a desired goal position, defined by $d \in \mathbb{R}$. Their dynamics are thus given by

$$\dot{x}_i = -\sum_{j \in \mathcal{N}_i} (x_i - x_j) + f(x_i, d), \qquad \forall i \in \mathcal{N}^l \tag{4}$$

where $f(x_i, d)$ is a term that pulls leader i towards d.

In this paper we specifically study the case where we have a linear goal attraction function $f(\xi, d) = -a(\xi - d)$, where $a > 0$ is a constant. Most of the results, however, are first derived and stated for a general function $f(\xi, d)$. The following Lemma guarantees the boundedness of solutions of the closed-loop system with linear attractive term $f(\xi, d)$:

Lemma 1: Let the closed-loop dynamics of (1) be given by (3),(4). Let Ω be the convex hull of the agents in $\mathcal{G}(t)$ and the goal position d, $\Omega(t) = \mathrm{Co}(\mathcal{G}(t) \cup d)$. Then the trajectories of all agents in $\mathcal{G}$ remain within $\Omega(0)$ for all $t \geq 0$.

Proof: We will show that for an arbitrary agent $i \in \mathcal{G}$, positioned on the boundary of Ω, the motion is either on the boundary of Ω or pointing inside the polytope Ω. If $i \in \mathcal{N}^f$ the motion is given by $\dot{x}_i = -\sum_{k \in \mathcal{N}_i}(x_i - x_k)$. If $N_i = 0$ the agent will not move at all and the proof is trivial. Now consider the case $N_i > 0$. By setting $\alpha = N_i^{-1}$ and rearranging the terms we can show: $\alpha \dot{x}_i = -x_i + \sum_{k \in \mathcal{N}_i} \frac{x_k}{N_i}$. Apparently the motion of follower i is directed towards the barycenter of the subgraph $\mathcal{N}_i \subseteq \mathcal{G}$, which, thanks to convexity, is known to lie either on the boundary or in the interior of Ω. From the definition of convexity we can also conclude that the motion of follower i must lie within Ω.

Now assume that $i \in \mathcal{N}^l$. Then $\dot{x}_i = -\sum_{k \in \mathcal{N}_i}(x_i - x_k) - a(x_i - d)$. Define $\beta = (N_i + a)^{-1}$. Then we get: $\beta \dot{x}_i = -x_i + \beta\big(N_i \sum_{k \in \mathcal{N}_i} \frac{x_k}{N_i} + ad\big)$. The motion of agent i is directed towards a convex combination of the barycenter of the subgraph $\mathcal{N}_i \subseteq \mathcal{G}$ and the goal d. By definition, this convex combination lies within the convex hull of $\mathcal{G} \cup d$, and therefore, by the convexity of Ω, the motion of agent i is within Ω. Since the motion of any agent on the boundary of Ω is either on the boundary of Ω or directed into the interior of Ω, we can conclude that no agent will ever enter outside the convex hull defined by the initial positions of the agents and the goal d. Hence, $\Omega(0)$ is an invariant set. $\diamondsuit$

III. The Complete Graph case

In this section, we assume that all agents are initially within the sensing zone of one another, i.e., at a distance less than Δ from one another. Hence, the initial graph $\mathcal{G}(t)$ is complete and of course, connected. In the sequel, we derive sufficient conditions for the graph to remain complete as the leaders drag all followers towards the desired target point d.

Denote $|\mathcal{N}^f| = N_f$, $|\mathcal{N}^l| = N_l$. Since the graph is complete, the dynamics of follower $i \in \mathcal{N}^f$, are given by:

$$\begin{aligned}
\dot{x}_i &= -\sum_{j \in \mathcal{N}^f} (x_i - x_j) - \sum_{j \in \mathcal{N}^l} (x_i - x_j) \\
&= -N_f x_i + \sum_{j \in \mathcal{N}^f} x_j - N_l x_i + \sum_{j \in \mathcal{N}^l} x_j \\
&= -(N_f + N_l) x_i + \sum_{j \in \mathcal{N}^f \bigcup \mathcal{N}^l} x_j,
\end{aligned}$$

so that

$$\dot{x}_i = -(N_f + N_l) x_i + \sum_{j \in \mathcal{N}} x_j, \qquad \forall i \in \mathcal{N}^f, \tag{5}$$

since $\mathcal{N} = \mathcal{N}^f \bigcup \mathcal{N}^l$. Similarly, the dynamics for leader i, $i \in \mathcal{N}^l$, are given by:

$$\dot{x}_i = -(N_f + N_l) x_i + \sum_{j \in \mathcal{N}} x_j + f(x_i, d), \qquad \forall i \in \mathcal{N}^l. \tag{6}$$

Note that with $f(\xi, d) = -a(\xi - d)$ the result of Lemma 1 holds and guarantees boundedness of trajectories.

We can now study how the distance between two arbitrary robots changes with time. Denote $\delta_{ij} = x_i - x_j$. For two arbitrary followers $f_1, f_2 \in \mathcal{N}^f$, we have $\dot{\delta}_{f_1 f_2} = -(N_f + N_l)\delta_{f_1 f_2}$ which yields $\delta_{f_1 f_2} \to 0$ and, of course, $|\delta_{f_1 f_2}| \to 0$. With $f(\xi, d) = -a(\xi - d)$ the same holds for the inter-leader distances with a rate equal to $N_f + N_l + a$. In the general case the convergence rate depends on the properties of $f(\xi, d)$, but to guarantee $|\delta_{l_1 l_2}| \to 0$ for two arbitrary leaders $l_1, l_2 \in \mathcal{N}^l$ it is sufficient to require $\dot{f}(\xi, d) < 0$ for all ξ. Note that the distance between two agents of the same type is monotonically decreasing, which means that if the distance is initially smaller than Δ, the agents will remain within sensor range of each other at all times.

For $i \in \mathcal{N}^l, j \in \mathcal{N}^f$ and $f(\xi, d) = -a(\xi - d)$, we have

$$\dot{\delta}_{ij} = -(N_f + N_l)\delta_{ij} - a(x_i - d), \tag{7}$$

Since the graph is complete we have $-\Delta \leq \delta_{ij}(0) \leq \Delta$. For all $i \in \mathcal{N}$ and $t \geq 0$, we have $|x_i(t) - d| \leq d_{\max} \triangleq \max\limits_{i \in \mathcal{N}} |x_i(0) - d|$, by virtue of Lemma 1, since all agents remain within the convex hull of their initial positions and the goal. By virtue of the Comparison Lemma, δ_{ij} satisfies $\hat{\delta}_{ij} \leq \delta_{ij} \leq \tilde{\delta}_{ij}$ where $\hat{\delta}_{ij}$ is the solution of $\dot{\hat{\delta}}_{ij} = -(N_f + N_l)\hat{\delta}_{ij} - a d_{\max}$ and $\tilde{\delta}_{ij}$ is the solution of $\dot{\tilde{\delta}}_{ij} = -(N_f + N_l)\tilde{\delta}_{ij} + a d_{\max}$, with initial conditions $\hat{\delta}_{ij}(0) = \tilde{\delta}_{ij}(0) = \delta_{ij}(0)$. We have $\hat{\delta}_{ij}(t) \geq e^{-(N_l + N_f)t}(-\Delta + \frac{a d_{\max}}{N_l + N_f}) - \frac{a d_{\max}}{N_l + N_f}$ for all $t \geq 0$ and $\tilde{\delta}_{ij}(t) \leq e^{-(N_l + N_f)t}(\Delta - \frac{a d_{\max}}{N_l + N_f}) + \frac{a d_{\max}}{N_l + N_f}$ for all $t \geq 0$. A sufficient condition for the two agents (leader and follower) to remain connected, i.e., $-\Delta \leq \delta_{ij}(t) \leq \Delta$ for all $t \geq 0$, is thus given by

$$N_l \geq \frac{a d_{\max}}{\Delta} - N_f. \tag{8}$$

The previous derivations are summarized as follows:

Theorem 2: Let the closed loop dynamics of (1) be given by (5),(6) with $f(\xi, d) = -a(\xi - d)$. Let $x(0) \in \Omega$ and assume that the communication graph $\mathcal{G}(t)$ is initially complete and that (8) holds. Then, $\mathcal{G}(t)$ remains complete for all $t \geq 0$, and $\lim_{t\to\infty} x_i(t) = d$ for all $i \in \mathcal{N}$.

Proof: Having shown that $\mathcal{G}(t)$ remains complete, it remains to show that all agents actually converge to d. Equations (3),(4) are written in stack vector form as $\dot{x} = -Lx - \Gamma(x - \mathbf{d})$, where $x = [x_1, \ldots, x_N]^T, \mathbf{d} = [d, ..., d]^T$, and the elements of the diagonal matrix $\Gamma = \mathrm{diag}(\Gamma_1, \ldots, \Gamma_N)$ are given by $\Gamma_i = 0$, if $i \in \mathcal{N}^f$ and $\Gamma_i = a$, if $i \in \mathcal{N}^l$.

Defining $z = x - \mathbf{d}$, we have $\dot{z} = -Lx - \Gamma z = -L(z + \mathbf{d}) - \Gamma z = -Lz - \Gamma z$, so that $\dot{z} = -(L + \Gamma)z$, where L is the Laplacian of $\mathcal{G}(t)$. The eigen-properties of the Laplacian matrix are well established in the cooperative control literature and are not recapped here. The reader is referred to [3] for a review of the Laplacian matrix properties. Note now that the graph corresponding to the matrix $L + \Gamma$ is strongly connected [4], since $\mathcal{G}(t)$ is connected. Moreover, since L is positive semidefinite with zero row sums and Γ is non-negative, $L+\Gamma$ is diagonally dominant. Since there exists at least one leader, there exists at least one row i of $L + \Gamma$ for which $(L + \Gamma)_{ii} > \sum_{j\neq i} \left|(L + \Gamma)_{ij}\right| = \sum_{j\neq i} |L_{ij}|$. Thus, combining Theorem 6.2.14 and Corollary 6.2.9 in [4], we conclude that $L+\Gamma$ is positive definite. Thus $\dot{z} = -(L+\Gamma)z$ yields $z = 0$ at steady state, which implies that $x_i = d$ for all agents $i \in \mathcal{N}$ at steady state. $\diamondsuit$

Remark: It should be pointed out that the fact that all agents reach d also holds in the relaxed case where the communication graph remains connected. This will be used in the result of the next section.

IV. The Incomplete Graph Case

In the previous section we considered a complete initial graph. We next analyze a special case of incomplete graphs. We assume that both the subset of leaders and the subset of followers initially make up complete graphs. However, we no longer assume that all followers are connected to all leaders. To describe this scenario we need to introduce some additional notation. Let $\mathcal{N}_i^l \subset \mathcal{N}^l$ be the subset of leaders that can be seen by follower i ($i \in \mathcal{N}^f$), $N_{li} = |\mathcal{N}_i^l|$, and let $\mathcal{N}_j^f \subset \mathcal{N}^f$ be the subset of followers that can be seen by leader j ($j \in \mathcal{N}^l$), $N_{fj} = |\mathcal{N}_j^f|$. Using these notations, the dynamics for an arbitrary follower $i \in \mathcal{N}^f$ can be written as

$$\dot{x}_i = -\sum_{k\in\mathcal{N}^f} (x_i - x_k) - \sum_{k\in\mathcal{N}_i^l} (x_i - x_k), \qquad (9)$$

while for an arbitrary leader $j \in \mathcal{N}^l$ we have

$$\dot{x}_j = -\sum_{k\in\mathcal{N}^l} (x_j - x_k) - \sum_{k\in\mathcal{N}_j^f} (x_j - x_k) + f(x_j, d). \quad (10)$$

We will now derive general conditions for the two complete subgraphs to remain connected. After that, we will determine a conservative bound for the number of links needed between a specific leader and the group of followers to guarantee that connectivity is maintained for the full graph consisting of both of leaders and followers.

To start with, we consider the connection between two arbitrary followers $i, j \in \mathcal{N}^f$. Without loss of generality we can assume $N_{li} \leq N_{lj}$. As before, we define $\delta_{ij} = x_i - x_j$. We also introduce $\mathcal{N}_{j1}^l$ and $\mathcal{N}_{j2}^l$, which are subsets of $\mathcal{N}_j^l$ such that $|\mathcal{N}_{j1}^l| = |\mathcal{N}_i^l|$, $\mathcal{N}_{j1}^l \cup \mathcal{N}_{j2}^l = \mathcal{N}_j^l$ and $\mathcal{N}_{j1}^l \cap \mathcal{N}_{j2}^l = \emptyset$. From (9) it follows that

$$\begin{aligned}
\dot{x}_i - \dot{x}_j &= -\sum_{k\in\mathcal{N}^f}(x_i - x_k) - \sum_{k\in\mathcal{N}_i^l}(x_i - x_k) \\
&\quad + \sum_{k\in\mathcal{N}^f}(x_j - x_k) + \sum_{k\in\mathcal{N}_{j1}^l\cup\mathcal{N}_{j2}^l}(x_j - x_k) \\
&= -N_f x_i - N_{li}x_i + N_f x_j + N_{li}x_j \qquad (11)\\
&\quad + \sum_{k\in\mathcal{N}_i^l} x_k - \sum_{k\in\mathcal{N}_{j1}^l} x_j + \sum_{k\in\mathcal{N}_{j2}^l}(x_j - x_k). \\
&= -(N_f + N_{li})(x_i - x_j) \\
&\quad + \sum_{k\in\mathcal{N}_i^l} x_k - \sum_{k\in\mathcal{N}_{j1}^l} x_j + \sum_{k\in\mathcal{N}_{j2}^l}(x_j - x_k).
\end{aligned}$$

All leaders are connected to each other so it is known that $|x_k - x_m| \leq \Delta$ for any two leaders $k \in \mathcal{N}_i^l$, $m \in \mathcal{N}_{j1}^l$. Also, the distance between follower j and any leader seen by that follower is bounded by Δ. It follows that

$$\begin{aligned}
\dot{\delta}_{ij} &\geq -(N_f + N_{li})(x_i - x_j) - N_{lj2}\Delta - N_{lj1}\Delta \qquad (12)\\
&= -(N_f + N_{li})(x_i - x_j) - N_{lj}\Delta, \qquad (13)
\end{aligned}$$

and

$$\dot{\delta}_{ij} \leq -(N_f + N_{li})(x_i - x_j) + N_{lj}\Delta. \qquad (14)$$

Our assumption is that initially $|x_i - x_j| \leq \Delta$. Sufficient conditions for the connection to be kept at all times are

$$\delta_{ij} = \Delta \rightarrow \dot{\delta}_{ij} \leq 0,$$

$$\delta_{ij} = -\Delta \rightarrow \dot{\delta}_{ij} \geq 0.$$

For our purposes it is sufficient to consider the worst case scenarios at the two extremes. Insertion in (13) and (14) gives two conditions which can be summarized as $N_f \geq N_{lj} - N_{li}$. This condition is satisfied regardless of the topology for every graph that has

$$N_f \geq N_l. \qquad (15)$$

Next, we find out what it takes to keep the leader subgraph complete. Consider two leaders $i, j \in \mathcal{N}^l$. As in the follower-follower case we define $\delta_{ij} = x_i - x_j$ and we assume (without loss of generality) that $N_{fi} \leq N_{fj}$. Initially $|x_i - x_j| \leq \Delta$, *i.e.* the leaders are connected, and we know from before that sufficient conditions for the connection between leader i and j to be kept at all times are $\dot{\delta}_{ij} \leq 0$ at $\delta_{ij} = \Delta$ and $\dot{\delta}_{ij} \geq 0$ at $\delta_{ij} = -\Delta$. Following the computations in the follower-follower case we obtain the following two conditions for the leaders to stay connected:

$$\begin{aligned}
\dot{\delta}_{ij} &\leq -(N_l + N_{fi})\Delta + N_{fj}\Delta + \max(f(x_i, d) - f(x_j, d)) \\
&\leq -(N_l - N_f)\Delta + \max(f(x_i, d) - f(x_j, d)) \leq 0,
\end{aligned}$$

$$\begin{aligned}\dot{\delta}_{ij} &\geq (N_f + N_{fi})\Delta - N_{fj}\Delta + \min(f(x_i,d) - f(x_j,d)) \\ &\geq (N_f - N_{fj})\Delta + \min(f(x_i,d) - f(x_j,d)) \geq 0.\end{aligned}$$

Both conditions above lead to the same constraint on the goal-attraction function $f(\xi, d)$, namely

$$\max \frac{f(\xi+\Delta,d) - f(\xi,d)}{\Delta} \leq N_l - N_f, \qquad (16)$$

for any ξ that lies on the trajectory of any of the leaders. Remember that the condition for the follower subgraph to remain complete was $N_f \geq N_l$. This means that $f(\xi, d)$ must be decreasing with increasing ξ over every interval Δ. An interpretation of this result is that the overall effect of $f(\xi, d)$ must be an attractive force that brings two arbitrary leaders closer to each other. Another interpretation is that if two leaders are moving in the same direction, the magnitude of the velocity of the leader lagging behind should be larger.

Remark: For the particular case $f(\xi, d) = -a(\xi - d)$, condition (16) is equivalent to

$$a \geq N_f - N_l. \qquad (17)$$

To complete the analysis we now consider the connection between the group of leaders and the group of followers. The two subgroups are now assumed to be internally fully connected and to satisfy the constraints (15) and (17). Using the same strategy and the same notations as before, we start the analysis by studying the link between an arbitrary leader i, where initially $N_{fi} \geq 1$, and one of the followers j that is initially connected to leader i, *i.e.*, $j \in \mathcal{N}_i^f$.

The dynamics of leader i are given by (10) and the dynamics of follower j are given by (9). Thus,

$$\begin{aligned}\dot{x}_i - \dot{x}_j &= -\sum_{k\in\mathcal{N}^l}(x_i - x_k) - \sum_{k\in\mathcal{N}_i^f}(x_i - x_k) + f(x_i,d) \\ &\quad + \sum_{k\in\mathcal{N}^f}(x_j - x_k) + \sum_{k\in\mathcal{N}_j^l}(x_j - x_k) \\ &= -\sum_{k\in\mathcal{N}_j^l}(x_i - x_k) - \sum_{k\in\mathcal{N}^l\setminus\mathcal{N}_j^l}(x_i - x_k) \\ &\quad - \sum_{k\in\mathcal{N}_i^f}(x_i - x_k) + f(x_i,d) + \sum_{k\in\mathcal{N}_i^f}(x_j - x_k) \\ &\quad + \sum_{k\in\mathcal{N}^f\setminus\mathcal{N}_i^f}(x_j - x_k) + \sum_{k\in\mathcal{N}_j^l}(x_j - x_k) \\ &= -N_{fi}(x_i - x_j) - N_{lj}(x_i - x_j) + f(x_i,d) \\ &\quad - \sum_{k\in\mathcal{N}^l\setminus\mathcal{N}_j^l}(x_i - x_k) + \sum_{k\in\mathcal{N}^f\setminus\mathcal{N}_i^f}(x_j - x_k).\end{aligned}$$

As before we analyze the two cases $\delta_{ij} = x_i - x_j = \Delta$ and $\delta_{ij} = -\Delta$. This leads to two conditions on $\dot{\delta}_{ij}$:

$$\dot{\delta}_{ij} \leq f(x_i,d) - (2(N_{lj} + N_{fi}) - N)\Delta \leq 0 \qquad (18)$$

$$\dot{\delta}_{ij} \geq f(x_i,d) + (2(N_{lj} + N_{fi}) - N)\Delta \geq 0. \qquad (19)$$

Combined, the two conditions above give one condition on the sum $N_{lj} + N_{fi}$ and one condition on the goal attraction function $f(\xi, d)$. To guarantee that the link between leader i and follower j hold we require that

$$N_{lj} + N_{fi} \geq \frac{N}{2} \qquad (20)$$

and

$$|f(x_i,d)| \leq (2(N_{lj} + N_{fi}) - N)\Delta \qquad \forall t \geq 0 \qquad (21)$$

for all $t > 0$. By Lemma 1, with $f(\xi, d) = -a(\xi - d)$ equations (20) and (21) are satisfied for

$$N_{lj} + N_{fi} \geq \frac{N}{2} + \frac{ad}{2\Delta} \qquad (22)$$

The derivations are summarized as follows:

Theorem 3: Assume that $f(\xi, d) = -a(\xi - d)$ and that the communication graph $\mathcal{G}(t)$ is initially constituted of two complete subgraphs, the subgraph of leaders and the subgraph of followers. Further assume that there initially exists at least one connection between the two complete subgraphs. Then, if the graph satisfies conditions (15), (17) and if (22) holds for all initial links (i, j) such that $i \in \mathcal{N}^l$ $j \in \mathcal{N}^f$, all connections in the graph will be maintained and all the agents will tend to d.

Note that all agents converge to d provided that the communication graph remains connected (as is the case in Theorem 3) by virtue of Theorem 2 and the remark after it.

Remark: In the worst scenario the number of leaders, N_l, will be bounded by $N_f - 2 \leq N_l \leq N_f$, while in the best scenario it will be enough with just one leader.

In fact, if we do not require that *all* connections between leaders and followers are maintained, but simply want to ensure that the two subgroups remain connected, then the following result is useful.

Lemma 4: Assume that (15) and (17) are satisfied. Let E^* be a subset of the initial links between the group of leaders and the group of followers and let the neighbor sets of the graph be defined such that $i \in \mathcal{N}^l$ is considered a neighbor of $j \in \mathcal{N}^f$, and vice versa, if and only if they are initially connected and the link $(i, j) \in E^*$. Then, if it is possible to find a subset E^*, $|E^*| \geq 1$, such that all the links in E^* satisfy condition (22), the group of leaders and the group of followers will remain connected with each other.

Proof: The proof follows directly from Theorem (3) and from the constraints (18) and (19). By definition, the links included in E^* are invariant, *i.e.* for each robot there exist a well defined lower bound on the number of neighbors. If the inequalities (18) and (19) are satisfied for all robots for the lower bounds on their number of neighbors, then the inequalities will hold even if additional non-invariant links are added and removed. $\diamondsuit$

V. SIMULATIONS

In this section we illustrate the results of sections III and IV in a series of simulations. The simulations are performed in MATLAB and we used the built-in MATLAB function ode45 to obtain the trajectories of the robots through numerical integration. In all simulations we use $f(\xi, d) = -a(\xi - d)$.

Before describing the simulations we need to point out two things. First, due to rounding errors simple numerical integration methods should not be used in stability analysis other than as a complement to theoretical analysis. As such, however, they present a powerful tool for examining the behavior of a system. Second, the stability conditions derived in sections III and IV guarantee maintained connectivity and convergence to the goal, but they are not necessary for either connectivity or convergence. As simulations will show, the robots often converge even if all conditions are not satisfied.

A. Complete graph

To illustrate the behavior of a group of robots with an initially complete graph we show three examples for a group of 7 robots. Two of the robots are leaders with dynamics given by (4) and the remaining five are followers with dynamics given by (3). The robots have a sensor range of $\Delta = 10$ and the initial positions are given so that all robots are within sensing range of each other at $t = 0$, *i.e.* the communication graph is complete. The initial positions for the robots, as well as the position for the goal $d = 20$, are the same in all three simulations, but the value of the goal attraction term a is changed and thereby the stability condition (8) is affected. In the first example, $a = 3$ and (8) is satisfied. In this case, all robots converge to the goal d (see Fig. 1) and the graph remains complete for all $t \geq 0$. In the second example we have $a = 8$ and condition (8) is no longer satisfied. As seen in Fig. 2, the increased attraction towards the goal causes the leaders to converge faster to d, but at the expense of temporarily broken connections to the followers. In this case the followers catch up with the leaders again and all agents converge to d, but if a is increased further, the followers will eventually not be able to catch up with the leaders once the links start to break. With $a = 10$, all connections between leaders and followers are broken at approximately $t = 0.14$ and as a consequence, the followers do not reach the goal (Fig. 3).

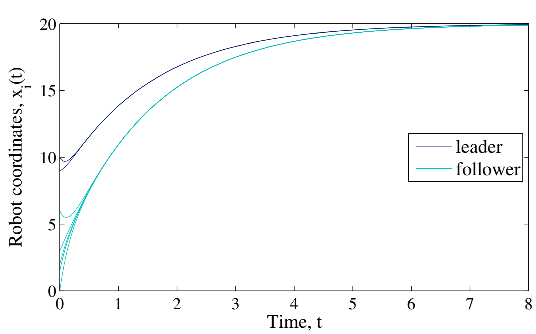

Fig. 1. Position coordinates as functions of time for the agents in a network where the communication graph is initially complete and the stability constraints are satisfied. As expected, the graph remains complete and all robots converge to the goal.

B. Incomplete graph

Let us now consider the case of an initially incomplete communication graph. We assume, as in previous sections,

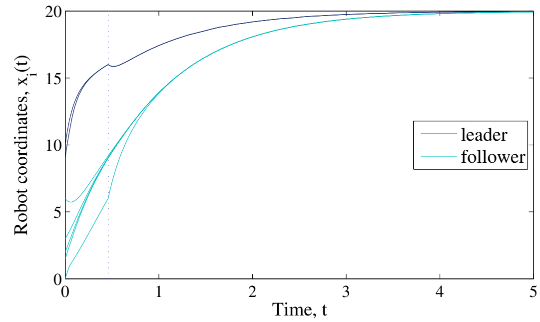

Fig. 2. Position coordinates as functions of time for agents in a network with an initially complete communication graph. At time $t = 0$ the stability constraints are not satisfied and some of the communication links are temporarily broken. At $t \approx 0.46$ (dotted line) all links have re-formed.

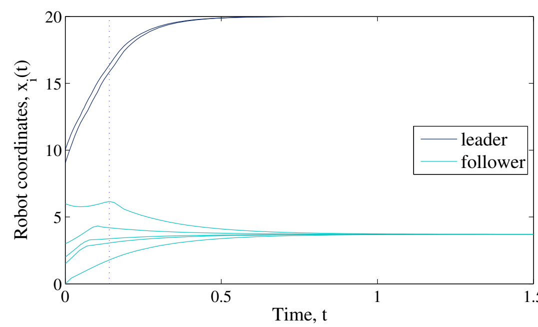

Fig. 3. Position coordinates as functions of time for agents in a network with an initially complete communication graph. The stability constraints are not satisfied and the contact between leaders and followers is broken at approximately $t = 0.14$ (dotted line).

that the subgraph of leaders and the subgraph of followers are complete but that the combined graph is not, *i.e.* some of the leaders and followers are not within sensing range of each other. We start with an example where sufficient constraints are satisfied to maintain the connection between leaders and followers. For the simulations we use a setup with nine robots, four leaders and five followers. The sensor range in this case is assumed to be $\Delta = 6$ and the initial positions of the robots are given by $x(0) = [10\ 8\ 7.5\ 7\ 5.5\ 4\ 3\ 2\ 0.5]$, where $\mathcal{N}^l = \{1, 2, 3, 4\}$ and $\mathcal{N}^f = \{5, 6, 7, 8, 9\}$. As in previous simulations, the goal is at $d = 20$. With $N_l = 4$ and $N_f = 5$ it is obvious that (15) is satisfied and by choosing $a = 1.1$ we make sure that (17) holds as well. With (15) and (17) satisfied we know that the leader and follower subgraphs will both remain complete, but we also need to consider the connection between the two subgroups. A closer analysis shows that (22) is not satisfied for all existing links between leaders and followers at $t = 0$. With the given values of Δ, d and a constraint (22) can be expressed as $N_{lj} + N_{fi} \geq 6.33$, and this is not satisfied by, for example, link $(1, 5)$. Since Theorem 3 does not apply we can not guarantee that all links will hold, but we may still be able to show that the connection between leaders and followers is maintained. To do this we turn to Lemma 4. If we define E^* to be the subset of links connecting leaders $i = 2, 3, 4$

to followers $j = 5, 6, 7, 8$ we can verify that (22) is satisfied for all links in E^* and by Lemma 4 we are able to guarantee that the full graph will at least stay connected. The stability result is confirmed by the simulation shown in Fig. 4. In fact, it turns out that even though we could not apply Theorem 3, all initial links hold for $t \geq 0$.

In the final example we use the same setup as before, but now we assume that the sensor range is decreased to $\Delta = 4$. The change of Δ do not affect conditions (15) and (17), which are still satisfied, but now it is not possible to find even a subset E^* of the links between leaders and followers that satisfy (22). As seen in Fig. 5, the existing links between the two subgroups are not sufficiently strong. At $t \approx 0.24$ the last connecting link breaks and the subgroups diverge.

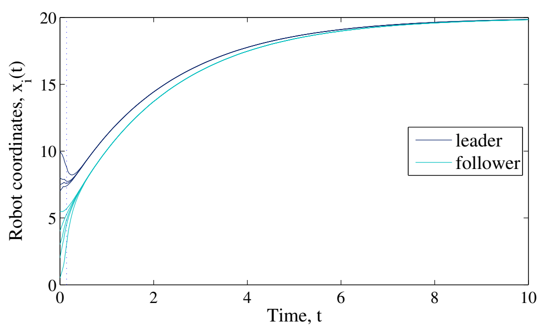

Fig. 4. Position coordinates as functions of time for the agents in a network with an initially incomplete communication graph. The stability constraints are satisfied and all robots converge to the goal. The dotted line indicates the time when the communication graph becomes complete.

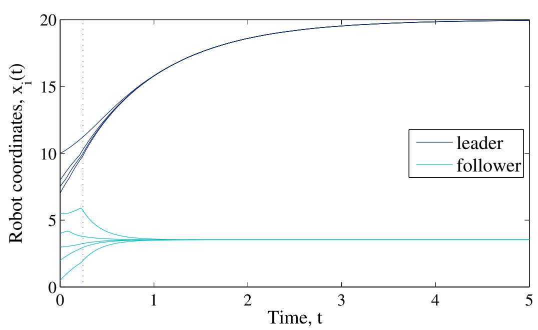

Fig. 5. Position coordinates as functions of time for the agents in a network with an initially incomplete communication graph. The stability constraints are not satisfied and the contact between leaders and followers is broken at approximately $t = 0.24$ (dotted line).

VI. Conclusions

In this paper we examined the leader-to-follower ratio needed to maintain connectivity and guarantee convergence of the whole group in leader-follower multi-agent networks with proximity based communication topology. First we studied the case where we had an initially complete communication graph. We obtained specific bounds on the number of leaders that drag the team to the desired goal while the connectedness of the interaction network is guaranteed. The results were then extended to the incomplete graph case where number and position of leader-follower communication links were also taken into account. The results were supported by illustrating computer simulations.

Further research involves extending the proposed framework to arbitrary dimensions, applying saturating leader goal attraction forces, and extending the results to the general incomplete graph case.

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