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Stability of node-based multipath routing and dual congestion control

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Abstract— This paper considers a network flow control problem where routing and input rates are controlled in a decentralized way across a network, to optimize a global welfare objective. We build on our recent work which combines "dual" congestion control for the traffic sources, with multipath routing at the router nodes, controlling the traffic split among outgoing links based on downstream congestion prices. The challenge is to obtain stabilization of the optimum point; in fact, controlling the split fractions following the price gradient has the correct equilibrium, but can lead to oscillatory instabilities. This suggests the use of derivative action to damp such oscillations. We study two alternatives in this regard; either anticipatory control of routing splits, which yields local stability in an arbitrary network topology, or anticipatory price generation, which yields a global result for the case of a network of parallel links. Proofs are based on a Lyapunov argument. Results are illustrated through simulations.

I. Introduction

Recent advances on Internet congestion control based on microeconomic modeling [6], [10], [11], [15] have led to the development of decentralized control laws running at traffic sources and network links, which serve the global objective of maximizing an overall utility.

A natural continuation of this success is to incorporate the degree of freedom of *routing* inside the network as well, to jointly carry out a "cross-layer" network optimization. If single-path routing is used as in the IP protocol, this combination is not easy: the underlying optimization problem is non-convex, and congestion-based route control oscillates [2], [17]. In contrast, multipath routing leads to a convex multicommodity optimization, a better candidate to combine with congestion control. Many proposals in this regard have *sources* controlling the rate of multiple paths to destination [6], [5], [8], [16]. A more scalable, node-centric alternative is to have routers take charge of the multipath function, by controlling the traffic split fractions to each destination among their outgoing links. This idea goes back to [4], [1] for inelastic source traffic; in that work the traffic split is adapted to follow the gradient of an overall cost function, interpreted as network delay. This approach can also include "primal" flow control, as shown in [18], which also includes power control for wireless nodes. Other cross-layer work for wireless networks with the node-centric view is [3], [9].

In [12] we proposed the use of congestion prices, generated by links and averaged recursively by nodes, as the feedback signal on which to base the gradient control of traffic split fractions. Primal and dual versions of the congestion price correspond to different optimization problems. Section II gives some background on this setup. In terms of stability, [12] establishes it for primal congestion control, but it does not hold in the dual case: as we see in Section III, harmonic oscillations can appear in these dynamics which are inherently of second order. This recognition motivates us to include derivative action in the control, aimed at damping these oscillations.

This paper contains stability studies of this kind of control. One alternative is to add a price anticipatory term in the control of traffic splits. This yields local asymptotic stability of the equilibrium; Section IV includes the proof for a network of parallel links, the general case is relayed to [13]. To obtain *global* results we must deal with a switching nonlinearity: the projections required to keep the vector of split fractions within the unit simplex. For this study we obtain stronger results with a second law, that includes the derivative term in the price generation mechanism. We prove in Section IV global asysmptotic stability of the welfare maximizing equilibrium, in the network of parallel links, through a Lyapunov argument. In Section V we supplement the theory with Matlab simulations that illustrate the dynamics, the above-mentioned projections, and the role of the derivative action in stabilization. Conclusions are given in Section VI, and some technical lemmas are proved in the Appendix.

II. Background

We describe here the combined framework for multipath routing and congestion control from [12].

A. Notation

Consider a network made up of a set of nodes $\mathcal{N}$, denoted by indices i, j, and a set of links $\mathcal{L}$ between them, denoted by l or by a directed pair of nodes (i, j).

The network supports various flows between source-destination pairs of nodes. The index $k \in \mathcal{K}$ denotes an individual flow or "commodity", and $s(k)$, $d(k)$ are the corresponding source and destination nodes. While these are unique for each k, we allow the traffic to follow multiple paths between source and destination.

Let y_l^k denote the rate (packets/sec) of flow k through link l, and x_i^k the total rate of this flow entering node i. At the source node, we only have the external rate x^k,

$$x_{s(k)}^k = x^k. \tag{1}$$

The flow balance equations at nodes are

$$x_j^k = \sum_{(i,j)\in\mathcal{L}} y_{(i,j)}^k, \quad j \neq s(k), \tag{2a}$$

$$x_i^k = \sum_{(i,j)\in\mathcal{L}} y_{(i,j)}^k, \quad i \neq d(k). \tag{2b}$$

The total rate on link l is

$$y_l = \sum_k y_l^k. \tag{3}$$

B. Welfare optimization objective

We associate with each commodity k an increasing, strictly concave utility function $U_k(x^k)$ that specifies the flow's demand for rate. We formulate the following cross-layer optimization problem.

Problem 1 (WELFARE): Maximize $\sum_k U_k(x^k)$, subject to link capacity constraints $y_l \leq c_l$, and flow balance constraints (1),(2),(3).

This convex program seeks the maximum achievable utility over all flows, if traffic is allowed to follow multiple, arbitrary routes between source and destination. We study decentralized control laws at sources and routers to solve this optimization.

C. Control variables

The source of flow k (the transport layer) controls the total rate x^k that it inputs to the network.

The router at node i controls the variable $\alpha_{(i,j)}^d$, that specifies the fraction of traffic with destination d, routed through outgoing link (i,j). We thus impose

$$y_{(i,j)}^k = \alpha_{(i,j)}^{d(k)} x_i^k, \qquad (i,j) \in \mathcal{L}. \tag{4}$$

The vector $\alpha_i^d := \{\alpha_{(i,j)}^d\}_{(i,j)\in\mathcal{L}}$, of dimention L_i (number of outgoing links at i) is in the unit simplex

$$\Delta_i = \{\alpha_{(i,j)}^d \geq 0 : \sum_{(i,j)\in\mathcal{L}} \alpha_{(i,j)}^d = 1\}.$$

D. Feedback signals

The primary feedback signal is a congestion measure or *price* p_l for each link $l \in \mathcal{L}$. Based on these link prices, nodes construct a price-to-destination q_i^d, $i \in \mathcal{N}$, representing the average price of sending packets from node i to destination d, under current routing patterns.

Node prices are thus defined to satisfy

$$q_d^d = 0,$$
$$q_i^d = \sum_{(i,j)\in\mathcal{L}} \alpha_{(i,j)}^d [p_{(i,j)} + q_j^d], \quad i \neq d. \tag{5}$$

Given link prices $p_{(i,j)}$, under mild assumptions of connectivity stated in [4], there exist unique solutions q_i^d

to the above recursive equations; more details are given in [13], which also contains a protocol that implements this recursion. At the source node of flow k, the price $q^k := q_{s(k)}^{d(k)}$ summarizes the congestion of the network.

E. Dual congestion control

The dual congestion control algorithm originating in [10] is based on the link price generation mechanism

$$\dot{p}_l = \gamma_l [y_l - c_l]_{p_l}^+. \tag{6}$$

The positive projection $[w_l]_{p_l}^+$ is defined to be zero if $w_l < 0$ and $p_l = 0$: in this case the projection is said to be *active*; otherwise, the result is w_l. Also, for column vectors w p, $[w]_p^+$ is the element-wise projection.

In Section IV-C we will consider an anticipatory variant of the dual price generation.

The source control in dual laws is static: based on the received price q^k, the source chooses the rate

$$x^k = f_k(q^k) \tag{7}$$

that instantaneously maximizes $U_k(x^k) - q^k x^k$. Hence, the *demand curve* f_k is the inverse function of the marginal utility $U_k'(x^k)$. f_k is strictly decreasing if U_k is strictly concave.

III. ROUTING CONTROL BASED ON PRICE GRADIENTS AND ITS INSTABILITY

To completely define the cross-layer decentralized control law, we must specify how to control the routing split vector $\alpha_i^d := \{\alpha_{(i,j)}^d\}_{(i,j)\in\mathcal{L}}$ as a function of

$$\pi_i^d := \{p_{(i,j)} + q_j^d\}_{(i,j)\in\mathcal{L}},$$

the vector of prices to destination d seen from node i.

A first choice for the control of α_i^d is to follow the negative price gradient: to transfer traffic gradually from more expensive to cheaper routes. One such law is

$$\dot{\alpha}_i^d = \beta_i E_{\alpha_i^d}[-\pi_i^d], \tag{8}$$

where $\beta_i > 0$ and $E_{\alpha_i^d}$ denotes a projection operation required to keep the trajectory within the simplex Δ_i. In the special case when α_i^d is *interior* to Δ_i ($\alpha_{(i,j)}^d > 0 \, \forall j$) the projection must simply enforce the balance of mass

$$\sum_{j:(i,j)\in\mathcal{L}} \dot{\alpha}_{(i,j)}^d = 0,$$

which means $\dot{\alpha}_i^d$ must be orthogonal to $\mathbf{1}$, the vector of all ones of dimension L_i. So in this case $E_{\alpha_i^d}$ is given by the orthogonal projection matrix

$$E = I - \frac{1}{L_i}\mathbf{1}\cdot\mathbf{1}^T, \tag{9}$$

where I is the identity matrix of dimension L_i.

Applying E to a vector subtracts the mean from each component. So, for an interior α_i^d , (8) is simply

$$\dot{\alpha}_{(i,j)}^d = \beta_i(\overline{\pi_i^d} - \pi_{(i,j)}^d);$$

this increases routing in links with lower-than-average prices, decreases it in the rest.

The definition of $E_{\alpha_i^d}$ for points α_i^d on the simplex boundary is postponed to Section IV-B.

Proposition 1: Consider the closed loop dynamics defined by differential equations (6), (8), and by relationships (4), (5), (7). At an equilibrium point, the source rates are at an optimum of Problem 1.

The proof parallels the one in [12] for primal laws: equilibrium prices are shown to be the Lagrange multipliers of a dual to Problem 1. Details are given in [13]. We now see that although the dynamics has the correct equilibrium, it is not necessarily attractive.

Example 1: Consider a simple network with two nodes (source and destination) and two parallel links, of capacity c_1, c_2. Each link generates a price according to (6). The traffic split can be described in this case by a single parameter $\alpha := \alpha_1$, with $\alpha_2 = 1 - \alpha$. An update that follows the negative price gradient has the form

$$\dot{\alpha} = \beta(p_2 - p_1),$$

with saturation to the interval $[0, 1]$. The equilibrium is $x^* = c_1 + c_2$, $\alpha^* = c_1/x^*$, with $p_1^* = p_2^* = q^*$ depending on the chosen utility function. To simplify the analysis, let us temporarily replace the source by an inelastic one with rate $x \equiv x^*$. Also, consider a trajectory for which the saturation constraints on α, p_1, p_2 remain inactive. Denoting $\delta\alpha = \alpha - \alpha^*$, $\delta p_i = p_i - p_i^*$, the dynamics becomes *linear:*

$$\begin{bmatrix} \delta\dot{\alpha} \\ \delta\dot{p}_1 \\ \delta\dot{p}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\beta & \beta \\ \gamma_1 x^* & 0 & 0 \\ -\gamma_2 x^* & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta\alpha \\ \delta p_1 \\ \delta p_2 \end{bmatrix}. \quad (10)$$

The eigenvalues of the preceding matrix are 0 and $\pm j\sqrt{\beta(\gamma_1 + \gamma_2)x^*}$. The 0 eigenvalue is a consequence of having introduced the inelastic source, which makes the equilibrium price indeterminate. The purely imaginary mode is of more concern: it reveals a harmonic oscillation of the price and split dynamics, that can have as large an amplitude as the saturation constraints allow.

If the elastic source is introduced back in the problem, the dynamics is no longer linear. Nevertheless, we can say the following: for the important case where $\gamma_l = 1/c_l$ (i.e., price represents queueing delay), the linearization around equilibrium replaces the mode at zero with a stable eigenvalue, but the imaginary modes remain. Moreover, through a Lyapunov analysis similar to Theorem 6 below (see Remark 1) we find that asymptotically the source rate must converge to x^* as above, with dynamics of α and p approaching the one in (10), and thus exhibiting possibly large oscillations. Note that it does not help to make the route adaptation "slow": if we reduce the parameter β, the frequency of oscillation is reduced, but the oscillations remain.

IV. Anticipatory Control and Its Stability

The preceding example reveals a limitation with controlling multipath routing based on the gradient of congestion price. The difficulty can be traced to the second-order nature of the dynamics (10), which behaves like a mass-spring system with no damping. How, then, do we introduce damping in this loop? A classical idea is to include a "derivative action" term[1] in either the price or the split equations.

A. Derivative action in routing splits and local stability

We consider first the alternative of adding the derivative term in the control of routing splits, essentially making this control *anticipate* future prices ($\nu_i > 0$):

$$\dot{\alpha}_i^d = \beta_i E_{\alpha_i^d}[-(\pi_i^d + \nu_i \dot{\pi}_i^d)]. \quad (11)$$

Note that the equilibrium point is unchanged with respect to the previous section, since the derivative terms vanish there. Hence, an equilibrium will still solve Problem 1; the issue is the stability of this equilibrium. We focus here on a simple case (deferring generalizations to [13]): a network of L parallel links between a two nodes, each connected to the single source and destination. Let

$$c = \begin{bmatrix} c_1 \\ \vdots \\ c_L \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_L \end{bmatrix}, \quad p = \begin{bmatrix} p_1 \\ \vdots \\ p_L \end{bmatrix}, \quad \alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_L \end{bmatrix}$$

be the vectors of link capacities, rates, prices and split ratios, and q, x the scalar source variables. We have

$$y = x\alpha, \qquad q = \alpha^T p, \qquad x = f(q), \quad (12)$$

where f from (7) is strictly decreasing, and T denotes transpose. The dynamics in this case are given by

$$\dot{\alpha} = \beta E_\alpha[-(p + \nu\dot{p})], \quad (13)$$
$$\dot{p} = \Gamma[y - c]_p^+, \quad (14)$$

where $\Gamma = \text{diag}\{\gamma_l\}_{l \in \mathcal{L}}$, $\nu > 0$. The equilibrium is

$$x^* = c_1 + \ldots + c_L = f(q^*), \qquad p^* = q^*\mathbf{1},$$
$$y_l^* = c_l, \quad \alpha_l^* = \frac{c_l}{x^*} \quad l = 1, \ldots, L. \quad (15)$$

We first rewrite the dynamics in incremental variables around equilibrium, $\delta x = x - x^*$ and so on. Without loss of generality take $q^* > 0$ and small δp under which no price saturation occurs. Then we have (exactly)

$$\delta\dot{p} = \Gamma[\delta y] = \Gamma[(x - x^*)\alpha + x^*(\alpha - \alpha^*)]$$
$$= \Gamma[\delta x\alpha + x^*\delta\alpha]. \quad (16)$$

If α is interior to the unit simplex Δ (which happens locally since α^* is interior), the projection in (13) is simply given by the matrix E as in (9). Furthermore, $Ep^* = 0$, so we locally rewrite (13) as

$$\delta\dot{\alpha} = -\beta E(\delta p + \nu\delta\dot{p}). \quad (17)$$

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[1]We acknowledge discussions with Jeff Shamma who has recently promoted the use of derivative action in dynamic games [14].

Also, noting that $\delta\alpha \perp \mathbf{1}$ and $p^* = q^*\mathbf{1}$, (12) yields

$$\delta q = \alpha^T \delta p + \delta\alpha^T p^* = \alpha^T \delta p. \tag{18}$$

Proposition 2: The equilibrium (15) is locally asymptotically stable under the dynamics (13-14), for $\nu > 0$.

Proof: Define the Lyapunov function candidate $V \geq 0$, vanishing only at equilibrium:

$$V(\delta\alpha, \delta p) = \frac{x^*}{2\beta}\|\delta\alpha + \beta\nu E\delta p\|^2 + \frac{1}{2}\delta p^T \Gamma^{-1}\delta p. \tag{19}$$

The derivative of $\delta\alpha + \beta\nu E\delta p$ is equal to $-\beta E\delta p$ from (17); therefore the first term in V has derivative

$$x^*(\delta\alpha + \beta\nu E\delta p)^T(-E\delta p) = -x^*\delta\alpha^T \delta p - x^*\beta\nu\|E\delta p\|^2.$$

Note for the above that $E\delta\alpha = \delta\alpha$. Now, using (16), the derivative of the second term in V is

$$\delta p^T[\alpha\delta x + x^*\delta\alpha] = (\alpha^T \delta p)\delta x + x^*\delta\alpha^T \delta p.$$

Combining both terms and using (18) yields

$$\dot{V} = -x^*\beta\nu\|E\delta p\|^2 + \delta q\delta x.$$

Now since $f(q)$ is strictly decreasing we have

$$\delta q\delta x = (q - q^*)(f(q) - f(q^*)) \leq 0,$$

so V decreases along trayectories.

The Lasalle principle (see [7]) implies convergence to an invariant set where $\dot{V} \equiv 0$. This implies $x \equiv x^*$ and $q \equiv q^*$. Also, due to the first term in $\dot{V}$ we have $E\delta p \equiv 0$ which means δp is parallel to $\mathbf{1}$, $\delta p(t) = \delta q(t)\mathbf{1}$. But since $\delta q \equiv 0$ we have $\delta p \equiv 0$. Finally, (16) implies $\delta\alpha \equiv 0$ so the invariant set is the equilibrium.
∎

The previous argument extends to a local stability result for a general network, with arbitrary topology and multiple commodities, superimposing Lyapunov terms similar to (19) for each node and each commodity. We state the general result, for the proof see [13].

Theorem 3: Consider the closed loop dynamics defined by the differential equations (6) and (11), together with the static relationships (4), (5), (7), for each i, d, k l in an arbitrary network. The equilibrium set (optimum of Problem 1) is locally attractive, for $\nu > 0$.

B. Projecting dynamics on the simplex

The remainder of the section focuses on *global* stability, for which it is essential to define the projection $E_\alpha[v]$ of equations (8),(13), for points on the boundary of Δ. Intuitively, $E_\alpha[v]$ must specify the direction that follows v most closely with motion within the simplex. Formally: for $a \in \mathbb{R}^L$, let $\Psi_\Delta(a) := \mathrm{argmin}_{b \in \Delta}|a - b|$ denote the point in Δ closest to a. Now define

$$E_\alpha[v] := \lim_{\epsilon \to 0+} \frac{\Psi_\Delta(\alpha + \epsilon v) - \alpha}{\epsilon}. \tag{20}$$

An illustration of the definition is given in Figure 1. Since the boundary of Δ is piecewise linear, the limit in

(20) is in fact achieved for small enough $\epsilon > 0$, for which $\alpha + \epsilon E_\alpha[v]$ becomes the point in the simplex closest to $\alpha + \epsilon v$. If α is interior to Δ, $\Psi_\Delta(\alpha + \epsilon v)$ is for small ϵ the orthogonal projection $\alpha + \epsilon Ev$, hence $E_\alpha[v]$ defaults to Ev. Considered globally, however, $E_\alpha[v]$ is not linear in v, and discontinuous (switching) in α.
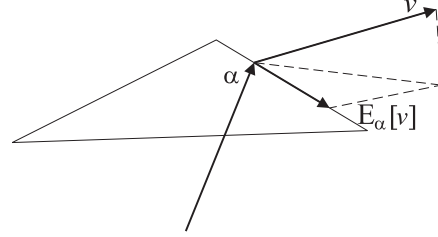

Fig. 1. Projection E_α.

This projection satisfies the following basic lemmas: proof is given in the Appendix.

Lemma 4: For any $b \in \Delta$, $v \in \mathbb{R}^L$, the inner product

$$\langle b - \alpha, v - E_\alpha[v]\rangle \leq 0. \tag{21}$$

Furthermore, for b interior to Δ, equality can only hold in (21) when $E_\alpha[v] = Ev$.

Lemma 5:

$$\|E_\alpha[v]\|^2 \leq \langle E_\alpha[v], v\rangle \quad \forall\alpha \in \Delta, v \in \mathbb{R}^L. \tag{22}$$

C. Global stability with derivative action in the prices

A natural question is whether the dynamics (13-14) is *globally* stabilizing. Simulation evidence indicates this is the case. Furthermore, the Lyapunov argument based on V in (19) is not completely local, in the sense that it can handle exactly the nonlinearities in (12), and extended to include the price projection in (14). So, for a trajectory that avoids the boundary of the simplex, V is decreasing. Unfortunately this need not occur for trajectories that hit the boundary, and apply the projection (20). So the global proof is still open.

We are able to give a positive result, nevertheless, by introducing derivative action in a different way in the problem. We will show that the following dynamics globally stabilizes the optimum equilibrium: keep the gradient control of routing splits,

$$\dot{\alpha} = \beta E_\alpha[-p], \tag{23}$$

but add an anticipative term in the price generation:

$$\dot{p} = \Gamma[y - c + \nu\dot{\alpha}]_p^+. \tag{24}$$

We express these in incremental variables:

$$\delta\dot{\alpha} = \beta E_\alpha[-\delta p]. \tag{25}$$

$$\delta\dot{p} = \Gamma[\delta x\, \alpha + x^*\delta\alpha + \nu\delta\dot{\alpha}]_p^+. \tag{26}$$

For (25) note that the projection E_α removes any component in the direction of $\mathbf{1}$, in particular $p^* = q^*\mathbf{1}$. The derivation of (26) is analogous to (16), although here the price projection is maintained. Also note that relationship (18) remains valid.

Theorem 6: Under the dynamics (23-24), the equilibrium (15) is globally asymptotically stable, for $\nu > 0$.

Proof: Define the Lyapunov function candidate $W \geq 0$, vanishing only at equilibrium:

$$W(\delta\alpha, \delta p) = \frac{x^*}{2\beta}\|\delta\alpha\|^2 + \frac{1}{2}\delta p^T\Gamma^{-1}\delta p =: W_1 + W_2.$$

The first term above has derivative

$$\dot{W}_1 = x^*\langle\delta\alpha, E_\alpha[-\delta p]\rangle. \tag{27}$$

The derivative of the second term is bounded as follows:

$$\begin{aligned}
\dot{W}_2 &= \delta p^T[\delta x\alpha + x^*\delta\alpha + \nu\beta E_\alpha[-\delta p]]_p^+ \\
&\leq \delta p^T(\delta x\alpha + x^*\delta\alpha + \nu\beta E_\alpha[-\delta p]) \\
&= \delta p^T(\delta x\alpha + x^*\delta\alpha) + \nu\beta\langle\delta p, E_\alpha[-\delta p]\rangle \\
&\leq \delta p^T(\delta x\alpha + x^*\delta\alpha) - \nu\beta\|E_\alpha[-\delta p]\|^2. \tag{28}
\end{aligned}$$

The first step above uses (26). The second step follows by noting that if a link l has an active price projection, $(\delta x\alpha + x^*\delta\alpha + \nu\beta E_\alpha[-\delta p])_l \leq 0$ and $p_l = 0$, hence $\delta p_l \leq 0$. The last step invokes Lemma 5 with $v = -\delta p$.

Combining (27-28) and using (18) yields

$$\dot{W} \leq x^*\langle\delta\alpha, \delta p + E_\alpha[-\delta p]\rangle - \nu\beta\|E_\alpha[-\delta p]\|^2 + \delta q\delta x. \tag{29}$$

Now invoke Lemma 4, with $b = \alpha^*$, $v = -\delta p$, to get

$$\begin{aligned}
\langle\delta\alpha, \delta p + E_\alpha[-\delta p]\rangle &= \langle-(b-\alpha), -v + E_\alpha[v]\rangle \\
&= \langle b-\alpha, v - E_\alpha[v]\rangle \leq 0. \tag{30}
\end{aligned}$$

Also, $\delta q\delta x \leq 0$ as before, so $\dot{W} \leq 0$ along trayectories.

Again we invoke Lasalle to claim convergence to an invariant set where $\dot{W} \equiv 0$. Under this condition, all terms in (29) must be identically zero. In particular, $E_\alpha[-\delta p] \equiv 0$, and since $f(\cdot)$ is strictly decreasing, $x \equiv x^*$ and $q \equiv q^*$. Also, since $b = \alpha^*$ used in (30) is interior to Δ, the second part of Lemma 4 implies that $E_\alpha[-\delta p] \equiv E[-\delta p] \equiv 0$.

Therefore, $\delta p(t)$ is parallel to $\mathbf{1}$, $\delta p(t) = \delta q(t)\mathbf{1}$, so $\delta p \equiv 0$, since $\delta q \equiv 0$. Thus prices are identically at equilibrium, $p \equiv p^* > 0$. Finally, note from (25) that $\dot{\alpha} \equiv 0$, so (26) (note the projection is inactive) implies $\delta\alpha \equiv 0$. The invariant trajectory is at equilibrium. ∎

Remark 1: If we set $\nu = 0$, i.e. there is no anticipatory term in the dynamics, the Lasalle argument still gives global convergence to $x \equiv x^*$, as claimed in the example of Section III for $L = 2$; however the final step fails and we cannot claim asymptotic stability.

V. SIMULATIONS

We present some Matlab simulations to illustrate the system dynamics, with and without the derivative action terms. We use for this purpose a three parallel link network, with one flow from source to destination.

We implemented an Euler discretization of (23-24) (m is the discrete time index, $[z]^+ = \max(z, 0)$):

$$\alpha_{m+1} = \alpha_m + \beta E_{\alpha_m}[-p_m],$$
$$p_{m+1} = [p_m + \Gamma(y_m - c) + \Gamma\nu(\alpha_m - \alpha_{m-1})]^+.$$

Rates are controlled by $x = \frac{1}{q}$, which amounts to using the utility function $U(x) = \log(x)$. We fix parameters $c = 1$, $\Gamma = 7.5.10^{-4}I$, $\beta = 0.02$. The equilibrium is

$$x^* = 3, \quad \alpha^* = \tfrac{1}{3}\mathbf{1}, \quad p^* = q^*\mathbf{1} = \tfrac{1}{3}\mathbf{1}. \tag{31}$$

For the first simulation we set $\nu = 0$. Figure 2 shows a trajectory of α in the 3-dimensional simplex Δ, and at each point we indicate the projected price vector $-Ep$. At the beginning the trajectory hits the $\alpha_1 = 0$ boundary of the simplex, and the projection E_α acts to keep it within Δ. It spends some time on the boundary, reaching the vertex $(0, 0, 1)$, later returning, and temporarily hitting the boundary $\alpha_3 = 0$. Eventually the trajectory settles into a limit cycle in the interior of Δ, similar to the earlier example with two links. Despite this oscillation in α, the source rate x and price q, shown in Figure 3, converge as predicted by the theory.
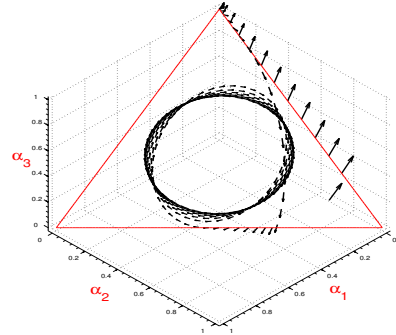

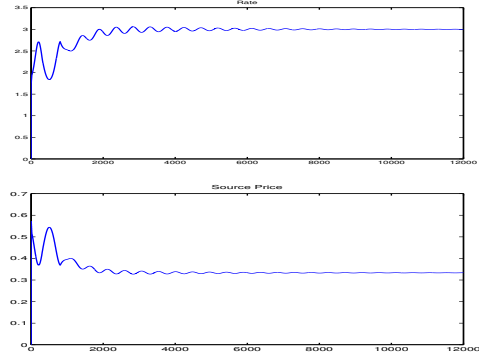Fig. 2. α's trajectory and negative price gradient, $\nu = 0$.

Fig. 3.   Source rate x (top) and price q (bottom), case $\nu = 0$.

We now include the derivative term in the price dynamics by setting $\nu = 50$. We observe in Figure 4 how this damps the α trajectory until it converges to α^*. Notice how the projection $-Ep$ is going to zero, implying prices are aligning with $\mathbf{1}\perp\Delta$. Finally, source rate x and source price q (not shown) still converge as in the previous simulation. Thus, equilibrium (31) is reached.

As a final remark, we note that qualitatively similar results are found with the control laws of Section IV-A, despite the lack of a global theorem for that case.
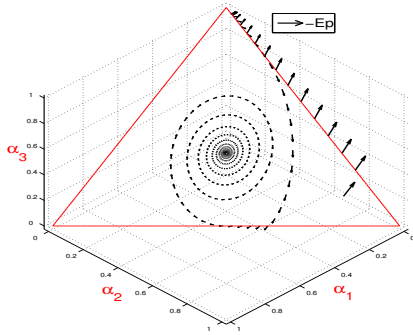
Fig. 4. α's trajectory and negative price gradient, $\nu = 50$

VI. Conclusions and Future work

The global stability of both primal and dual congestion control for fixed routing has been known for some time. We have studied its combination with controlled multipath routing. While adapting the traffic splits α based on the price gradient is stabilizing when combined with primal congestion control [12], in the dual case the dynamics can be oscillatory. We have studied stabilization based on derivative action in the control.

Two alternative places to add the anticipative term are the control of routing splits, or the link price control. Both give local asymptotic stability around the optimum, but only for the latter have we obtained a global result, in a simple topology. Global stability for the alternative of Section IV-A, which has the most general local results and is the most attractive for implementation in a large network (see [13]), remains open for future research.

Appendix: proof of Lemmas

A. Proof of Lemma 4

We use the following property of the approximation operation Ψ_Δ (see [14] and references therein, and the illustration in Figure 5):

$$\langle b - \Psi_\Delta(a), a - \Psi_\Delta(a) \rangle \leq 0 \quad \forall b \in \Delta, a \in \mathbb{R}^L. \quad (32)$$
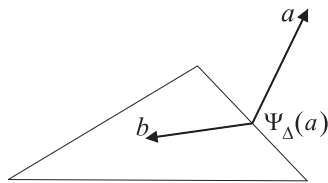

Fig. 5.   Property of approximation Ψ_Δ.

Apply it to $a = \alpha + \epsilon v$, and any $b \in \Delta$, and divide the second term by $\epsilon > 0$ preserving inequality. Then,

$$\left\langle b - \Psi_\Delta(\alpha + \epsilon v), \frac{\alpha + \epsilon v - \Psi_\Delta(\alpha + \epsilon v)}{\epsilon} \right\rangle \leq 0.$$

Taking limit with $\epsilon \to 0+$, the first term converges to $b - \Psi_\Delta(\alpha) = b - \alpha$, and the second to $v - E_\alpha[v]$ from (20). So we have (21).

For the second part, note that for fixed v, α, the inner product in (21) is *affine* in b. Assume it is zero at a point interior to the simplex: for the inequality to hold in a neighborhood of this point requires that it be identically zero in $b \in \Delta$ (otherwise, it would change sign in the neighborhood). But then the vector $v - E_\alpha[v]$ must be orthogonal to the plane of the simplex, so $Ev = E_\alpha[v]$. ∎

B. Proof of Lemma 5

Apply (32) to $a = \alpha + \epsilon v$, $b = \alpha$, and divide both terms by $\epsilon > 0$ (preserving inequality) to get

$$\left\langle \frac{\alpha - \Psi_\Delta(\alpha + \epsilon v)}{\epsilon}, \frac{\alpha + \epsilon v - \Psi_\Delta(\alpha + \epsilon v)}{\epsilon} \right\rangle \leq 0.$$

Taking limit with $\epsilon \to 0+$ and using (20) yields

$$\langle -E_\alpha[v], v - E_\alpha[v] \rangle \leq 0,$$

which implies (22). ∎

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