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Reduced-Order Observer for Sliding Mode Control of Nonlinear Non-Affine Systems

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Abstract— **In this paper nonlinear non-affine systems, for which the state vector is not completely available, are considered. It is assumed that the system's mathematical model is perfectly known and conditions hold, which assure the global injectivity of any required state transformation. The methodology aims at reducing chattering while ruling out possible ambiguous behaviors and considers an augmented state (the state and its first time derivative) and a new control, which is the time derivative of the original one. The proposed procedure combines sliding mode controller/observer and Luenberger like observer. The designed reduced-order observer relies on second order sliding mode differentiators just to provide the necessary, otherwise unavailable, artificial outputs exploited to steer to zero a lower dimensional estimation error vector under a simplified set of convergence conditions.**

I. Introduction

In this paper we consider the sliding mode control of non-affine, nonlinear systems with known dynamics and non completely available state. The proposed methodology, according to the twofold aim of reducing chattering and ruling out possible ambiguous behaviors [1], consists in an augmentation of the state space by considering both the states and their first time derivatives.

In the new representation the control vector is the derivative of the original one.

It is designed an observer to make available the augmented state of the new system.

A possible approach is a generalization to nonlinear systems of the well known Luenberger observer for linear ones. The considered problem, which is non trivial even in the case of affine systems, has been dealt within the sliding mode context by the Authors [2] and [3].

In recent literature a common method for the class of systems transformable into the Brunowsky canonical form, is to estimate the unavailable states by means of differentiators [4], [5] and high-gain observers [6], [7], [8], [9]; this strategy has proven to be effective even in presence of large uncertainties in the mathematical model. In spite of this great advantage, the differentiation procedure results not to be reliable as the order of the involved output derivative increases ([5], page 933), therefore the methodology appears not suitable for large scale systems.

The two approaches feature different properties, which could be regarded, in some situations, as complementary.

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The first method is differentiator free, but requires perfect modeling and additional constraints and conditions to be satisfied in order to fulfill both the approximability property and the convergence of the estimation error; the second one suffers from the previously outlined accuracy problem when the output derivatives to be estimated is high.

The proposed strategy is a compromise between the two approaches aimed at building reduced-order observers using differentiators just to provide the necessary, otherwise unavailable, artificial outputs exploited to steer to zero a lower dimensional estimation error vector under an often simplified set of convergence conditions.

II. The Variable Structure Control System

Let us consider the variable structure control system

$$\dot{\eta} = \varphi\left(t, \eta, u\right), \qquad t \geq 0, \tag{1}$$

with the control vector $u \in R^m$, the state variable $\eta \in R^n$ and with control constraint

$$u \in U. \tag{2}$$

The output vector $y \in R^k$ is expressed by the following equation

$$y = h\left(\eta\right), \tag{3}$$

where $h \in C^1$.
We are given a fixed sliding manifold

$$\sigma\left(\eta\right) = 0, \tag{4}$$

with $\sigma\left(\eta\right) \in R^m$, which fulfills prescribed control aims. We want to control the state variables $\eta\left(t\right)$, $t \geq 0$, of the control system (1) in order to guarantee the sliding property

$$\sigma\left[\eta\left(t\right)\right] = 0$$

for every t sufficiently large.

Let us assume that $n \geq m$, Ω is an open set of R^n,

$$\sigma = \sigma\left(\eta\right) : \Omega \to R^m,$$

σ is $C^1\left(\Omega\right)$, and the $m \times n$ Jacobian matrix

$$\frac{\partial \sigma}{\partial \eta}\left(\eta\right) \qquad \text{has maximum rank } m \tag{5}$$

for $\eta \in \Omega$. We denote by $\sigma_1, \ldots, \sigma_m$ the components of σ and define the Jacobian matrix $\sigma_\eta \in R^{m \times n}$ also through

(5) (and similarly for φ_η, h_η). A prime denotes a transpose. Moreover the control region

$$U \text{ is a nonempty closed set in } R^m.$$

The dynamics φ in (1)

$$\varphi : [0, +\infty) \times \Omega \times U \to R^n$$

is a Carathéodory mapping.

Consider again the control system (1) with control constraint (2) and sliding manifold (4). In order to reduce chattering due to the discontinuous nature of the control u, we consider the augmented control system

$$\dot{\eta} = \varphi(t, \eta, u), \quad \dot{u} = v, \quad t \geq 0, \tag{6}$$

with control variable $v \in V \subseteq R^m$ and u available. If the new control v is discontinuous, then the vector u turns out to be (absolutely) continuous.

Assume that φ, σ are both of class C^2 everywhere. Then for almost every t

$$\dot{\sigma} = \sigma_\eta(\eta)\, \varphi(t, \eta, u). \tag{7}$$

We fix a constant $m \times m$ matrix

$$\Lambda = \operatorname{diag}(\lambda_i), \quad \lambda_i > 0, \quad i = 1, \ldots, m,$$

and consider

$$s = \dot{\sigma} + \Lambda\sigma. \tag{8}$$

We want to control, by the signal v, the state variables $(\eta'(t), u'(t))'$, $t \geq 0$, of the augmented system in order to guarantee the sliding property

$$s(t) = 0 \tag{9}$$

for every t sufficiently large. If $s = 0$, the original sliding output $\sigma[\eta(t)] \to 0$ as $t \to +\infty$ exponentially fast. This means that $\sigma[\eta(t)]$ is arbitrarily close to 0 for t sufficiently large.

If suitable first order approximability properties are fulfilled by the given control system (see [10], [11], [12] and [13]), then $\eta(t)$ is arbitrarily close to the ideal sliding state on the bounded time intervals, thus η fulfills approximately the control aims guaranteed by the choice of the sliding manifold $\sigma(\eta) = 0$, by using the continuous original control variable u. In this way chattering is counteracted in the given control system.

III. The full order observer

We have defined a new variable structure control system

$$\begin{aligned} &\dot{\eta} = \varphi(t, \eta, u), \qquad u \in U, \\ &\dot{u} = v, \\ &y = h(\eta), \\ &s(t) = 0, \end{aligned} \tag{10}$$

with $\eta \in R^n$, $y \in R^k$, u, v, $s \in R^m$ and $m \leq n$.

By setting $\dot{\eta} = \zeta$ we have

$$\begin{bmatrix} \dot{\eta} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} \zeta \\ \Phi(t, \eta, u) \end{bmatrix} + \begin{bmatrix} 0 \\ \Gamma(t, \eta, u) \end{bmatrix} v, \tag{11}$$

where $\Phi = \varphi_t + \varphi_\eta \varphi$ and $\Gamma = \varphi_u$.

Let the augmented state vector $x = (\eta', \zeta')' \in R^{2n}$, since $u(t)$ is available, we can write

$$\dot{x} = A(t, x) + B(t, x)\, v = f(t, x, v). \tag{12}$$

If x were available, as well as u, the control v could be chosen discontinuous on $s(t)$, according to standard techniques, such that $s = 0$ in finite time.

We consider the following observed variable structure control system

$$\dot{x} = f(t, x, v), \quad t \geq 0, \quad \text{state equation}, \tag{13}$$

$$\dot{u} = v, \quad u \in U, \quad \text{control equation and constraint}, \tag{14}$$

$$y = h(x), \quad \text{output equation}, \tag{15}$$

$$s(x) = 0, \quad \text{sliding manifold}. \tag{16}$$

The full-order observer has the form

$$\dot{z} = f(t, z, v) + P[y(t) - h(z)]. \tag{17}$$

We have x, $z \in R^{2n}$, $y \in R^k$, u, v, $s \in R^m$ and $m \leq n$. The functions f, h, s are continuously differentiable in x, with f measurable in t and continuous in (x, v).

Given the constant matrix $P \in R^{n \times k}$ we consider the following conditions.

Condition 1: For every $t \geq 0$, y, z there exists a unique solution

$$v^*(t, y, z)$$

of the equation

$$s_x(z)\{f(t, z, \cdot) + P[y - h(z)]\} = 0, \tag{18}$$

where P is as in (17).

Such a mapping v^* is by definition *the observer's equivalent control* corresponding to the output y.

Condition 2: Whenever $s(x) = 0$,

$$\operatorname{rank} s_x(x) = m. \tag{19}$$

Condition 2 is satisfied by system (13)–(17) since (5).

According to [2], since Condition 2 holds, we can find a positive integer p such that for every x_0 with $s(x_0) = 0$ and some $\delta > 0$, the ball of center x_0 and radius δ can be written as a disjoint union of subsets of the surfaces $s_j(x) = 0$, $j = 1, \ldots, m$, and of p open connected regions $C_1, \ldots, C_p$.

Given $P \in R^{n \times k}$ and Condition 2, we shall consider the set Z of all output feedback $v = v(t, y, z)$ such that the following properties hold:

1) v is measurable in t, continuous in (y, z) if $s_j(z) \neq 0$, $j = 1, \ldots, m$;
2) given any z_0 with $s(z_0) = 0$, denote by v^j the restriction of v to the set of those points (t, y, z) such that $z \in C_j$, $j = 1, \ldots, p$ (C_j as defined above): then for every (t, y) there exists a finite
$$\lim_{z \to z_0} v^j(t, y, z) = v^j(t, y, z_0);$$
3) if $(x', z')'$ is any Filippov solution in $[0, +\infty)$ to (13), (15), (17) corresponding to v (see [14], [15], or [16],

pp. 101–102) such that $s[z(0)] = 0$, then $s[z(t)] = 0$, $t \geq 0$.

We consider solutions in $[0, +\infty)$ (either in the Filippov or a.e. sense) to (13), (15), (17) corresponding to solutions to the observer's equivalent control, i.e. solutions to

$$\dot{x} = f\{t, x, v^*[t, h(x), z]\}, \tag{20}$$

$$\dot{z} = f\{t, z, v^*[t, h(x), z]\} + P[h(x) - h(z)]. \tag{21}$$

Existence of a.e. solutions in $[0, +\infty)$ to (13), (15), corresponding to the observer's equivalent control and to the bounded continuous output y is obtained if the set V: $v \in V$, is compact, f is independent of t and bounded, f, h, s, are continuous, h bounded on the manifold (16), if $f(x, V)$ is always convex and if Condition 1 holds. This follows from [16], Theorem 1, p. 191. Of course different assumptions (linearity, growth conditions, etc.) suffice for global existence.

Condition 3: Assume Condition 1 (for any given P) and Condition 2. Suppose V is closed and for every t, x, z with $s(z) = 0$,

$$\left\{(f(t, x, v), f(t, z, v))' : \ v \in V\right\} \text{ is convex in } R^{4n}. \tag{22}$$

Remark 1: Assumption (22) is stronger than convexity of $f(t, x, V)$ for all t, x. On the other hand, (22) is verified (for any s) if

$$f(t, x, v) = A(t, x) + B(t, x)\, g(t, v) \tag{23}$$

with $g(t, V)$ convex for all t. The f affine in v is a particular case of (23), to which (22) applies if V is convex.

From previous remark system (13)–(17) satisfies Condition 3 since f is given by (12). Therefore the following theorem applies.

Theorem 1: Assume Condition 3. Let $(x', z')'$ be a Filippov solution in $[0, +\infty)$ to (13), (15), (17) with $s[z(0)] = 0$ corresponding to any $\overline{u} \in Z$. Then $(x', z')'$ solves (20) and (21) a.e. in $[0, +\infty)$.

Proof. See [2].

Theorem 1 shows that, under a convexity condition, every Filippov solution to the state-observer coupled system corresponding to any feedback law in Z, is an a.e. solution to (20) and (21). This is a key property for employing the equivalent control in order to achieve asymptotically (16).

Remark 2: The equivalent control is physically meaningful if the approximability condition is verified, as introduced in [17], [10] for nonlinear systems. See [10] for a discussion about such a problem. Under some smoothness conditions, (23) implies approximability ([10], Theorem 5). Therefore using the observer's equivalent control is justified for significant classes of systems (13)-(17). Moreover one obtains all the (Filippov) solutions corresponding to piecewise-continuous output feedback.

Condition 4: There exist matrices $P \in R^{n \times k}$, $Q \in R^{n \times n}$ and positive numbers α, ε, ω such that the eigenvalues of the symmetric matrix Q are between α and ω, and those of the symmetric part of $Q(f_x - Ph_x)$ are $\leq -\varepsilon$ everywhere.

Theorem 2: Assume Condition 4, Condition 1 with P given by Condition 4 and suppose $|s_x(x)| \leq M$ everywhere, for some constant M. Then

$$|s[x(t)]| \leq M\left(\frac{\omega}{\alpha}\right)^{\frac{1}{2}} |x(0) - z(0)| \exp(ct), \quad t \geq 0, \tag{24}$$

where $\omega c = -\varepsilon$, for every a.e. solution $(x', z')'$ in $[0, +\infty)$ to (20) and (21), such that $s[z(0)]$.

Proof. See [2].

Remark 3: The proof of Theorem 2 shows that under Condition 4,

$$|s[x(t)] - s[z(t)]| \leq M\left(\frac{\omega}{\alpha}\right)^{\frac{1}{2}} |x(0) - z(0)| \exp(ct), \ t \geq 0,$$

without assuming neither Condition 1 nor $s[z(0)] = 0$.

Discussion:

1) For linear time invariant systems

$$f(t, x, v) = Ax + Bv, \quad h(x) = Cx,$$

Condition 4 holds if (C, A) is detectable.

2) Assume f, h continuously differentiable in $[0, +\infty) \times F$, F open connected. If there exists P such that

$$Lf_x - Ph_x$$

is a stability matrix for all $w = (t, x, v)$, then known results [18], Section 12.4, imply the following necessary and sufficient condition for the existence of Q fulfilling Condition 4: for some symmetric equi-negative definite matrix S and all w,

$$\int_0^{+\infty} \exp(aL')\left[a\left(L'_w S + SL_w\right) + S_w\right] \exp(aL)\, da = 0. \tag{25}$$

Theorem 3: Under the assumptions of Theorem 2, suppose moreover boundedness of f_x, h_x and of $f(t, x, v)$ when $s(x) = 0$. Then there exists a constant K such that for a.e. $t \geq 0$,

$$\left|\left(\frac{d}{dt}\right) s[x(t)]\right| \leq K\, |x(0) - z(0)| \exp(ct), \tag{26}$$

c as in Theorem 2, for every a.e. solution $(x', z')'$ to (20) and (21) in $[0, +\infty)$ such that $s[z(0)] = 0$.

Proof. See [2].

Remark 4: The exponential decay of s (with a possibly bigger c than in (26)) may be obtained by weakening the boundedness assumption about f,

Discussion: Condition 4 can be explicitly written as

$$Q(f_x - Ph_x) + (f_x - Ph_x)' Q = H, \tag{27}$$

where H is such that $\lambda_{\min}(H) \leq -\varepsilon < 0$ and

$$f_x = \begin{bmatrix} 0 & I \\ \varphi_t + \varphi_{\eta\eta}\zeta & \varphi_\eta \end{bmatrix},$$

$$P = \begin{bmatrix} P_{11} \\ P_{21} \end{bmatrix},$$

$$h_x = \begin{bmatrix} h_\eta & 0 \end{bmatrix}.$$

In equation (27) the positive $2n \times 2n$ matrix Q is unknown, while the entries of the $2n \times k$ matrix P play the role of design parameters. The solution depends on the jacobian f_x and h_x, which are time varying and state dependent. Therefore in general the treatment has only a local validity provided some observability conditions are locally fulfilled.

It is necessary to identify tools to make the observer design more flexible and of lower dimension.

The first possibility is that of introducing differentiators to increase the dimension of the output vector. This fact increases the number of columns of P, which is the main degree of freedom in the full-order observer design. This practice could be taken to the extreme by suitably choosing (among possible alternatives) a combination of outputs and their higher order derivatives until a vector related to the original state vector by a diffeomorphism is attained, even eliminating the need of an observer. This approach could require differentiators of order higher than the one realistically implementable.

The alternative way of exploiting a possible increment of the dimension of the output vector by differentiation to make the solution of (27) easier, is that of introducing reduced order observer according to the next section.

IV. THE REDUCED ORDER OBSERVER

We consider again the variable structure control system (13)–(16)

$$\begin{aligned} &\dot{x} = f(t,x,v), \qquad \dot{u} = v, \quad u \in U, \\ &y = h(x), \\ &s(x) = 0, \end{aligned}$$

with $x \in R^{2n}$, $y \in R^k$, u, v, $s \in R^m$ and $m \leq n$.

Let $q \in R^{2n-k}$

$$q = g(x), \tag{28}$$

be such that the map

$$\theta(x) = \begin{bmatrix} g(x) \\ h(x) \end{bmatrix} \text{ is one-to-one and invertible.} \tag{29}$$

By definitions and assumption (29), we have

$$\begin{aligned} q &= g\left[\theta^{-1}(q,y)\right] \\ y &= h\left[\theta^{-1}(q,y)\right] \end{aligned} \tag{30}$$

We consider the state vector $(q', y')'$ and the related state equations

$$\dot{q} = g_x f(t,x,v) = \psi(t,q,y,v), \tag{31}$$

$$\dot{y} = h_x f(t,x,v) = \gamma(t,q,y,v), \tag{32}$$

while the sliding output can be expressed as

$$s(x) = s\left[\theta^{-1}(q,y)\right] = \varsigma(q,y). \tag{33}$$

If q were available, as well as y and u, the control v could be chosen discontinuous on $\varsigma(q,y)$, according to standard techniques, such that $\varsigma = 0$ in finite time.

In the considered case neither q is available, nor an output depending on it. In order to artificially construct such an output, we introduce a second order sliding mode differentiator [19], [4], [20], [5], to obtain in finite time $\dot{y}$.

The proposed reduced-order observer has the form

$$\dot{r} = \psi(t,r,y,v) + N\left[p(t) - \gamma(t,r,y,v)\right], \tag{34}$$

where $p = \dot{y} = \gamma(t,q,y,v)$ is from the second order sliding mode differentiator.

Given the constant matrix $N \in R^{(2n-k)\times k}$ we shall consider the following conditions.

Condition 5: For every $t \geq 0$, r, y, p there exists a unique solution

$$v_*(t,r,y,p) \in V$$

of the equation

$$\begin{aligned} &\varsigma_q(r,y)\left\{\psi(t,r,y,\cdot) + N\left[p - \gamma(t,r,y,\cdot)\right]\right\} \\ &\quad + \varsigma_y(r,y)\,\gamma(t,r,y,\cdot) = 0, \end{aligned} \tag{35}$$

where N is as in (34).

Such a mapping v_* is by definition *the reduced-order observer's equivalent control* corresponding to the output p.

Condition 6: Whenever $\varsigma(q,y) = 0$,

$$\operatorname{rank}\left[\varsigma_q(q,y),\ \varsigma_y(q,y)\right] = m. \tag{36}$$

We consider solutions in $[0,+\infty)$ (either in the Filippov or a.e. sense) to (31), (34) corresponding to solutions to the reduced-order observer's equivalent control, i.e. solutions to

$$\dot{q} = \psi(t,q,y,v_*(t,r,y,p)), \tag{37}$$

$$\dot{r} = \psi(t,r,y,v_*(t,r,y,p)) + N\left[p - \gamma(t,r,y,v_*(t,r,y,p))\right]. \tag{38}$$

Condition 7: Assume Condition 5 (for any given N) and Condition 6. Suppose V is closed and for every t, q, y, r with $\varsigma(r,y) = 0$,

$$\left\{(\psi(t,q,y,v), \gamma(t,q,y,v), \psi(t,r,y,v))' :\ v \in V\right\} \tag{39}$$

is convex in R^{4n-k}. Let $(q', r')'$ be a Filippov solution in $[0,+\infty)$ to (31), (34) with $\varsigma[r(0), y(0)] = 0$. Then $(q', r')'$ solves (37) and (38) a.e. in $[0,+\infty)$.

Let $\psi_q(t,q,y,v)$ and $\gamma_q(t,q,y,v)$ be the jacobian matrices with respect to q of the vector fields $\psi(t,q,y,v)$ and $\gamma(t,q,y,v)$ respectively.

Condition 8: There exist matrices $N \in R^{(2n-k)\times k}$, $W \in R^{(2n-k)\times(2n-k)}$ and positive numbers μ, κ, ν such that the eigenvalues of the symmetric matrix W are between μ and κ, and those of the symmetric part of $W(\psi_q - N\gamma_q)$ are $\leq -\nu$ everywhere.

From Remark 1 system (31) and (34) satisfies Condition 7 since f in (31) and (33) is given by (12). Therefore the following applies.

Theorem 4: Assume Condition 8, Condition 5 with N given by Condition 8 and suppose $|\varsigma_q(q,y)| \leq D$ everywhere, for some constant D. Then

$$|\varsigma[q(t), y(t)]| \leq D\left(\frac{\nu}{\mu}\right)^{\frac{1}{2}} |q(0) - r(0)| \exp(bt), \quad t \geq 0, \tag{40}$$

where $\nu b = -\kappa$, for every a.e. solution $(q', r')'$ in $[0, +\infty)$ to (37) and (38), such that $\varsigma [r(0), y(0)] = 0$.

Proof. Set

$$V_1(t) = [q(t) - r(t)]' W [q(t) - r(t)], \quad t \geq 0.$$

Then for a.e. $t \geq 0$,

$$\dot{V}_1(t) = 2[q(t) - r(t)]' W \\ \{\psi(t, q, y, v_*) - \psi(t, r, y, v_*) - N[p(t) - \gamma(t, r, y, v_*)]\},$$

where

$$v_*(t) = v_*(t, r, y, p).$$

Therefore

$$\dot{V}_1(t) \leq \\ 2[q(t) - r(t)]' W \left[\int_0^1 (\psi_q - N\gamma_q)\, da\right] [q(t) - r(t)],$$

where ψ_q and γ_q are evaluated at $(t, \beta(a,t), y, v_*)$ and $\beta(a,t) = aq(t) + (1-a) r(t)$.

We have

$$\dot{V}_1 \leq \\ 2\int_0^1 [q-r]' \left[W(\psi_q - N\gamma_q) + (\psi_q - N\gamma_q)' W'\right] [q-r]\, da,$$

then

$$\dot{V}_1 \leq -2\int_0^1 \nu (q-r)'(q-r)\, da \leq -2\frac{\nu}{\kappa} V_1(t) \leq -2bV_1(t).$$

If we set

$$w(t) = V_1(t)\exp(-2bt),$$

we have that $\dot{w}(t) \leq 0$, thus giving

$$V_1(t) = V_1(0)\exp(-2bt);$$

Then

$$\begin{aligned} \mu |q-r|^2 &\leq V_1(t) \\ &\leq \nu |q(0) - r(0)|^2 \exp(-2bt). \end{aligned}$$

We can conclude

$$|q - r| \leq \frac{\nu}{\mu} |q(0) - r(0)| \exp(-bt),$$

$$|\sigma(q,y) - \sigma(r,y)| \leq D|q-r|$$

and therefore (40). □

Remark 5: Theorem 4 holds even in the case in which N is chosen to be not constant; matrix N can be allowed to depend on t and any available signal depending on time, including the measurable input and output vectors.

Note that both Condition 5 and Condition 8 are, in general, difficult to be fulfilled globally; more realistic semi-global or local solutions are to be found. In this sense Condition 8 implies the solution of a Lyapunov-Krasovsky equation of reduced order and therefore easier to be verified with respect to the one implied by Condition 4. We have further to consider the fact that the choice of $q = g(x)$ is rather arbitrary and in practical situations an extra degree of freedom is offered to the designer by ψ_q, as it is shown by the following example.

V. Example

We consider the following variable structure control system (1), the state equation is

$$\dot{\eta} = \varphi(t, \eta, u) = \begin{bmatrix} (1+\eta_1^2)(\eta_2 - \eta_1) - c_1\eta_1 \\ (1+\eta_1^2)(\eta_1 - \eta_2) + \varphi_3(\eta_1)\varphi_4(u) \end{bmatrix}, \tag{41}$$

where $\eta \in R^2$; the control constraint is $u \in U \subset R$, the output is η_1 and the sliding manifold (4) is $s(\eta) = \eta_2 - \eta_1 = 0$.

We set $\dot{\eta} = \zeta$, derive (41) with respect to time and obtain the system in the form (11)

$$\begin{bmatrix} \dot{\eta} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} \zeta \\ \Phi(t, \eta, \zeta, u) \end{bmatrix} + \begin{bmatrix} 0 \\ \Gamma(t, \eta, \zeta, u) \end{bmatrix} v, \quad \text{where} \tag{42}$$

$$\begin{aligned} \Phi_1 &= \left[2\eta_1(\eta_2 - \eta_1) - (1+\eta_1^2) - c_1\right]\zeta_1 + (1+\eta_1^2)\zeta_2, \\ \Phi_2 &= \left[2\eta_1(\eta_1 - \eta_2) + (1+\eta_1^2) + \varphi_{3\,\eta_1}(\eta_1)\varphi_4(u)\right]\zeta_1 \\ &\quad - (1+\eta_1^2)\zeta_2, \\ \Gamma_1 &= 0, \qquad \Gamma_2 = \varphi_3(\eta_1)\varphi_{4\,u}(u), \end{aligned}$$

$\zeta \in R^2$ and the signal $v = \dot{u} \in V \subset R$ is regarded as the new control.

Then, by setting the augmented state vector $x = (\eta', \zeta')' \in R^4$, since $u(t)$ is available, we can express the system in the form (12)

$$\dot{x} = A(t,x) + B(t,x)v = f(t,x,v). \tag{43}$$

We assume that the measurable state component is $\eta_1 = x_1$ and, with the aim of exploiting all the available signals, we construct a second order sliding mode differentiator and in finite time we have $p = \zeta_1 = x_3$.

We can now define the output equation on the basis of the available signals

$$y = h(x) = \begin{bmatrix} \eta_1 \\ \zeta_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} \tag{44}$$

We consider the following variable structure control system (13)–(16), where the state equation is given by (43), the control equation is $\dot{u} = v$, $u \in U \subseteq R$, $v \in V \subseteq R$, the output equation is (44), and the sliding manifold is chosen as

$$\sigma(x) = (x_4 - x_3) + \Lambda(x_2 - x_1) = 0, \tag{45}$$

with $\sigma \in R$.

The vector $q \in R^2$ is defined as

$$q = g(x) = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}, \tag{46}$$

and the map $\theta(x)$

$$\theta(x) = \begin{bmatrix} g(x) \\ h(x) \end{bmatrix} = \begin{bmatrix} x_2 \\ x_4 \\ x_1 \\ x_3 \end{bmatrix} \tag{47}$$

is one-to-one and invertible.

We consider the state vector $[q', y']'$ and the related state equations

$$\begin{aligned} \dot{q} &= \psi(t, q, y, v), \\ \dot{y} &= \gamma(t, q, y, v), \end{aligned} \tag{48}$$

where

$$\begin{aligned}\psi_1 &= q_2 = \left(1+y_1^2\right)\left(y_1-q_1\right)+\varphi_3\left(y_1\right)\varphi_4\left(u\right),\\ \psi_2 &= \left[2y_1\left(y_1-q_1\right)+\left(1+y_1^2\right)+\varphi_{3\,y_1}\left(y_1\right)\varphi_4\left(u\right)\right]y_2\\ &\quad -\left(1+y_1^2\right)q_2+\varphi_3\left(y_1\right)\varphi_{4\,u}\left(u\right)v,\\ \gamma_1 &= y_2 = \left(1+y_1^2\right)\left(q_1-y_1\right)-c_1y_1,\\ \gamma_2 &= \left[2y_1\left(q_1-y_1\right)-\left(1+y_1^2\right)-c_1\right]y_2+\left(1+y_1^2\right)q_2,\end{aligned}$$

while the sliding output $\varsigma\left(q,y\right)$ is

$$\varsigma\left(q,y\right)=\left(q_2-y_2\right)+\Lambda\left(q_1-y_1\right). \tag{49}$$

The reduced-order observer (34) for system (43)–(45) takes the form

$$\begin{aligned}\dot{r} &= \psi\left(t,r,y,v\right)+N\left[p\left(t\right)-\gamma_1\left(t,r,y,v\right)\right]=\\ &= \left[\begin{array}{c} r_2\\ \psi_2\left(t,r,y,v\right)\end{array}\right]+N\left[y_2-\left(1+y_1^2\right)\left(r_1-y_1\right)-c_1y_1\right],\end{aligned} \tag{50}$$

where $N \in R^2$.

We have the two jacobian matrices

$$\psi_q\left(t,q,y,v\right)=\left[\begin{array}{cc} 0 & 1\\ -2y_1y_2 & -\left(1+y_1^2\right)\end{array}\right] \tag{51}$$

and

$$\gamma_{1\,q}\left(t,q,y,u\right)=\left[\begin{array}{cc}\left(1+y_1^2\right) & 0\end{array}\right]. \tag{52}$$

Let $\Theta = W\left(\psi_q - N\gamma_{1\,q}\right)$ with $W=\left[\begin{array}{cc}1 & 0\\ 0 & 1\end{array}\right]$ and $N=\left[\begin{array}{c}N_1\\ N_2\end{array}\right]$, we obtain $\Theta+\Theta'=$

$$\left[\begin{array}{cc} 2N_1\left(1+y_1^2\right) & 1-2y_1y_2+N_2\left(1+y_1^2\right)\\ 1-2y_1y_2+N_2\left(1+y_1^2\right) & -2\left(1+y_1^2\right)\end{array}\right]. \tag{53}$$

According to Theorem 4, the eigenvalues of $\Theta+\Theta'$, with $\Theta = W\left(\psi_q - N\gamma_{1\,q}\right)$, $W>0$, must be negative to guarantee the asymptotic convergence of r to q.

Remembering Remark 5, we make matrix (53) globally negative definite, independently of v and on the chosen sliding manifold, by choosing $N_1 < -\frac{1}{4}$ and $N_2 = \frac{2y_1y_2}{\left(1+y_1^2\right)}$.

VI. CONCLUSIONS

In this paper nonlinear non-affine systems, for which the state vector is not completely available, have been considered.

We assume that the system's mathematical model is perfectly known and conditions hold, which assure the global injectivity of any required state transformation.

The methodology attains chattering reduction, while ruling out possible ambiguous behaviors and considers an augmented state (the state and its first time derivative) and a new control, which is the time derivative of the original one.

The proposed procedure combines sliding mode controller/observer and Luenberger like observer. This combination could be significant for large scale high dimensional systems.

In the paper we have considered the role of the output derivative as an auxiliary output for the reduced-order observer.

The output derivative is made available by a second order sliding mode differentiator. As a result new degrees of freedom are offered to the designer.

In future research possible generalization of the proposed approach, ranging from the use of higher order differentiators to the introduction of uncertainties can be envisaged.

The proposed approach can be reasonably applied to other applications involving sliding mode observers, see e.g. [21], [22], [23].

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