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Semi-Global Asymptotic Stability of a Class of Sampled-data Nonlinear Systems in Output Feedback Form

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Abstract— **The paper considers sampled-data output feedback control of a class of nonlinear systems in output feedback form. The underlying continuous-time controller is designed based on backstepping technique, employing a linear dynamic filter, and globally asymptotically stabilises the system. Rigorous analysis shows that when implemented with a sampling and zero-order hold device, the sampled-data version of the continuous-time controller semi-globally asymptotically stabilises the original system to be controlled, in the sense that the sampling period can be arbitrarily small.**

I. INTRODUCTION

Over decades, sampled-data control has played an ever-increasing important role in control engineering practice, due to wide applications of digital computers. Compared with that of linear systems, of which the latest results, see [1] and the references therein, sampled-data control of nonlinear systems remains a more challenging problem. This partly results from the fact that an exact, discrete-time model of a nonlinear system is generally not available for design [4].

The approaches to nonlinear sampled-data control mainly fall into two categories: approximate discrete-time model based approach, and emulation approach. For detailed results based on approximate discrete-time models, refer to [2], [3], [4]. A typical result of approximate discrete-time model-based design, is that the resultant sampled-data controller ensures practical stability of a nonlinear continuous-time plant. Emulation design exploits the advantages of existing continuous-time control design, and most of the results in this regard take the spirit of fast sampling. That means, if the sampling period is small enough, the sampled-data version of a corresponding continuous-time controller, implemented using a sampling and hold device, can still stabilise the system to some extent, as expected. Although not necessarily providing constructive methods for sampled-data controller design, emulation design is still of considerable practical importance due to simplicity of its implementation.

Previous efforts of emulation in the literature mainly attempted to establish certain stability preservation under sampling. For general nonlinear systems, the effect of fast sampling on static controllers was investigated in [10]; An ISS stability result was shown in [12] both for static controllers and for dynamic ones; asymptotic controllability and observability of the underlying continuous-time system was proved to imply semi-global practical asymptotic stabilization by a sampled-data output feedback controller [8]; a similar result via a different approach, gap metric theory, was presented in [9]. A sampled-data scheme for adaptive control of a class of nonlinear systems was presented in [14]. When it comes to some particular nonlinear systems, results in [6] and [7] have shown that the sampled-data version of an output feedback controller using high-gain observers ensures local practical stability of the overall nonlinear sampled-data system, if the underlying continuous-time system is locally asymptotic stable, and specifically, local exponential stability if it is local exponential stable and the continuous-time controller uses state feedback.

For a class of nonlinear systems in output feedback form, it is known that a continuous-time output feedback controller based on a filtered transformation [13] can be designed to guarantee global asymptotic stability. However, stability analysis still remains an issue for such a system when a sampled-data version of the continuous-time control is applied, which is considered here in this paper. The analysis of the present paper shows that if the sampling period can be chosen arbitrarily small, then the sampled-data version of the existing continuous-time controller will still semi-globally asymptotically stabilise the original continuous-time system to be controlled. Note that this stability result is obtained under local Lipschitz conditions. In addition, backstepping technique is applied to obtain the continuous-time controller, and therefore, our result can also be interpreted as a step toward understanding the effect of sampling on backstepping controllers. This is important because backstepping, as one of few systematic control design schemes for nonlinear systems, has been intensively used in literature.

II. PROBLEM STATEMENT

We consider sampled-data control of one class of single-input-single-output nonlinear systems which can be transformed into the following output feedback form

$$\begin{aligned} \dot{x} &= A_c x + \phi(y) + bu \\ y &= Cx \end{aligned} \tag{1}$$

with

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_\rho \\ \vdots \\ b_n \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix},$$

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where $x \in R^n$ is the state vector, $u, y \in R$ are the input and the output respectively, $b \in R^n$ is an known vector, while $\phi(y)$ is a smooth vector field with each element being a nonlinear function of y and satisfying $\phi(0) = 0$.

Assumption 1: The system is of minimum phase, i.e., the polynomial $\mathcal{B}(s) = \sum_{i=\rho}^{n} b_i s^{n-i}$ is Hurwitz.

For the above system, the sampled-data controller is given as follows

$$u_d(t) = u_c\big(y(mT), \xi(mT)\big), \quad \forall t \in [mT, mT+T) \quad (2)$$

$$\xi(mT) = e^{\Lambda T}\xi((m-1)T) + b_f \cdot u_d\big(y((m-1)T), \xi((m-1)T)\big)\int_0^T e^{\Lambda\tau}d\tau \quad (3)$$

where u_c is the continuous-time controller designed in next section, $y(mT)$ is obtained by sampling $y(t)$ at each sampling instant, $\xi(mT)$ is the discrete-time implementation of the filter introduced in next section, T is the fixed sampling period, $m = 0, 1, 2, \cdots$.

The problem considered here is to prove that the sampled-data controller (2)-(3) will ensure semi-global asymptotic stability of system (1), if T can be very small.

III. Preliminary Results

In this section we briefly describe the continuous-time control design and the associated stability result for this particular system, which is the foundation of our analysis for sampled-data case.

A. State Transformation

For system (1) with relative degree $\rho \geq 2$, we can introduce the following $(\rho-1)$th order filter

$$\begin{bmatrix} \dot\xi_1 \\ \dot\xi_2 \\ \vdots \\ \dot\xi_{\rho-1} \end{bmatrix} = \begin{bmatrix} -\lambda_1 & 1 & 0\dots & 0 \\ 0 & -\lambda_2 & 1\dots & 0 \\ \vdots & \vdots & \vdots\,\vdots & \vdots \\ 0 & 0 & 0\dots & -\lambda_{\rho-1} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{\rho-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u \triangleq \Lambda\xi + b_f u \quad (4)$$

where $\lambda_i > 0$, $i = 1, \cdots, \rho-1$, are the design parameters. With the vectors $\bar d_i \in R^n$, defined recursively by $\bar d_{\rho-1} = b$ and $\bar d_i = A_c\bar d_{i+1} + \lambda_{i+1}\bar d_{i+1}$ for $i = \rho-2, \cdots, 1$, the following filtered transformation

$$\eta = x - \sum_{i=1}^{\rho-1}\bar d_i\xi_i \quad (5)$$

can transform system (1) into

$$\begin{aligned} \dot\eta &= A_c\eta + \phi(y) + d\xi_1 \\ y &= C\eta \end{aligned} \quad (6)$$

where $d = [A_c\bar d_1 + \lambda_1\bar d_1]$. It can be shown that $d_1 = b_\rho$ and

$$\sum_{i=1}^{n} d_i s^{n-i} = \mathcal{B}(s)\prod_{i=1}^{\rho-1}(s+\lambda_i) \quad (7)$$

Since all λ_i are positive, d is a Hurwitz vector with $d_1 = b_\rho = 1$ (here we assume $b_\rho = 1$ without loss of generaility). Therefore with ξ_1 being the input, system (6) is of minimum phase and relative degree one. To extract the internal dynamics of (6), introduced is the following state transform

$$\begin{aligned} z_1 &= \eta_2 - d_2\eta_1 \\ &\vdots \\ z_{n-1} &= \eta_n - d_n\eta_1 \\ y &= \eta_1 \end{aligned} \quad (8)$$

where $z \in R^{n-1}$. In the new coordinates, system (6) can be written as

$$\begin{aligned} \dot z &= Dz + \phi_z(y) \\ \dot y &= z_1 + \phi_y(y) + \xi_1 \end{aligned} \quad (9)$$

where D is the companion matrix of $d[1]$ and given by

$$D = \begin{bmatrix} -d_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -d_n & 0 & \cdots & 0 \end{bmatrix}$$

and

$$\begin{aligned} \phi_z(y) &= y\begin{bmatrix} d_3 - d_2^2 \\ d_4 - d_3d_2 \\ \vdots \\ d_n - d_{n-1}d_2 \\ -d_nd_2 \end{bmatrix} + \begin{bmatrix} \phi_2 - \phi_1d_2 \\ \phi_3 - \phi_1d_3 \\ \vdots \\ \phi_{n-1} - \phi_1d_{n-1} \\ \phi_n - \phi_1d_n \end{bmatrix} \\ \phi_y(y) &= \phi_1 + yd_2 \end{aligned}$$

where ϕ_i denotes the ith component of the vector ϕ. Both ϕ_y and ϕ_z are locally Lipschitz in their arguments. Note that due to $\phi(0) = 0$, we have $\phi_y = y\bar\phi_y(y)$ and $\phi_z(y) = y\bar\phi_z(y)$. Finally the model for control design is the extended system consisting of (9) and (4).

B. The Continuous-Time Control

If system (1) is of relative degree one, then we are actually dealing with the following system

$$\begin{aligned} \dot z &= Dz + \phi_z(y) \\ \dot y &= z_1 + \phi_y(y) + u_{c1} \end{aligned} \quad (10)$$

and the continuous-time control u_{c1} is designed as

$$u_{c1} = -\phi_y - ky - y\bar\phi_z^{\mathrm{T}}P^2\bar\phi_z \quad (11)$$

where k is a positive real and P is the symmetric positive definite solution of the Lyapunov equation

$$D^{\mathrm{T}}P + PD = -(\gamma+2)I \quad (12)$$

with γ being a positive real.

In the case of relative degree $\rho \geq 2$, backstepping technique will be employed to find the final control u_c from the desirable value of ξ. By introducing $\tilde\xi_1 = \xi_1 - \hat\xi_1$ with $\hat\xi_1 = u_{c1}$, we have

$$\dot{\tilde\xi}_1 = -\lambda_1\xi_1 - \frac{\partial\hat\xi_1}{\partial y}\dot y + \xi_2 \quad (13)$$

Now we deal with the subsystem consisting of (9) and (13), which can be stabilized by the stabilizing function

$$\hat{\xi}_2 = -y + \lambda_1\hat{\xi}_1 + \frac{\partial\hat{\xi}_1}{\partial y}(\phi_y + \xi_1) - \tilde{\xi}_1(\frac{\partial\hat{\xi}_1}{\partial y})^2 \tag{14}$$

Continuing in this way, we can easily obtain a sequence of $\hat{\xi}_i$ $(3 \leq i \leq \rho)$

$$\begin{aligned}\hat{\xi}_i &= -\tilde{\xi}_{i-2} + \lambda_{i-1}\hat{\xi}_{i-1} + \frac{\partial\hat{\xi}_{i-1}}{\partial y}(\phi_y + \xi_1) \\ &\quad -\left(\frac{4}{3}\right)^{i-2}\tilde{\xi}_{i-1}(\frac{\partial\hat{\xi}_{i-1}}{\partial y})^2 + \sum_{j=1}^{i-2}\frac{\partial\hat{\xi}_{i-1}}{\partial\xi_j}\dot{\xi}_j\end{aligned} \tag{15}$$

and a sequence of $\tilde{\xi}_i$ $(3 \leq i \leq \rho-1)$

$$\dot{\tilde{\xi}}_i = -\lambda_i\tilde{\xi}_i - \lambda_i\xi_i - \sum_{j=1}^{i-1}\frac{\partial\hat{\xi}_i}{\partial\xi_j}\dot{\xi}_j - \frac{\partial\hat{\xi}_i}{\partial y}(\phi_y + \xi_1) + \xi_{i+1} \tag{16}$$

Note that the item $(4/3)^{i-2}$ is added for presentation convenience, which eventually results in the form of (18) if P is chosen as in (12). Finally we set the continuous-time control as $u_c = \hat{\xi}_\rho$.

C. Stability of the Continuous-Time System

We shall establish the asymptotical stability of the origin of system (1) forced by the continuous-time control u_c. To that end, we start with the Lyapunov function

$$V_c = z^{\mathrm{T}}Pz + \frac{1}{2}y^2 + \frac{1}{2}\sum_{i=1}^{\rho-1}\tilde{\xi}_i^2 \tag{17}$$

and from $(11) \sim (16)$, its time derivative satisfies

$$\dot{V}_c \leq -ky^2 - \gamma z^{\mathrm{T}}z - \lambda_0\sum_{i=1}^{\rho-1}\tilde{\xi}_i^2 \tag{18}$$

where $\lambda_0 := \min(\lambda_1, \cdots, \lambda_{\rho-1})$. This proves by the theorem 4.10 in [11] that $(y, z, \tilde{\xi}) = 0$ is an exponentially stable equilibrium point of the extended system of (9) and (4), which implies that $(y, z, \xi) = 0$ is an globally asymptotic equilibrium point, and so is the origin $(x, \xi) = 0$.

IV. MAIN RESULTS

The following lemma is needed for stability analysis in sampled-data case.

Lemma 1: Let $V : R^n \to R^+$ be a continuously differentiable, radially unbounded, positive definite function. Define $\mathcal{D} := \{\chi \in R^n | V(\chi) \leq r\}$ with $r > 0$. Suppose

$$\dot{V} \leq -\alpha V + \beta V_m, \quad \forall t \in (mT, (m+1)T], \tag{19}$$

hold for all $\chi(mT) \in \mathcal{D}$, where α, β are any given positive reals with $\alpha > \beta$, $T > 0$ the fixed sampling period and $V_m := V(\chi(mT))$. If $\chi(0) \in \mathcal{D}$, then the following holds:

$$\lim_{t\to\infty}\chi(t) = 0 \tag{20}$$

Proof. Since $\chi(0) \in \mathcal{D}$, then (19) holds for $t \in (0, T]$ with the following form

$$\dot{V} \leq -\alpha V + \beta V(\chi(0)).$$

Using comparison lemma [11] it is easy to get from the above that for $t \in (0, T]$,

$$\begin{aligned}V(\chi(t)) &\leq e^{-\alpha t}V_0 + \frac{1 - e^{-\alpha t}}{\alpha}\beta V_0 \\ &= q(t)V_0,\end{aligned} \tag{21}$$

where

$$q(t) := \left(e^{-\alpha t} + \frac{\beta}{\alpha}(1 - e^{-\alpha t})\right)$$

Since $\alpha > \beta > 0$, then $q(t) \in (0, 1)$, $\forall t \in (0, T]$. Then we have

$$V(\chi(t)) < V_0, \ \forall t \in (0, T]. \tag{22}$$

Particularly, letting $t = T$ in (21) leads to

$$V_1 \leq q(T)V_0 \tag{23}$$

which means that $V_1 \in \mathcal{D}$. Therefore (19) holds for $t \in (T, 2T]$. By induction, we have

$$V(\chi(t)) < V_m, \ \forall t \in (mT, (m+1)T]. \tag{24}$$

which states inter-sample behaviour of the sampled-data system concerned, and in particular

$$V_{m+1} \leq q(T)V_m \tag{25}$$

indicating that V decreases at two consecutive sampling points with a fixed ratio. From (25),

$$V_m \leq q(T)V_{m-1} \leq q^m(T)V_0 \tag{26}$$

which implies that $\lim_{m\to\infty}V_m = 0$. The conclusion then follows from (24), which completes the proof.

A. Stability of the Sampled-Data System

It is easy to see from (3) that $\xi(mT)$ is the exact, discrete-time model of the filter

$$\dot{\xi} = -\Lambda\xi + b_f u_d \tag{27}$$

due to the fact that u_d remains constant during each interval and the dynamics of ξ shown in (27) is linear. Then (3) and (27) are virtually equivalent at each sampling instant. This indicates that we can use (27) instead of (3) for stability analysis of the sampled-data system.

Let $\chi := [z; y; \tilde{\xi}]$ and we have the following result:

Theorem 1: For the extended system consisting of (9), (4) and the sampled-data controller u_{d1} shown in (2) and (3), and a given neighbourhood of the origin $B_r := \{\chi \in R^n | \ ||\chi|| \leq r\}$ with r any given positive real, there exists a constant $T_1 > 0$ such that, for all $0 < T < T_1$ and for all $\chi(0) \in B_r$, the system is asymptotically stable.

Proof. We still choose $V(\chi) = z^{\mathrm{T}}Pz + \frac{1}{2}y^2 + \frac{1}{2}\sum_{i=1}^{\rho-1}\tilde{\xi}_i^2$ as the Lyapunov function candidate for the sampled-data system. We start with some sets used throughout the proof. Define $c := \max_{\chi\in B_r}V(\chi)$ and the set $\Omega_c := \{x \in R^n | V(\chi) \leq c\}$. There exist two $\mathcal{K}$ functions ψ_1, ψ_2 such that $\psi_1(||\chi||) \leq V(\chi) \leq \psi_2(||\chi||)$. Let $l = \psi_1^{-1}(c) + \nu$ with ν a positive number, and define $B_l := \{\chi \in R^n | \ ||\chi|| \leq l\}$. Then $B_r \subset \Omega_c \subset B_l$. The constants L_1 and L_2 are Lipschitz constants of the functions ϕ_y, ϕ_z with respect to B_l.

These local Lipschitz conditions establishes that for the overall sampled-data system with $\chi(0) \in \Omega_c$, there exists a unique solution $\chi(t)$ over some interval $[0, t_1)$. Notice that t_1 might be finite. However, our analysis below will show that the solution can be extended one sampling interval after another, and thus exists for all $t \geq 0$ with the property that $\lim_{t\to\infty} \chi(t) = 0$. Particularly we intend to formulate the time derivative of the Lyapunov function V into the form shown in (19) (or (45)), which is shown below.

Consider the case when $t = 0$, $\chi(0) \in B_r \subset \Omega_c$. First, there exists a $T_1^* > 0$, and for all $T \in (0, T_1^*)$, the following holds:

$$\chi(t) \in B_l,\ \forall t \in [0,T], \chi(0) \in \Omega_c \tag{28}$$

The existence of T_1^* is ensured by continuous dependency of the solution $\chi(t)$ on the initial conditions.

Next we shall derive the bounds for $||\xi(t) - \xi(0)||$ and $|y(t) - y(0)|$ during $t \in [0, T]$ with $0 < T < T_1^*$. We have from (27)

$$\xi(t) = e^{\Lambda t}\xi(0) + \int_0^t e^{\Lambda(t-\tau)} b_f u_d d\tau \tag{29}$$

Since Λ is a Hurwitz matrix, there exist positive reals k_3, k_4, σ_2 such that $\|e^{\Lambda t}\| \leq k_3 e^{-\sigma_2 t}$ and $\|e^{\Lambda t} - I\| \leq k_4 e^{-\sigma_2 t}$, where I is the identity matrix. Then, using the Lipschitz property of u_c with respect to the set B_l and the fact that $u_d(0, 0) = 0$, it can be obtained from (29) that

$$\begin{aligned}\int_0^t ||\xi(\tau)||d\tau \quad &\leq \quad \frac{k_3\|\xi(0)\|}{\sigma_2}\big(1 - e^{-\sigma_2 t}\big) \\ &\quad + \frac{k_3 L_u}{\sigma_2}\big(|y(0)| + \|\xi(0)\|\big)t \end{aligned}\tag{30}$$

and

$$\begin{aligned}\|\xi(t) - \xi(0)\| \quad &\leq \quad k_4\|\xi(0)\|(1 - e^{-\sigma_2 t}) \\ &\quad + |u_d\big(y(0), \xi(0)\big)| \int_0^t k_3 e^{-\sigma_2(t-\tau)} d\tau \\ &\leq \quad \delta_1(T)|y(0)| + \delta_2(T)||\xi(0)|| \end{aligned}\tag{31}$$

where $\delta_1(T) = \sigma_2^{-1} k_3 L_u\big(1 - e^{-\sigma_2 t}\big)$, $\delta_2(T) = (k_4 + \sigma_2^{-1} k_3 L_u)\big(1 - e^{-\sigma_2 t}\big)$ and L_u is a Lipschitz constant of u_c.

Now calculate the estimate of $|y(t) - y(0)|$ during the interval $[0, T]$, provided that $\chi(0) \in B_r \subset \Omega_c$ and $T \in (0, T_1^*)$. From (9),

$$\begin{aligned} y(t) \quad &= \quad y(0) + \int_0^t z_1(\tau)d\tau + \int_0^t \xi_1(\tau)d\tau \\ &\quad + \int_0^t \Big(\phi_y(y) - \phi_y\big(y(0)\big)\Big)d\tau \\ &\quad + \int_0^t \phi_y\big(y(0)\big)d\tau \end{aligned}\tag{32}$$

It can then be shown that

$$\begin{aligned} |y(t) - y(0)| \quad &\leq \quad \underbrace{\int_0^t \|z(\tau)\|d\tau}_{\Delta_1} + \underbrace{\int_0^t \|\xi(\tau)\|d\tau}_{\Delta_2} \\ &\quad + \int_0^t L_1|y(\tau) - y(0)|d\tau \\ &\quad + \int_0^t L_1|y(0)|d\tau \end{aligned}\tag{33}$$

where Δ_2 is already shown in (30) and Δ_1 is computed as follows. From the first equation of system (9), we obtain

$$z(t) = e^{Dt}z(0) + \int_0^t e^{Dt}\phi_z\big(y(\tau)\big)d\tau \tag{34}$$

Since D is a Hurwitz matrix, there exist positive reals k_2, σ_1 such that $\|e^{Dt}\| \leq k_2 e^{-\sigma_1 t}$. Thus, from (34)

$$\begin{aligned} \|z(t)\| \quad &\leq \quad k_2 e^{-\sigma_1 t}\|z(0)\| \\ &\quad + \int_0^t k_2 e^{-\sigma_1(t-\tau)}\|\phi_z\big(y(\tau)\big) - \phi_z\big(y(0)\big)\|d\tau \\ &\quad + \int_0^t k_2 e^{-\sigma_1(t-\tau)}\|\phi_z\big(y(0)\big)\|d\tau \\ &\leq \quad k_2 e^{-\sigma_1 t}\|z(0)\| \\ &\quad + L_2\int_0^t k_2 e^{-\sigma_1(t-\tau)}|y(\tau) - y(0)|d\tau \\ &\quad + L_2\int_0^t k_2 e^{-\sigma_1(t-\tau)}|y(0)|d\tau \end{aligned}\tag{35}$$

Then the following inequality holds

$$\begin{aligned} \Delta_1 \quad &\leq \quad \frac{k_2\|z(0)\|}{\sigma_1}\big(1 - e^{-\sigma_1 t}\big) + \frac{k_2 L_2}{\sigma_1}|y(0)|t \\ &\quad + \frac{k_2 L_2}{\sigma_1}\int_0^t |y(\tau) - y(0)|d\tau \end{aligned}\tag{36}$$

With (36), (30) and (33), it follows that

$$\begin{aligned} |y(t) - y(0)| \quad &\leq \quad A_1(1 - e^{-\sigma_1 t}) + A_2(1 - e^{-\sigma_2 t}) \\ &\quad + B_2 t + H\int_0^t |y(\tau) - y(0)|d\tau \end{aligned}\tag{37}$$

where $A_1 = \sigma_1^{-1} k_2\|z(0)\|$, $H = \sigma_1^{-1} k_2 L_2 + L_1$, $A_2 = \sigma_2^{-1} k_3||\xi(0)||$ and $B_2 = L_1|y(0)| + \sigma_1^{-1} k_2 L_2|y(0)| + \sigma_2^{-1} k_3 L_u|y(0)| + \sigma_2^{-1} k_3 L_u||\xi(0)||$. Define $A_3 := A_1 + A_2$ and $\sigma_0 := \max(\sigma_1, \sigma_2)$, and we have

$$\begin{aligned} |y(t) - y(0)| \quad &\leq \quad A_3(1 - e^{-\sigma_0 t}) \\ &\quad + B_2 t + H\int_0^t |y(\tau) - y(0)|d\tau \end{aligned}\tag{38}$$

Applying Gronwall-Bellman inequality [11] to (38) produces

$$\begin{aligned} |y(t) - y(0)| \quad &\leq \quad A_3\big(1 - e^{-\sigma_0 t}\big) + \frac{B_2}{H}\big(e^{Ht} - 1\big) \\ &\quad + A_3\big(\sigma_0 e^{Ht} + He^{-\sigma_0 t} \\ &\quad - (H + \sigma_0)\big)(H + \sigma_0)^{-1} \end{aligned}\tag{39}$$

Setting $t = T$ in the right side of (39) leads to

$$\begin{aligned} |y(t) - y(0)| \quad &\leq \quad \delta_3(T)|y(0)| + \delta_4(T)||z(0)|| \\ &\quad + \delta_5(T)||\xi(0)|| \end{aligned}\tag{40}$$

where

$$\begin{aligned}\delta_3(T) &= H^{-1}(L_1+\sigma_1^{-1}k_2L_2+\sigma_2k_3^{-1}L_u)\big(e^{HT}-1\big)\\ \delta_4(T) &= \sigma_1^{-1}k_2\big(\sigma_0e^{HT}+He^{-\sigma_0T}-(H+\sigma_0)\big)\cdot\\ &\quad (H+\sigma_0)^{-1}+\sigma_1^{-1}k_2\big(1-e^{-\sigma_0T}\big)\\ \delta_5(T) &= \sigma_2^{-1}k_3(1-e^{-\sigma_0T})+\sigma_2^{-1}k_3L_u(e^{HT}-1)\\ &\quad +\sigma_2^{-1}k_3L_u\big(\sigma_0e^{HT}+He^{-\sigma_0T}-(H+\sigma_0)\big)\cdot\\ &\quad (H+\sigma_0)^{-1}\end{aligned}$$

Note that $||\xi(0)||$ appears in (31) and (40) while for the analysis using Lyapunov function to carry on, $||\tilde{\xi}(0)||$ is needed. Therefore it is necessary to find out an expression of $||\xi(0)||$ that exclusively involves $||\tilde{\xi}(0)||$, $|y(0)|$, which is shown below.

Notice that due to the special structure of the filter (4) and the backstepping technique, each stabilising function has the property that $\hat{\xi}_1 = \hat{\xi}_1(y)$, $\hat{\xi}_1(0) = 0$, and $\hat{\xi}_i = \hat{\xi}_i(y, \xi_1, \cdots, \xi_{i-1})$ and $\hat{\xi}_i(0, \cdots, 0) = 0$, $i = 2, \cdots, \rho-1$. From $\tilde{\xi}_1 = \xi_1 - \hat{\xi}_1$ we have

$$|\xi_1(0)| \leq |\tilde{\xi}_1(0)| + |\hat{\xi}_1(0)| \leq |\tilde{\xi}_1(0)| + \mathcal{L}_1|y(0)| \quad (41)$$

where with a bit abuse of notation, $\mathcal{L}_1$ is the Lipschitz constant of $\hat{\xi}_1$ with respect to the set B_l. It follows from (41) that $|\xi_1(0)|$ is bounded given bounded $|\tilde{\xi}_1(0)|$ and $|y(0)|$, which in return implies that so is $|\xi_2(0)|$, as we have

$$|\xi_2(0)| \leq |\tilde{\xi}_2(0)| + |\hat{\xi}_2(0)| \leq |\tilde{\xi}_2(0)| + \mathcal{L}_2|y(0)| + \mathcal{L}_2|\xi_1(0)|$$

Continuing this way we can prove that all $|\xi_i(0)|$ will be bounded given $[z(0), y(0), \tilde{\xi}(0)]^T \in B_l$. Thus, a constant $\mathcal{L}_0$ can be found such that the following holds

$$||\xi(0)|| \leq \mathcal{L}_0(||\tilde{\xi}(0)|| + |y(0)|). \quad (42)$$

which implies that if $[z(0), y(0), \tilde{\xi}(0)]^T \in B_l$, then $[z(0), y(0), \xi(0)]^T$ will be confined in a bounded set, denoted by B_l'.

Next we shall study the behaviour of the sampled-data system during the interval $t \in (0, T]$ with $\chi(0) \in B_r \subset \Omega_c$. When $t \in (0, T]$, the time derivative of the Lyapunov function $V(\chi)$ satisfies

$$\begin{aligned}\dot{V} &= -(\gamma+2)\|z\|^2 + 2z^{\mathrm{T}}P\phi_z + y(z_1+\phi_y+\xi_1)\\ &\quad +\sum_{i=1}^{\rho-2}\tilde{\xi}_i\dot{\tilde{\xi}}_i + \tilde{\xi}_{\rho-1}(u_d-u_c)\big) +\\ &\quad \tilde{\xi}_{\rho-1}\big(-\lambda_{\rho-1}\xi_{\rho-1}+u_c-\frac{\partial\hat{\xi}_{\rho-1}}{\partial y}(z_1+\phi_y+\xi_1)\\ &\leq -ky^2-\gamma||z||^2-\lambda_0\sum_{i=1}^{\rho}\tilde{\xi}_i^2+||\tilde{\xi}|||u_d-u_c| \quad (43)\end{aligned}$$

In addition, we have

$$\begin{aligned}||\xi|||u_d-u_c| &\leq L_u||\xi||(|y-y(0)|+||\xi-\xi(0)||)\\ &\leq L_u||\xi||(\delta_1(T)+\delta_3(T))|y(0)|\\ &\quad +L_u||\xi||\delta_4(T)||z(0)||\\ &\quad +L_u||\xi||(\delta_2(T)+\delta_5(T))||\xi(0)||\\ &\leq \varepsilon_1(T)|y(0)|^2+\varepsilon_2(T)||z(0)||^2\\ &\quad +\varepsilon_3(T)||\xi(0)||^2+\varepsilon_4(T)||\tilde{\xi}||^2 \quad (44)\end{aligned}$$

where $\varepsilon_1(T) = \frac{L_u}{2}\big(\delta_1(T)+\delta_3(T)\big)$, $\varepsilon_2 = \frac{L_u}{2}\delta_4(T)$, $\varepsilon_3 = \frac{L_u}{2}(\delta_2(T)+\delta_5(T))$, $\varepsilon_4 = \frac{L_u}{2}\sum_{i=1}^{5}\delta_i(T)$, L_u is a Lipschitz constant of u_c with respect to the set B_l'.

From (42) – (44) we then have

$$\begin{aligned}\dot{V} &\leq -ky^2-\gamma||z||^2-(\lambda_0-\varepsilon_4)\tilde{\xi}^2\\ &\quad +(\varepsilon_1(T)+2\mathcal{L}_0^2\varepsilon_3(T))|y(0)|^2+\varepsilon_2(T)||z(0)||^2\\ &\quad +2\mathcal{L}_0^2\varepsilon_3(T)||\tilde{\xi}(0)||^2\\ &= -\alpha_1(T)V+\beta_1(T)V(z(0),y(0),\tilde{\xi}(0)) \quad (45)\end{aligned}$$

where

$$\alpha_1(T) = \min\left\{2k, \frac{\gamma}{\lambda_{max}(P)}, 2(\lambda_0-\varepsilon_4(T))\right\}$$

$$\beta_1(T) = \max\left\{2(\varepsilon_1(T)+2\mathcal{L}_0^2\varepsilon_3(T)), \frac{\varepsilon_2(T)}{\lambda_{min}(P)}, 4\mathcal{L}_0^2\varepsilon_3(T)\right\}$$

Note from (31), (40) and (44) that each ε_i $(1 \leq i \leq 4)$ is a continuous function of T with $\varepsilon_i(0) = 0$. Define $e_1(T) := \alpha_1(T) - \beta_1(T)$ and we have $e_1(0) > 0$ as $\alpha_1(0) > 0$ while $\beta_1(0) = 0$. It can also be established from (45) that $e_1(T)$ is a decreasingly continuous function of T, which asserts by the continuity of $e_1(T)$ the existence of T_2^* so that for $0 < T < T_2^*$, $e_1(T) > 0$, that is, $0 < \beta_1(T) < \alpha_1(T)$.

Lastly, set $T_1 = \min(T_1^*, T_2^*)$, and from lemma 1 it is known that $V_1 \leq c$, ie, $\chi(T) \in \Omega_c$, and subsequently, all the above analysis can be repeated for every interval $[mT, mT+T]$. Applying lemma 1 completes the proof.

Remark 1: Theorem 1 only declares the existence of a certain upper limit of sampling period, but states no information regarding the effects of control parameters and initial sets of the system on the upper limit. Those effects are still subject to further investigation.

Remark 2: If $\rho = 1$, then the controller reduces to a static output feedback controller. If $\rho \geq 2$, the dynamic controller uses a particular linear filter, which brings in convenience in control implementation, in contrast to other observer-based approaches, for instance, [6].

V. SIMULATION

Consider the following system with relative degree $\rho = 2$

$$\begin{aligned}\dot{x}_1 &= x_2+y^2\\ \dot{x}_2 &= u\\ y &= x_1 \quad (46)\end{aligned}$$

The filter $\dot{\xi} = -\lambda\xi + u$ is introduced so that the filtered transformation $\eta_1 = x_1$ and $\eta_2 = x_2 - \xi$, and the state transformation $z = \eta_2 - \lambda\eta_1$ can render the system into the following form

$$\begin{aligned}\dot{z} &= -\lambda z+y^2-\lambda^2y-\lambda y^2\\ \dot{y} &= z+(\lambda+y)y+\xi\end{aligned}$$

Finally, the stabilizing function $\hat{\xi} = -ky-(\lambda+y)y-\frac{1}{2}\lambda^2(y+\lambda)^2y^2$ and the control u_c can be obtained using (14). For simulation, we choose $\lambda = 3$, $k = 4$.

Simulations are carried out by Simulink using zero-order hold blocks for the case where the initial values are $x_1(0) =$

1 and $x_2(0) = 200$. Results shown in Fig.1 and Fig.2 indicate that the sampled-data system is asymptotically stable when $T = 0.0001s$, which is confirmed by a closer look at the convergence of V shown in Fig.3. Further simulations show that the overall system is unstable if $T = 0.0005s$. In summary, the example illustrates that for a range of sampling period T, the sampled-data controller designed in the former sections can asymptotically stabilise the sampled-data system.

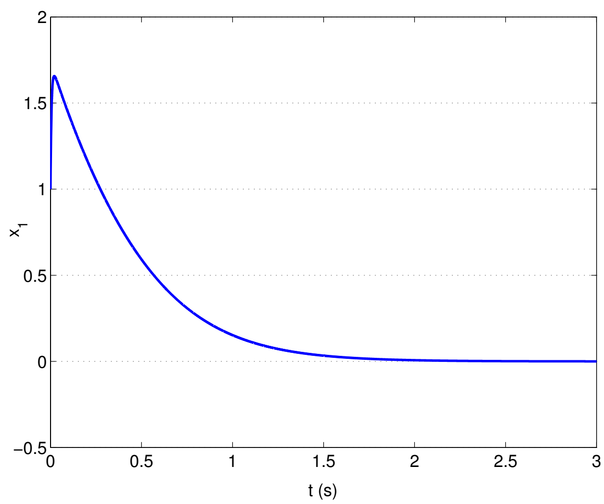

Fig. 1. The time response of x_1 for $T = 0.0001$

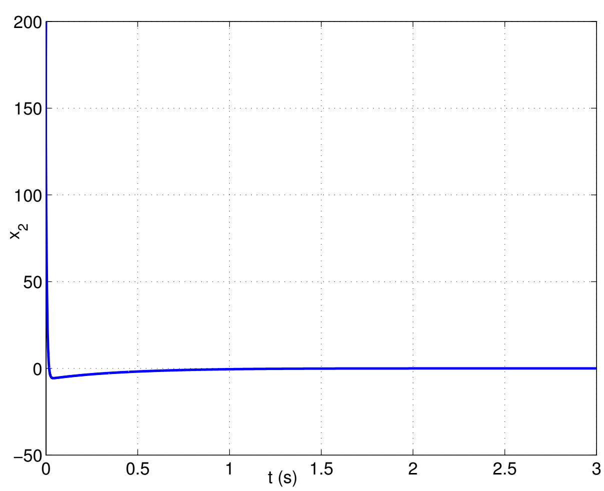

Fig. 2. The time response of x_2 for $T = 0.0001$

VI. Conclusion

We have presented an analysis for sampled-data output feedback control of one class of nonlinear systems in output feedback form under fast-sampling principle. It has been shown that the sampled-data version of continuous-time controllers will still semi-globally asymptotically stabilise the system, provided that the sampling period T is small enough.

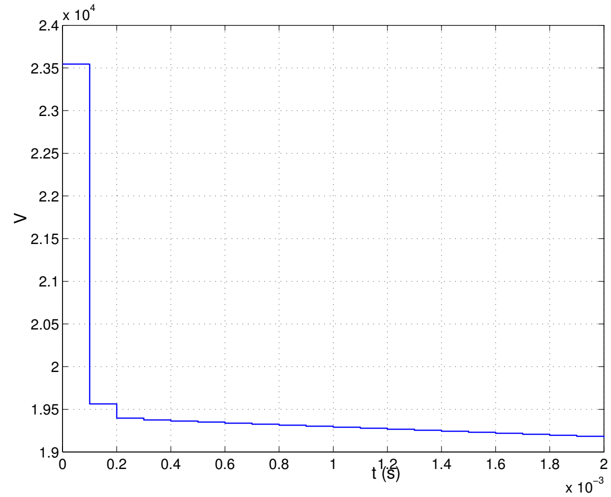

Fig. 3. The convergence of V for $T = 0.0001$

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