Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Coordination on Lie groups

Alain Sarlette, Silvère Bonnabel and Rodolphe Sepulchre

***Abstract*— This paper studies the coordinated motion of a group of agents evolving on a Lie group. Left- or right-invariance with respect to the absolute position on the group lead to two different characterizations of relative positions and two associated definitions of coordination (fixed relative positions). Conditions for each type of coordination are derived in the associated Lie algebra. This allows to formulate the coordination problem on Lie groups as consensus in a vector space. Total coordination occurs when both types of coordination hold simultaneously. The discussion in this paper provides a common geometric framework for previously published coordination control laws on $SO(3)$, $SE(2)$ and $SE(3)$. The theory is illustrated on the group of planar rigid motion $SE(2)$.**

I. INTRODUCTION

Recently, many efforts have been devoted to the design and analysis of control laws that coordinate swarms of identical autonomous agents — see e.g. oscillator synchronization [31], [30], flocking mechanisms [7], [2], vehicle formations [5], [4], [22], [9], [10], [11], spacecraft formations [17], [34], [13], [12], [16], [25], mechanical system networks [29], [6], [20] and mobile sensor networks [26], [27], [28], [14], [32] to name a few. Despite the success of researchers in studying specific cases, a general systematic method is still missing for the design of coordinating control laws. One frequent difficulty is nonlinearity arising from the fact that the configuration space of the agents is not (isomorphic to) a vector space. Examples include coordination of phase variables (S^1), planar vehicles ($SE(2)$), or rigid bodies in space (satellites or underwater vehicles, $SE(3)$). In all these examples, however, the configuration space is a Lie group.

The main contribution of this paper is to give geometric definitions of coordination on a finite dimensional Lie group, and to characterize the coordinated trajectories. This geometric background can help or even replace arguments based on physical intuition in complex situations. Coordination on $SE(3)$ (3-dimensional rigid body motion) for instance is already quite difficult to solve based on physical intuition [26]. In contrast, the present geometric framework readily applies to general Lie groups. The characterization of coordinated trajectories is an important step towards the design of stabilizing control laws, that will be addressed in a forthcoming paper.

Coordination is defined as follows: let g_k and g_j be the positions of agents k and j. *Left-invariant* (resp. right-invariant) coordination is achieved when $g_k^{-1}g_j$ (resp. $g_j g_k^{-1}$) are constant over time (for any k, j). *Total coordination* is when both left and right-invariant coordination are achieved.

The conditions for left- or right-invariant coordination can be expressed in the associated Lie algebra and coordination on Lie groups boils down to *consensus* (equal values) in a vector space. Total coordination corresponds to two consensus conditions. Then the relative positions of the totally coordinated agents are restricted to a subgroup.

The paper is organized as follows. Sections II and III provide the theoretical framework for left-/right-invariant coordination and for total coordination respectively. Section IV applies these theoretical concepts to $SE(2)$, providing a clear geometric interpretation to results of [27], [10].

Department of Electrical Engineering and Computer Science (Montefiore Institute, B28), University of Liège, 4000 Liège, Belgium. alain.sarlette@ulg.ac.be, bonnabel@montefiore.ulg.ac.be, r.sepulchre@ulg.ac.be

II. LEFT- AND RIGHT-INVARIANT COORDINATION

This paper uses the notations of Arnold [1]. Let G be a Lie group, and $\mathfrak{g}$ its Lie algebra. Let $L_h : g \mapsto hg$ denote left multipication, and $R_h : g \mapsto gh$ right multiplication on G. Let $L_{h*} : TG_g \to TG_{hg}$ and $R_{h*} : TG_g \to TG_{gh}$ for all $g \in G$ be the induced maps on tangent spaces. Let $Ad_g : \mathfrak{g} \to \mathfrak{g}$, $Ad_g = R_{g^{-1}*}L_{g*}$. Consider a set of N "agents" evolving on G, with $g_k \in G$ denoting the position of agent k. Let $\mathfrak{g} \ni \xi_k^l(\tau) = L_{g^{-1}(\tau)*}(\frac{d}{dt}g_k(t)|_{t=\tau})$ resp. $\mathfrak{g} \ni \xi_k^r(\tau) = R_{g^{-1}(\tau)*}(\frac{d}{dt}g_k(t)|_{t=\tau})$ be its left- and right-invariant velocities. An important equality is

$$\xi_k^r = Ad_{g_k}\xi_k^l \ .$$

Definition 1: *The* left-invariant relative position *of agent j with respect to agent k is $\lambda_{jk} = g_k^{-1}g_j$. The* right-invariant relative position *of agent j with respect to agent k is $\rho_{jk} = g_j\, g_k^{-1}$.*

Indeed, λ_{jk} (resp. ρ_{jk}) is invariant under left (resp. right) multiplication since $(hg_k)^{-1}(hg_j) = g_k^{-1}g_j \ \forall h \in G$.

Note that the left-/right-invariant relative positions are the *joint invariants* associated to the left-/right-invariant action of G on $G \times G ... \times G$ (N copies). In [3], two copies of the state space (true and estimated space) are used to build symmetry-preserving observers.

In this paper, *coordination* of a set of agents is achieved when the *relative positions among all agents remain constant*. This corresponds to the intuition of a "formation", in which the swarm of points moves as if it was a single extended body. Some authors refer to coordination as the situation where all agents are located *at the same point on G*; the present paper prefers the denomination *synchronization* or *consensus* for this situation.

978-1-4244-3124-3/08/$25.00 ©2008 IEEE

The two different definitions of relative position lead to two different types of coordination.

Definition 2: Left-invariant *(resp. right-invariant)* coordination *is achieved when the* λ_{jk} *(resp.* ρ_{jk}*) are constant in time for all pairs of agents* j, k *in the swarm.*

Proposition 1: *Left-invariant coordination corresponds to equal right-invariant velocities* $\xi_j^r = \xi_k^r \ \forall j,k$. *Right-invariant coordination corresponds to equal left-invariant velocities* $\xi_j^l = \xi_k^l \ \forall j,k$.

Proof: For λ_{jk}, $\frac{d}{dt}(g_k^{-1}g_j) = L_{g_k^{-1}*}\frac{d}{dt}g_j + R_{g_j*}\frac{d}{dt}g_k^{-1}$. But if $\frac{d}{dt}g_k = L_{g_k*}\xi_k^l$, then $\frac{d}{dt}g_k^{-1} = -L_{g_k^{-1}*}Ad_{g_k}\xi_k^l$. Thus

$$\begin{aligned}\frac{d}{dt}(g_k^{-1}g_j) &= L_{g_k^{-1}g_j*}\xi_j^l - L_{g_k^{-1}*}R_{g_j*}Ad_{g_k}\xi_k^l \\ &= L_{g_k^{-1}g_j*}Ad_{g_j}^{-1}(Ad_{g_j}\xi_j^l - Ad_{g_k}\xi_k^l)\,.\end{aligned}$$

Since $L_{g_k^{-1}g_j*}$ and $Ad_{g_j^{-1}}$ are invertible operators, left-invariant coordination $\frac{d}{dt}(\lambda_{jk}) = 0$ is equivalent to $Ad_{g_j}\,\xi_j^l = Ad_{g_k}\,\xi_k^l$ or equivalently $\xi_j^r = \xi_k^r$. The proof for right-invariant coordination is strictly analogous. △

Proposition 1 shows that coordination on the Lie group G is equivalent to consensus on the vector space $\mathfrak{g}$. This formulation is of interest since consensus in vector spaces is a well-studied subject [19], [18], [33], [24], [2], [23], [21].

III. TOTAL COORDINATION

Definition 3: Total coordination *is achieved when left-invariant and right-invariant coordination are achieved simultaneously:* $\xi_j^l = \xi_k^l$ *and* $\xi_j^r = \xi_k^r \ \forall j,k$.

Proposition 2: *Total coordination is equivalent to*

$$\forall k = 1...N, \quad \xi_k^l = \xi^l \in \bigcap_{i,j} \ker(Ad_{\lambda_{ij}} - \mathrm{Id})\ .$$

Proof: Right-invariant coordination requires $\xi_k^l = \xi_j^l \ \forall j,k$; denote the common value of the ξ_k^l by ξ^l. Left-invariant coordination requires $Ad_{g_k}\,\xi_k^l = Ad_{g_j}\,\xi_j^l \Leftrightarrow \xi_k^l = Ad_{\lambda_{jk}}\,\xi_j^l$ $\forall j,k$, which with $\xi_k^l = \xi^l \ \forall k$ yields $\xi^l = Ad_{\lambda_{jk}}\,\xi^l$. △

Proposition 2 shows that total coordination puts no constraints on the relative positions when the group is abelian, since $Ad_{\lambda_{ij}} = \mathrm{Id}$ in this case. In contrast, on a general Lie group, the admissible relative positions belong to a subgroup characterized as follows.

Proposition 3: *Let* $CM_\xi := \{g \in G : Ad_g\,\xi = \xi\}$.
a) *For every* $\xi \in \mathfrak{g}$, CM_ξ *is a subgroup of* G.
b) *The Lie algebra of* CM_ξ *is the kernel of* $ad_\xi = [\xi,\,]$, *i.e.* $\mathfrak{cm}_\xi = \{\eta \in \mathfrak{g} : [\xi,\eta] = 0\}$. *In particular,* CM_ξ *has dimension at least 1 since* $\xi \in \mathfrak{cm}_\xi$.

Proof: a) $Ad_e\,\xi = \xi \ \forall \xi$ since Ad_e is the identity operator. $Ad_g\,\xi = \xi$ implies $Ad_{g^{-1}}\,\xi = \xi$ by simple inversion of the relation. Moreover, if $Ad_{g_1}\,\xi = \xi$ and $Ad_{g_2}\,\xi = \xi$, then $Ad_{g_1g_2}\,\xi = Ad_{g_1}\,Ad_{g_2}\,\xi = Ad_{g_1}\,\xi = \xi$. Thus CM_ξ satisfies all group axioms and must be a subgroup of G.

b) The Lie algebra $\mathfrak{cm}_\xi$ of CM_ξ is its tangent space at e. Consider $g(t) \in CM(\xi)$ with $g(\tau) = e$ and $\frac{d}{dt}g(t)|_\tau = \eta$. Thus $\mathfrak{cm}_\xi$ is characterized by all such η. For constant ξ, $Ad_g(t)\xi = \xi$ implies $\frac{d}{dt}(Ad_g(t))\xi = 0$. But a basic Lie group property is $\frac{d}{dt}(Ad_g(t))|_\tau = ad_\eta$. Therefore $[\eta,\,\xi] = 0$ is necessary. It is also sufficient since, for any η such that $[\eta,\,\xi] = 0$, the curve $g(t) = \exp(\eta t)$ generated as the group exponential of η belongs to CM_ξ. △

CM_ξ and $\mathfrak{cm}_\xi$ are called the isotropy subgroup and isotropy Lie algebra of ξ; these are classical mathematical objects in group theory [15]. From Propositions 2 and 3, one method to obtain a totally coordinated motion on a Lie group is to

1) choose ξ^l in the vector space $\mathfrak{g}$ and
2) position the agents such that $\lambda_{jk} \in CM_{\xi^l} \ \forall j,k$.

Then indeed, $\xi_k^l = \xi^l \ \forall k$ ensures right-invariant coordination, and $\lambda_{jk} \in CM_{\xi^l}$ implies $Ad_{\lambda_{jk}}\xi_k^l = \xi_k^l = \xi_j^l$ such that $\xi_k^r = Ad_{g_k}\xi_k^l = Ad_{g_j}\xi_j^l = \xi_j^r$ and left-invariant coordination is achieved as well.

Remark: It may be interesting to examine coordinated trajectories with varying velocity ξ^l.

a) When ξ^l is varying during a totally coordinated motion, Propositions 2 and 3 must be satisfied at each time instant; since the λ_{jk} are fixed, this implies $\lambda_{jk} \in \bigcap_t CM_{\xi^l(t)}$.
b) When the λ_{jk} are varying, the swarm is not totally coordinated (by definition). However, it is still possible to maintain right-invariant coordination. Then the $\lambda_{jk}(t)$ can evolve in the *conjugation class* of the initial λ_{jk}, i.e. $\lambda_{jk}(t) = h(t)\lambda_{jk}(0)h^{-1}(t) \ \forall j,k$, for some $h \in G$.

Similar observations can be made with right-invariant velocities and relative positions.

IV. COORDINATED MOTION ON $SE(2)$

As an illustration of the theory, coordinated trajectories on the group $SE(2)$ are characterized. Left-invariant coordination for the particular example of $SE(2)$ was already formulated in Lie group notation in [10]. The properties of $SE(2)$ are well-known and can be found even in control textbooks like [8]. The *special Euclidean group in the plane* $G = SE(2)$ describes all planar rigid body motions (translations and rotations). An element of $SE(2)$ can be written $g = (r,\theta) \in \mathbb{R}^2 \times S^1$ where r denotes position and θ orientation. Then

- $g_1g_2 = (r_1 + Q_{\theta_1}r_2, \theta_1 + \theta_2)$ where Q_θ is the rotation of angle θ ;
- $e = (0,0)$ and $g^{-1} = (-Q_{-\theta}r, -\theta)$;
- $\xi = (v,\omega) \in \mathfrak{se}(2) = \mathbb{R}^2 \times \mathbb{R}$;
- $L_{g*}(v,\omega) = (Q_\theta v, \omega)$ and $R_{g*}(v,\omega) = (v + \omega Q_{\pi/2}r, \omega)$;
- $Ad_g\,(v,\omega) = (Q_\theta v - \omega Q_{\pi/2}r, \omega)$;
- $[(v_1,\omega_1),(v_2,\omega_2)] = (\omega_1 Q_{\pi/2}v_2 - \omega_2 Q_{\pi/2}v_1, 0)$;

- v^l is the body's linear velocity expressed in body frame, $\omega^l = \omega^r$ is its rotation rate.

A. Coordination on $SE(2)$

The theory is applied step by step.

- **Relative positions**: $g_k^{-1} g_j = (Q_{-\theta_k}(r_j - r_k), \theta_j - \theta_k)$ (left-invariant) and $g_j g_k^{-1} = (r_j - Q_{\theta_j - \theta_k} r_k, \theta_j - \theta_k)$ (right-invariant).
- **Left-invariant coordination**: $\xi_k^l = Ad_{g_k^{-1} g_j}\, \xi_j^l$ writes $(v_k^l, \omega_k^l) = (Q_{\theta_j - \theta_k} v_j^l - \omega_j^l Q_{\pi/2} Q_{-\theta_k}(r_j - r_k), \omega_j^l)$ or equivalently $\xi_k^r = \xi_j^r \Leftrightarrow (v_k^r, \omega_k^r) = (v_j^r, \omega_j^r)$.
  The agents move like a single rigid body: relative orientations and relative positions on the plane do not change (Figures 1.a and 2.a).
- **Right-invariant coordination**: $\xi_k^r = Ad_{g_k g_j^{-1}}\, \xi_j^r$ writes $(v_k^r, \omega_k^r) = (Q_{\theta_k - \theta_j} v_j^r - \omega_j^r Q_{\pi/2}(r_k - Q_{\theta_k - \theta_j} r_j), \omega_j^r)$ or equivalently $\xi_k^l = \xi_j^l \Leftrightarrow (v_k^l, \omega_k^l) = (v_j^l, \omega_j^l)$.
  The agents move with the same velocity expressed in body frame (Figure 3).
- **Total coordination**: the swarm moves like a single rigid body *and* each agent has the same velocity expressed in body frame. Propositions 2 and 3 lead to the following characterization of totally coordinated motions.
  $[\xi^l, \eta] = 0$ (Proposition 3.b) $\Leftrightarrow \omega^l v_\eta = \omega_\eta v^l$ and $Ad_g\, \xi^l = \xi^l$ (Proposition 3) $\Leftrightarrow (Q_\theta - \mathrm{Id}) v^l = \omega^l Q_{\pi/2} r$.
  This implies three different cases:
  (o) $\omega^l = v^l = 0 \Rightarrow \mathfrak{cm}_{\xi^l} = \mathfrak{se}(2)$ and $CM_{\xi^l} = SE(2)$;
  (i) $\omega^l = 0$, $v^l \neq 0 \Rightarrow \mathfrak{cm}_{\xi^l} = \{(v, 0) : v \in \mathbb{R}^2\}$ and $CM_{\xi^l} = \{(r, 0) : r \in \mathbb{R}^2\}$;
  (ii) $\omega^l \neq 0$, any $v^l \Rightarrow \mathfrak{cm}_{\xi^l} = \{(\frac{\omega}{\omega^l} v^l, \omega) : \omega \in \mathbb{R}\}$. Define $\mathcal{C}$, the circle of radius $\frac{\|v^l\|_2}{|\omega^l|}$ containing the origin, tangent to v^l at the origin and such that v^l and ω^l imply rotation in the same direction. Then solving $Ad_g \xi = \xi$ for g and making a few calculations shows that $CM_{\xi^l} = \{(r, \theta) : r \in \mathcal{C}$ and $Q_\theta v^l$ tangent to $\mathcal{C}$ at $r\}$. This is consistent with an intuitive analysis of possibilities for circular motion with unitary linear velocity and fixed relative positions and orientations in the plane.

  The dimension of $\mathfrak{cm}_{\xi^l}$ ($\Leftrightarrow$ of CM_{ξ^l}) is (o) 3, (i) 2 or (ii) 1. In case (o), the configuration is arbitrary but at rest. In case (i), the agents have the same orientation and move on parallel straight lines (Figure 1.b). In case (ii), they move on the same circle and have the same orientation with respect to their local radius (Figure 2.b).

B. Link with previous work on $SE(2)$

Left-invariant coordination is studied in [10], [27], [28] under the constraint of *steering control*: $v_k^l = \mathbf{e}_1\ \forall k$ ($\mathbf{e}_1 \in \mathbb{R}^2$ denoting any fixed vector) and only ω_k^l can be controlled. Since $\omega_k^l = \omega_k^r$, *left-invariant coordination on $SE(2)$ with steering control is equivalent to total coordination*. Steering control imposes $\xi^l = (\mathbf{e}_1, \omega)$, allowing relative positions in CM_{ξ^l} as for (i) or (ii) above. The authors of [10] show indeed that these parallel and circular motions are the only possible trajectories for left-invariant coordination with steering control (this illustrates that left-invariant coordination under velocity constraints can impose restrictions on achievable relative positions). Note that right-invariant coordination in this setting would simply require the agents to agree on a common rotation rate ω; this can be solved with a classical linear consensus algorithm.

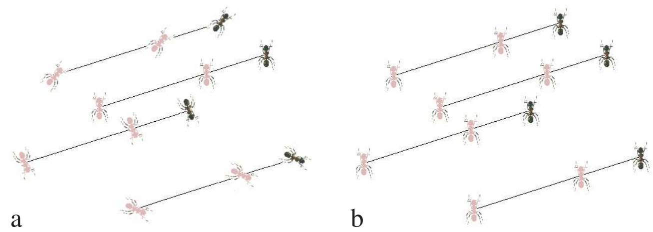

Fig. 1. Coordinated swarms with constant velocity, $\omega_k = 0$. a: left-invariant coordination. b: total coordination. (light color: intermediate positions and orientations in time).

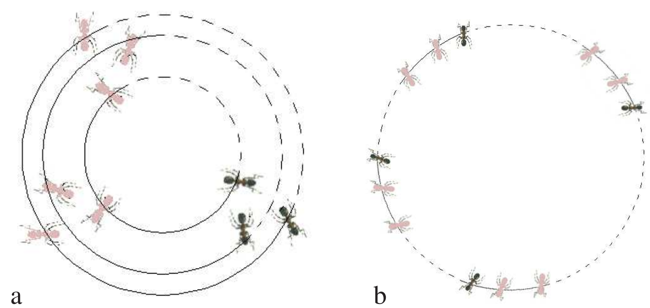

Fig. 2. Coordinated swarms with constant velocity, $\omega_k \neq 0$. a: left-invariant coordination. b: total coordination. (light color: intermediate positions and orientations in time).

Fig. 3. Right-invariant coordinated swarm with varying velocity. (light color: intermediate positions and orientations in time).

The authors in [27] propose the following control laws in order to asymptotically stabilize coordinated motions under steering control on $SE(2)$; in order to focus on collective issues, the individual dynamics are reduced to first order integrators, instead of considering the full mechanical rigid body dynamics (Newton and Euler equations).

(i) Parallel linear motion: being in CM_{ξ^l} with $\omega^l = 0$ requires equal orientations. Building on the literature about Kuramoto oscillators, [27] take a Lyapunov function $V = \frac{1}{4N} \sum_{j,k} \|e^{i\theta_k} - e^{i\theta_j}\|^2$. The gradient control law

$$\omega_k^l = \sum_j \sin(\theta_j - \theta_k) \quad , \; k = 1...N. \qquad (1)$$

drives all θ_k towards a common value on S^1. Then $\omega_k^l = \omega^l = 0$ and coordination is achieved.
The proposed Lyapunov function can be seen as measuring the disagreement between vectors $Q_{\theta_k}\mathbf{e}_1$; this corresponds to the disagreement between right-invariant velocities ξ_k^r *assuming that* $\omega_k^l = \omega^l = 0$ *and* $v_k^l = v^l = \mathbf{e}_1$.

(ii) Circular motion: Denote by $s_k = Q_{\theta_k} Q_{-\pi/2}\, \mathbf{e}_1 - \omega^l r_k$ the center of curvature of k's trajectory multiplied by ω^l. Then synchronizing the s_k ensures that all agents are on the same circle, as required for being in CM_{ξ^l} with $\omega^l \neq 0$. In [27], this motivates the Lyapunov function $V = \frac{1}{4N} \sum_{j,k} \|s_k - s_j\|^2$. This leads to control law

$$\omega_k = \omega + \sum_j \, (s_j - s_k) \cdot (Q_{\theta_k}\, \mathbf{e}_1) \qquad (2)$$

where $\cdot$ denotes the scalar product in $\mathbb{R}^2$. For all initial conditions, (2) drives the agents to the same circle of diameter $2/\omega$, with θ_k such that velocity is tangent to the circle; moreover ω_k is asymptotically equal to ω so coordination is achieved.
The present framework shows that $s_k = Q_{-\pi/2} v_k^r$ if $\omega_k^l = \omega^l$: the proposed Lyapunov function again measures the disagreement between right-invariant velocities ξ_k^r *assuming that* $\xi_k^l = \xi^l$.

Further variants of these controls have been developed; an adaptation when each agent only uses information from a restricted set of other agents is proposed in [28]. A future paper will discuss the design of control laws like (1),(2) on general Lie groups.

V. CONCLUSION

The present paper studies coordination on Lie groups: a set of agents has to move such that their relative positions remain constant. A geometric framework is presented and its implications are analyzed. In particular, a direct link between coordination and conditions on velocities is highlighted before examining how this influences achievable situations of coordination. This is further illustrated on $SE(2)$.

This paper formalizes general geometric principles behind the work of [10], [27], [28] on $SE(2)$ and [11], [26] on $SE(3)$; on $SO(3)$, left- and right-invariant velocities correspond to angular velocity expressed in body frame and in inertial frame respectively. Design methods for coordination control laws on general Lie groups will be addressed in a future paper.

VI. ACKNOWLEDGMENTS

This paper presents research results of the Belgian Network DYSCO (Dynamical Systems, Control, and Optimization), funded by the Interuniversity Attraction Poles Programme, initiated by the Belgian State, Science Policy Office. The scientific responsibility rests with its authors. The first author is supported as an FNRS fellow (Belgian Fund for Scientific Research).

REFERENCES

[1] V. Arnold. *Mathematical methods of classical mechanics*. Springer, 1989.
[2] V.D. Blondel, J.M. Hendrickx, A. Olshevsky, and J.N. Tsitsiklis. Convergence in multiagent coordination, consensus and flocking. *Proc. 44th IEEE Conf. Decision and Control*, 2005.
[3] S. Bonnabel, Ph. Martin, and P. Rouchon. Symmetry-preserving observers. *to be published in IEEE Trans. Automatic Control*, 2008.
[4] J.P. Desai, J.P. Ostrowski, and V. Kumar. Modeling and control of formations of nonholonomic mobile robots. *IEEE Trans. Robotics and Automation*, 17(6):905–908, 2001.
[5] J. Fax and R.M. Murray. Information flow and cooperative control of vehicle formations. *IEEE Trans. Automatic Control*, 49(9):1465–1476, 2004.
[6] H. Hanssmann, N.E. Leonard, and T.R. Smith. Symmetry and reduction for coordinated rigid bodies. *Eur. J. Control*, 12(2):176–194, 2006.
[7] A. Jadbabaie, J. Lin, and A.S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Trans. Automatic Control*, 48(6):988–1001, 2003.
[8] V. Jurdjevic. *Geometric control theory*. Cambridge University Press, 1997.
[9] E. Justh and P. Krishnaprasad. A simple control law for UAV formation flying. Technical report, TR 2002-38, ISR, University of Maryland, 2002.
[10] E.W. Justh and P.S. Krishnaprasad. Equilibria and steering laws for planar formations. *Systems and Control Letters*, 52:25–38, 2004.
[11] E.W. Justh and P.S. Krishnaprasad. Natural frames and interacting particles in three dimensions. *Proc. 44th IEEE Conf. Decision and Control*, pages 2841–2846, 2005.
[12] T.R. Krogstad and J.T. Gravdahl. Coordinated attitude control of satellites in formation. In *Group Coordination and Cooperative Control*, volume 336 of *Lect. N. Control and Information Sci.*, chapter 9, pages 153–170. Springer, 2006.
[13] J.R. Lawton and R.W. Beard. Synchronized multiple spacecraft rotations. *Automatica*, 38:1359–1364, 2002.
[14] N.E. Leonard, D. Paley, F. Lekien, R. Sepulchre, D. Frantantoni, and R. Davis. Collective motion, sensor networks and ocean sampling. *Proc. IEEE*, 95(1):48–74, January 2007.
[15] J.E. Marsden and T.S. Ratiu. *Introduction to mechanics and symmetry*. Springer, 1994.
[16] C.R. McInnes. Distributed control for on-orbit assembly. *Adv. Astronautical Sci.*, 90:2079–2092, 1996.
[17] M. Mesbahi and F.Y. Hadaegh. Formation flying control of multiple spacecraft via graphs, matrix inequalities, and switching. *J. Guidance, Control and Dynamics*, 24(2):369–377, 2001.
[18] L. Moreau. Stability of continuous-time distributed consensus algorithms. *Proc. 43rd IEEE Conf. Decision and Control*, pages 3998–4003, 2004.
[19] L. Moreau. Stability of multi-agent systems with time-dependent communication links. *IEEE Trans. Automatic Control*, 50(2):169–182, 2005.
[20] S. Nair and N.E. Leonard (advisor). Stabilization and synchronization of networked mechanical systems. *PhD Thesis, Princeton University*, 2006.
[21] R. Olfati-Saber, J.A. Fax, and R.M. Murray. Consensus and cooperation in networked multi-agent systems. *Proc. IEEE*, 95(1):215–233, 2007.
[22] R. Olfati-Saber and R.M. Murray. Graph rigidity and distributed formation stabilization of multi-vehicle systems. *Proc. 41st IEEE Conf. Decision and Control*, 2002.

[23] R. Olfati-Saber and R.M. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Trans. Automatic Control*, 49(9):1520–1533, 2004.

[24] A. Olshevsky and J.N. Tsitsiklis. Convergence rates in distributed consensus and averaging. *Proc. 45th IEEE Conf. Decision and Control*, pages 3387–3392, 2006.

[25] Wei Ren. Distributed attitude consensus among multiple networked spacecraft. *Proc. American Control Conference*, 2006.

[26] L. Scardovi, R. Sepulchre, and N.E. Leonard. Stabilization of collective motion in three dimensions: A consensus approach. *Proc. 46th IEEE Conf. Decision and Control*, pages 2931–2936, 2007.

[27] R. Sepulchre, D. Paley, and N.E. Leonard. Stabilization of planar collective motion with all-to-all communication. *IEEE Trans. Automatic Control*, 52(5):811–824, 2007.

[28] R. Sepulchre, D. Paley, and N.E. Leonard. Stabilization of planar collective motion with limited communication. *IEEE Trans. Automatic Control*, 53(3):706–719, 2008.

[29] T.R. Smith, H. Hanssmann, and N.E. Leonard. Orientation control of multiple underwater vehicles with symmetry-breaking potentials. *Proc. 40th IEEE Conf. Decision and Control*, pages 4598–4603, 2001.

[30] S.H. Strogatz. From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled nonlinear oscillators. *Physica D*, 143:1–20, 2000.

[31] S.H. Strogatz. *Sync: The emerging science of spontaneous order*. Hyperion, 2003.

[32] D.T. Swain, N.E. Leonard, I.D. Couzin, A. Kao, and R. Sepulchre. Alternating spatial patterns for coordinated group motion. *Proc. 46th IEEE Conf. Decision and Control*, pages 2925–2930, 2007.

[33] J.N. Tsitsiklis and M. Athans (advisor). Problems in decentralized decision making and computation. *PhD Thesis, MIT*, 1984.

[34] M.C. VanDyke and C.D. Hall. Decentralized coordinated attitude control of a formation of spacecraft. *J. Guidance, Control and Dynamics*, 29(5):1101–1109, 2006.