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# Gain-Scheduled $\mathscr{H}_\infty$-Control of Discrete-Time Polytopic Time-Varying Systems

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*Abstract*— **This paper presents synthesis procedures for the design of both robust and gain-scheduled $\mathscr{H}_\infty$ static output feedback controllers for discrete-time linear systems with time-varying parameters. The parameters are assumed to vary inside a polytope and have known bounds on their rate of variation. The geometric properties of the polytopic domain are exploited to derive a finite set of linear matrix inequalities that consider the bounds on the rate of variation of the parameters. A numerical example illustrates the proposed approach.**

## I. INTRODUCTION

For more than a decade, analysis and control design for linear parameter-varying (LPV) models have received a lot of attention from the control community. This stems from the fact that LPV models are useful to describe the dynamics of linear systems affected by time-varying parameters as well as to represent nonlinear systems in terms of a family of linear models [1]. In the LPV control framework, the scheduling parameters that govern the variation of the dynamics of the system are usually unknown, but supposed to be measured or estimated in real-time [2]. There is a continuing effort towards the design of LPV controllers, scheduled as a function of the varying parameters, to achieve higher performance while still guaranteeing stability for all possible parameter variations (see, for instance, [1, 3–8]).

One practical way (e.g. [9–11]) to compute a gain-scheduled controller for a given LPV system consists of the following steps. First determine a family of linear time-invariant (LTI) models by selecting different operating conditions of the system and then design a local controller for each one of the LTI models. Next, based on the values of the parameters (measured or estimated on-line), schedule the local controllers using some interpolation method. The final step consists of checking the closed-loop stability and performance using extensive simulation. Although the system performance can be improved by means of increasing the number of local models (at the expense of increasing the computational burden) this approach may be unreliable, since the closed-loop stability and performance are only verified through simulations.

To overcome these difficulties, several analysis and synthesis results for LPV systems have been proposed based on different types of Lyapunov functions that are able to guarantee closed-loop loop stability and performance. However, many of these approaches (e.g. [12, 13] among others) use the notion of quadratic stability where the Lyapunov matrix is constant. This generally leads to conservative results for practical applications since it allows arbitrarily fast variation of the scheduling parameters.

To alleviate some of the conservatism associated with the quadratic stability-based approaches, many works using parameter-dependent Lyapunov functions have been published (e.g. [14–21]). For instance, an affine Lyapunov matrix is used in [20] to derive linear matrix inequality (LMI) conditions for the synthesis of gain-scheduled $\mathscr{H}_\infty$ state feedback controllers for linear discrete-time systems with time-varying parameters belonging to a polytope. Although less conservative than the approaches based on quadratic stability, these LMI conditions still allow arbitrarily fast parameter variations which are not realistic for practical applications. In [14], a piecewise-constant Lyapunov function is used to derive synthesis conditions for gain-scheduled $\mathscr{H}_\infty$ static state feedback controllers for discrete-time multi-affine LPV systems with bounded parameter variation. In [21], on the other hand, LMI conditions are presented for the synthesis of stabilizing gain scheduled full state feedback controllers for polytopic time-varying systems considering bounds on the rate of variation of the scheduling parameters.

The aim of this work is to provide LMI conditions for the synthesis of gain-scheduled $\mathscr{H}_\infty$ static output feedback controllers for discrete-time linear systems with time-varying parameters belonging to a polytope with a prescribed bound on the rate of variation. In this way, this paper extends the results in [21] by allowing static output feedback and considering $\mathscr{H}_\infty$ performance. The approach is restricted to the case where the measurement equation is unaffected by the control inputs, the exogenous disturbance inputs and the scheduling parameter.

The paper is organized as follows. Section III provides preliminary material concerning the modeling of the polytopic domain. Section IV provides a finite set of LMIs to calculate an upper bound on the $\mathscr{H}_\infty$ performance, while Section V extends this result to the synthesis of an $\mathscr{H}_\infty$ static output feedback controller. A numerical example is presented in Section VI that shows the benefits of the proposed approach. The conclusions are presented in Section VII.

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## II. NOTATION

The $\ell_2^n$ space of square-summable sequences on the set of nonnegative integers $\mathbb{Z}_+$ is given by

$$\ell_2^n \triangleq \left\{ f : \mathbb{Z}_+ \to \mathbb{R}^n \;\Big|\; \sum_{k=0}^{\infty} f[k]^T f[k] < \infty \right\}.$$

The 2-norm is defined as $\|x[k]\|_2^2 = \sum_{k=0}^{\infty} x[k]^T x[k]$. The identity matrix of size $r \times r$ is denoted as $I_r$. The notation $\mathbf{0}_{n,m}$ indicates an $n \times m$ matrix of zeros. The convex hull of a set $X$ is denoted by $co\{X\}$.

## III. PRELIMINARIES

Consider the discrete-time polytopic linear time-varying system

$$\begin{aligned} x[k+1] &= A(\alpha[k])\; x[k] + B_w(\alpha[k])\; w[k] + B_u(\alpha[k])\; u[k] \\ z[k] &= C_z(\alpha[k])\; x[k] + D_w(\alpha[k])\; w[k] + D_u(\alpha[k])\; u[k] \end{aligned} \tag{1}$$

where $x[k] \in \mathbb{R}^n$ is the state, $w[k] \in \mathbb{R}^r$ the exogenous input, $u[k] \in \mathbb{R}^m$ the control input and $z[k] \in \mathbb{R}^p$ the system output. The system matrices $A(\alpha[k]) \in \mathbb{R}^{n\times n}$, $B_w(\alpha[k]) \in \mathbb{R}^{n\times r}$, $B_u(\alpha[k]) \in \mathbb{R}^{n\times m}$, $C_z(\alpha[k]) \in \mathbb{R}^{p\times n}$, $D_w(\alpha[k]) \in \mathbb{R}^{p\times r}$ and $D_u(\alpha[k]) \in \mathbb{R}^{p\times m}$ belong to the polytope

$$\begin{aligned} \mathscr{D} = \{\, & (A, B_w, B_u, C_z, D_w, D_u)(\alpha[k]) : \\ & (A, B_w, B_u, C_z, D_w, D_u)(\alpha[k]) \\ & = \sum_{i=1}^{N} \alpha_i[k] \,(A, B_w, B_u, C_z, D_w, D_u)_i \,,\; \alpha[k] \in \Lambda_N \}, \end{aligned}$$

where, for all $k \geq 0$, the vector of time-varying parameters $\alpha[k]$ belongs to the unit simplex

$$\Lambda_N = \left\{ \xi \in \mathbb{R}^N : \sum_{i=1}^{N} \xi_i = 1, \xi_i \geq 0, i = 1, \ldots, N \right\}. \tag{2}$$

For all $k \geq 0$, the rate of variation of the parameters

$$\Delta\alpha_i[k] = \alpha_i[k+1] - \alpha_i[k], \quad i = 1, \ldots, N \tag{3}$$

is assumed to be limited by an *a priori* known bound $b$ such that

$$-b\alpha_i[k] \leq \Delta\alpha_i[k] \leq b(1 - \alpha_i[k]),\; i = 1, \ldots, N \tag{4}$$

with $b \in \mathbb{R}, \quad b \in [0,1]$. As recently discussed in [21], less conservative results can be obtained by explicitly taking into account that $\Delta\alpha_i[k]$ satisfies (4). Following [21], the space where the vector $\Delta\alpha[k]$ can assume values can be modeled by the set

$$\begin{aligned} \Gamma_b = \Big\{ & \delta \in \mathbb{R}^N : \delta \in co\left\{h^1, \ldots, h^N\right\}, \\ & \sum_{i=1}^{N} h_i^j = 0, \quad j = 1, \ldots, N, \quad h \in \mathbb{R}^N \Big\}. \end{aligned}$$

The first step to construct the vectors $h^j$ is to observe that due to (2) and (3), one has

$$\sum_{i=1}^{N} \Delta\alpha_i[k] = 0.$$

Solving this equality under the extreme values of (4), one has the following vectors (solutions) $h^j$ (depending on both $b$ and $\alpha[k]$)

$$\left[h^1 \cdots h^N\right] = b \begin{bmatrix} 1-\alpha_1[k] & -\alpha_1[k] & \ldots & -\alpha_1[k] \\ -\alpha_2[k] & 1-\alpha_2[k] & \ldots & -\alpha_2[k] \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_N[k] & -\alpha_N[k] & \ldots & 1-\alpha_N[k] \end{bmatrix}$$

By taking the convex combination of the $N$ columns $h^j$ using $\beta[k] \in \Lambda_N$, the following expression is obtained

$$\Delta\alpha_j[k] = b(\beta_j[k] - \alpha_j[k]). \tag{5}$$

The next section presents LMI conditions, based on affine parameter-dependent Lyapunov functions, that provide, for all $\alpha \in \Lambda_N$ and $\Delta\alpha \in \Gamma_b$, an upper bound $\eta$ on the $\mathscr{H}_\infty$ performance of system (1).

## IV. GUARANTEED $\mathscr{H}_\infty$ PERFORMANCE

The aim of this section is to provide through a finite set of LMIs an $\mathscr{H}_\infty$ guaranteed cost for system (1) in open-loop, such that for any input $w[k] \in \ell_2^r$, the system output $z[k] \in \ell_2^p$ satisfies

$$\|z[k]\|_2 < \eta \;\|w[k]\|_2, \qquad 0 < \eta < \infty,$$

for any possible variation of the parameter $\alpha[k] \in \Lambda_N$ with prescribed bound $b$ on its rate of variation. This main analysis result is presented in Theorem 2.

Consider the discrete-time linear system (1) in open-loop. Using the well known bounded-real lemma [22], the $\mathscr{H}_\infty$ performance for system (1) can be characterized using a parameter-dependent LMI as described in the next lemma.

**Lemma 1** *The system* (1) *in open-loop has an $\mathscr{H}_\infty$ performance bounded by $\eta$ if, for all $\alpha[k] \in \Lambda_N$, there exists a symmetric positive definite matrix $P(\alpha[k])$ such that*

$$\begin{bmatrix} P(\alpha[k+1]) & \star & \star & \star \\ P(\alpha[k])A(\alpha[k])^T & P(\alpha[k]) & \star & \star \\ B_w(\alpha[k])^T & 0 & \eta I & \star \\ 0 & C_z(\alpha[k])P(\alpha[k]) & D_w(\alpha[k]) & \eta I \end{bmatrix} > 0. \tag{6}$$

The above characterization of the $\mathscr{H}_\infty$ performance can be extended by introducing additional instrumental matrix variables, in a similar way as done in [23] for the time-invariant polytopic case. This property is presented in the next theorem.

**Theorem 1** *The system* (1) *in open-loop has an $\mathscr{H}_\infty$ performance bounded by $\eta$ if, for all $\alpha[k] \in \Lambda_N$, there exist a matrix $G(\alpha[k])$ and a symmetric positive definite matrix $P(\alpha[k])$ such that*

$$\begin{bmatrix} P(\alpha[k+1]) & \star & \star & \star \\ G(\alpha[k])^T A(\alpha[k])^T & G(\alpha[k]) + G(\alpha[k])^T - P(\alpha[k]) & \star & \star \\ B_w(\alpha[k])^T & 0 & \eta I & \star \\ 0 & C_z(\alpha[k])G(\alpha[k]) & D_w(\alpha[k]) & \eta I \end{bmatrix} = \Theta(\alpha) > 0. \tag{7}$$

*Proof:*

Actually, (6) and (7) are equivalent. To prove that (6) implies (7), choose $G(\alpha[k]) = G(\alpha[k])^T = P(\alpha[k])$.

Conversely, assume that (7) is feasible. Hence $G(\alpha[k]) + G(\alpha[k])^T - P(\alpha[k]) > 0$, which implies that $G(\alpha[k])$ is nonsingular for all $\alpha[k] \in \Lambda_N$. Since $P(\alpha[k])$ is positive definite for all $\alpha[k] \in \Lambda_N$ the inequality

$$(P(\alpha[k]) - G(\alpha[k]))^T P(\alpha[k])^{-1} (P(\alpha[k]) - G(\alpha[k])) > 0$$

holds. Therefore

$$G(\alpha[k])P(\alpha[k])^{-1}G(\alpha[k])^T > G(\alpha[k]) + G(\alpha[k])^T - P(\alpha[k])$$

and the LMI (7) becomes

$$\begin{bmatrix} P(\alpha[k+1]) & \star & \star & \star \\ G(\alpha[k])^T A(\alpha[k])^T & \Theta_{22} & \star & \star \\ B_w(\alpha[k])^T & 0 & \eta I & \star \\ 0 & C_z(\alpha[k])G(\alpha[k]) & D_w(\alpha[k]) & \eta I \end{bmatrix} > 0. \quad (8)$$

with $\Theta_{22} = G(\alpha[k])^T P(\alpha[k])^{-1} G(\alpha[k])$.

Since $G(\alpha[k])$ is nonsingular, the LMI (8) can be multiplied by

$$T := \operatorname{diag}\left\{I, G(\alpha[k])^{-1}P(\alpha[k]), I, I\right\}$$

on the right and by $T^T$ on the left to recover the LMI (6). This concludes the proof. ∎

For analysis, the conditions in Lemma 1 and Theorem 1 are equivalent. However, for synthesis, the introduction of $G(\alpha[k])$ yields less conservative results, as shown in [24].

It is worth to emphasize that the conditions of Theorem 1, which consist in evaluating the parameter dependent LMI for all $\alpha[k]$ in the unit simplex $\Lambda_N$, leads to an infinite dimensional problem. However, by imposing on the Lyapunov matrix $P(\alpha[k])$ the following affine parameter-dependent structure

$$P(\alpha[k]) = \sum_{i=1}^{N} \alpha_i[k] P_i, \quad \alpha[k] \in \Lambda_N, \quad (9)$$

a finite set of LMIs in terms of the vertices of the polytope $\mathscr{D}$ can be obtained, as shown in the next theorem.

**Theorem 2** *The system* (1) *in open-loop has an $\mathscr{H}_\infty$ performance bounded by $\eta$ if there exist matrices $G_i \in \mathbb{R}^{n\times n}$ and symmetric matrices $P_i \in \mathbb{R}^{n\times n}$ such that*

$$\begin{bmatrix} (1-b)P_i + bP_\ell & \star & \star & \star \\ G_i^T A_i^T & G_i + G_i^T - P_i & \star & \star \\ B_{w,i}^T & 0 & \eta I & \star \\ 0 & C_{z,i}G_i & D_{w,i} & \eta I \end{bmatrix} = \Theta_{i\ell} > 0, \quad (10)$$

*holds for $i = 1,\dots,N$ and $\ell = 1,\dots,N$ and*

$$\begin{bmatrix} \Xi_{11} & \star & \star & \star \\ G_j^T A_i^T + G_i^T A_j^T & \Xi_{22} & \star & \star \\ B_{w,i}^T + B_{w,j}^T & 0 & 2\eta I & \star \\ 0 & C_{z,i}G_j + C_{z,j}G_i & D_{w,i} + D_{w,j} & 2\eta I \end{bmatrix} = \Theta_{ij\ell} > 0, \quad (11)$$

*with*

$$\Xi_{11} = (1-b)P_i + (1-b)P_j + 2bP_\ell$$
$$\Xi_{22} = G_i + G_i^T + G_j + G_j^T - P_i - P_j$$

*holds for $\ell = 1,\dots,N$, $i = 1,\dots,N-1$ and $j = i+1,\dots,N$.*

*Proof:*

First note that using (5) and (9), it can be shown that

$$P(\alpha[k+1]) = \sum_{i=1}^{N} ((1-b)\alpha_i[k] + b\beta_i[k])P_i.$$

Now, multiply (10) by $\alpha_i^2 \beta_\ell$ and sum for $i = 1,\dots,N$ and $\ell = 1,\dots,N$. Likewise, multiply (11) by $\alpha_i\alpha_j\beta_\ell$ and sum for $\ell = 1,\dots,N$, $i = 1,\dots,N-1$ and $j = i+1,\dots,N$. Adding the resulting two expressions yields

$$\Theta(\alpha) = \sum_{i=1}^{N}\sum_{\ell=1}^{N} \alpha_i^2 \beta_\ell \Theta_{i\ell} + \sum_{\ell=1}^{N}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \alpha_i\alpha_j\beta_\ell\Theta_{ij\ell}.$$

Therefore, the conditions of Theorem 1 are satisfied since the set of LMIs (10)–(11) ensures that $\Theta(\alpha) > 0$. ∎

## V. GAIN-SCHEDULED STATIC OUTPUT FEEDBACK

In this section, the analysis result presented in Theorem 2 is extended to provide a finite set of LMI conditions for the synthesis of a gain-scheduled $\mathscr{H}_\infty$ static output feedback controller for system (1). The goal is to provide a parameter-dependent control law

$$u[k] = K(\alpha[k])y[k], \qquad \text{with} \quad K(\alpha[k]) \in \mathbb{R}^{m\times q},$$

such that the closed-loop system

$$\begin{aligned} x[k+1] &= A_{cl}(\alpha[k])x[k] + B_w(\alpha[k])w[k] \\ z[k] &= C_{cl}(\alpha[k])x[k] + D_w(\alpha[k])w[k], \end{aligned} \quad (12)$$

with

$$A_{cl}(\alpha[k]) = A(\alpha[k]) + B_u(\alpha[k])K(\alpha[k])C_y$$
$$C_{cl}(\alpha[k]) = C_z(\alpha[k]) + D_u(\alpha[k])K(\alpha[k])C_y$$

is asymptotically stable with a bound $\eta$ on the closed-loop $\mathscr{H}_\infty$ performance, guaranteed for all possible variation of the parameter $\alpha[k] \in \Lambda_N$.

It is assumed that the first $q$ states of the system can be measured in real-time for feedback without corruption by the exogenous input $w[k]$, that is, $y[k] = C_y\, x[k]$, where $y[k] \in \mathbb{R}^q$ is the measured output. The matrix $C_y$ is independent of the time-varying parameters and has the structure

$$C_y = \begin{bmatrix} I_q & \mathbf{0}_{q,n-q} \end{bmatrix}, \quad (13)$$

If this is not the case, one can use a similarity transformation as in [25], whenever the output matrix is not affected by the time-varying parameter.

A solution to this $\mathscr{H}_\infty$ static output feedback design problem, in terms of a finite set of LMIs, is provided by the next theorem.

$$\begin{bmatrix} (1-b)P_i+(1-b)P_j+2bP_\ell & \star & \star & \star \\ G_j^TA_i^T+G_i^TA_j^T+Z_j^TB_{u,i}^T+Z_i^TB_{u,j}^T & G_i+G_i^T+G_j+G_j^T-P_i-P_j & \star & \star \\ B_{w,i}^T+B_{w,j}^T & 0 & 2\eta I & \star \\ 0 & C_{z,i}G_j+C_{z,j}G_i+D_{u,i}Z_j+D_{u,j}Z_i & D_{w,i}+D_{w,j} & 2\eta I \end{bmatrix} = \Psi_{ij\ell} > 0 \quad (15)$$

$$\begin{bmatrix} P(\alpha_{[k+1]}) & \star & \star & \star \\ G(\alpha_{[k]})^TA(\alpha_{[k]})^T+Z(\alpha_{[k]})^TB_u(\alpha_{[k]})^T & G(\alpha_{[k]})+G(\alpha_{[k]})^T-P(\alpha_{[k]}) & \star & \star \\ B_w(\alpha_{[k]})^T & 0 & \eta I & \star \\ 0 & C_z(\alpha_{[k]})G(\alpha_{[k]})+D_u(\alpha_{[k]})Z(\alpha_{[k]}) & D_w(\alpha_{[k]}) & \eta I \end{bmatrix} = \Psi(\alpha) > 0 \quad (18)$$

**Theorem 3** *If there exist matrices $G_{i,1} \in \mathbb{R}^{q\times q}$, $G_{i,2} \in \mathbb{R}^{n-q\times q}$, $G_{i,3} \in \mathbb{R}^{n-q\times n-q}$, $Z_{i,1} \in \mathbb{R}^{m\times q}$, and symmetric matrices $P_i \in \mathbb{R}^{n\times n}$ such that the following LMIs*

$$\begin{bmatrix} (1-b)P_i+bP_\ell & \star & \star & \star \\ G_i^TA_i^T+Z_i^TB_{u,i}^T & G_i+G_i^T-P_i & \star & \star \\ B_{w,i}^T & 0 & \eta I & \star \\ 0 & C_{z,i}G_i+D_{u,i}Z_i & D_{w,i} & \eta I \end{bmatrix} = \Psi_{i\ell} > 0, \quad (14)$$

*hold for $i=1,\dots,N$ and $\ell=1,\dots,N$ and the LMIs $\Psi_{ij\ell}$ (15) on the top of the page hold for $\ell=1,\dots,N$, $i=1,\dots,N-1$ and $j=i+1,\dots,N$, with*

$$G_i = \begin{bmatrix} G_{i,1} & 0 \\ G_{i,2} & G_{i,3} \end{bmatrix} \quad \text{and} \quad Z_i = \begin{bmatrix} Z_{i,1} & 0 \end{bmatrix} \quad (16)$$

*then the parameter-dependent static output feedback gain*

$$K(\alpha_{[k]}) = \hat{Z}(\alpha_{[k]})\hat{G}(\alpha_{[k]})^{-1}, \quad (17)$$

*with*

$$\hat{Z}(\alpha_{[k]}) = \sum_{i=1}^{N} \alpha_{i[k]}Z_{i,1}, \quad \text{and} \quad \hat{G}(\alpha_{[k]}) = \sum_{i=1}^{N} \alpha_{i[k]}G_{i,1},$$

*stabilizes the system (1) with a guaranteed $\mathscr{H}_\infty$ performance bounded by $\eta$ for all $\alpha \in \Lambda_N$ and $\Delta\alpha \in \Gamma_b$.*

*Proof:*
Multiply (14) by $\alpha_i^2\beta_\ell$ and sum for $i=1,\dots,N$ and $\ell=1,\dots,N$. Multiply (15) by $\alpha_i\alpha_j\beta_\ell$ and sum for $\ell=1,\dots,N$, $i=1,\dots,N-1$ and $j=i+1,\dots,N$. Adding the resulting two expressions yields

$$\Psi(\alpha) = \sum_{i=1}^{N}\sum_{\ell=1}^{N} \alpha_i^2\beta_\ell\Psi_{i\ell} + \sum_{\ell=1}^{N}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \alpha_i\alpha_j\beta_\ell\Psi_{ij\ell},$$

which is the LMI (18) displayed on the top of the page. Now, using (16) and (17) and considering the specific structure (13) for $C_y$, the LMI $\Psi(\alpha)$ can be written as

$$\begin{bmatrix} P(\alpha_{[k+1]}) & \star & \star & \star \\ G(\alpha_{[k]})^TA_{cl}(\alpha_{[k]})^T & \Psi_{22} & \star & \star \\ B_w(\alpha_{[k]})^T & 0 & \eta I & \star \\ 0 & C_{cl}(\alpha_{[k]})G(\alpha_{[k]}) & D_w(\alpha_{[k]}) & \eta I \end{bmatrix} > 0$$

with $\Psi_{22} = G(\alpha_{[k]}) + G(\alpha_{[k]})^T - P(\alpha_{[k]})$. Therefore, as a result of Theorem 1, feasibility of the LMIs (14) and (15) ensures that the closed-loop system (12) is asymptotically stable with an upper bound $\eta$ on its $\mathscr{H}_\infty$ performance.

■

Some remarks are in order now. First, if all states are available for feedback, that is, $y_{[k]} = x_{[k]}$, the LMIs in Theorem 3 provide conditions for the existence of a gain-scheduled static state feedback control law $u_{[k]} = K(\alpha_{[k]})x_{[k]}$. Moreover, if the bound $b$ on the rate of variation is $b=1$, the set of LMIs from Theorem 3 reduces, upon a similarity transformation, to the LMIs given in Theorem 2 from [20]. Second, the particular case of a robust $\mathscr{H}_\infty$ static output feedback controller

$$u_{[k]} = K\ y_{[k]}$$

is easily derived from Theorem 3, as shown in the next corollary.

**Corollary 1** *If there exist matrices $G_1 \in \mathbb{R}^{q\times q}$, $G_2 \in \mathbb{R}^{n-q\times q}$, $G_3 \in \mathbb{R}^{n-q\times n-q}$, $Z_1 \in \mathbb{R}^{m\times q}$, and symmetric matrices $P_i \in \mathbb{R}^{n\times n}$ such that (14) holds for $i=1,\dots,N$ and $\ell=1,\dots,N$ with*

$$G_i = G = \begin{bmatrix} G_{11} & 0 \\ G_{21} & G_{22} \end{bmatrix} \quad \text{and} \quad Z_i = Z = \begin{bmatrix} Z_{11} & 0 \end{bmatrix}$$

*the control gain $K = Z_{11}G_{11}^{-1}$ provides a robust static output feedback controller that stabilizes (1) with guaranteed $\mathscr{H}_\infty$ performance bounded by $\eta$.*

If all states are available for feedback and the bound $b$ on the rate of variation is $b=0$, the above Corollary 1 reduces to the robust $\mathscr{H}_\infty$ state-feedback case given by Theorem 10 from [24].

There are two main advantages of introducing the slack variables. First, if the slack variables would not be introduced, the same structure imposed on $G_i$ in (16) should have to be imposed on the Lyapunov variables $P_i$, which is obviously more restrictive. Second, in Corollary 1, a robust static output feedback controller is found by enforcing $G_i = G$ and $Z_i = Z$, for all $i$. The variables $P_i$, however, are free. To obtain a robust static output feedback controller without introduction of the slack variables, all Lyapunov variables should have to be the same, that is, all $P_i$ should have to be $P_i = P$.

## VI. NUMERICAL EXAMPLE

Consider the polytopic time-varying system (1) for $n=3$ and $N=2$ with the following system matrices:

$$\begin{bmatrix} A_1 \,\vdots\, A_2 \end{bmatrix} = \mu \left[\begin{array}{ccc:ccc} 1 & 0 & -2 & 0 & 0 & -1 \\ 2 & -1 & 1 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 & -2 & -1 \end{array}\right], B_{w,1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$B_{w,2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, B_{u,i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, C_{z,i} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^T, D_{u,i} = D_{w,i} = 0,$$

$i = 1,2$, with $\mu \in \mathbb{R}$ a given nonnegative scalar. These system matrices are borrowed from [21]. The aim in this example is to determine the maximum bound $b_{\max}$ on the rate of parameter variation as a function of the scalar $\mu$ such that the system can be stabilized by an $\mathscr{H}_\infty$ static output feedback controller. Both gain-scheduled and robust output feedback controllers are designed for three different cases of the measurement equation $y[k] = C_y x[k]$. Case 1: only the first state is assumed to be measured. Case 2: the first two states are available. Case 3: all states are available for feedback.

Figure 1 shows $b_{\max}$ as a function of $\mu$. The labeling is as follows. Case 1: dashed lines, Case 2: dotted lines and Case 3: dash-dotted lines. For each one of these three cases, the robust controllers are denoted by $R_1$, $R_2$ and $R_3$ (using thin lines). Likewise, the gain-scheduled controllers are denoted by $G_1$, $G_2$ and $G_3$ (using thick lines).

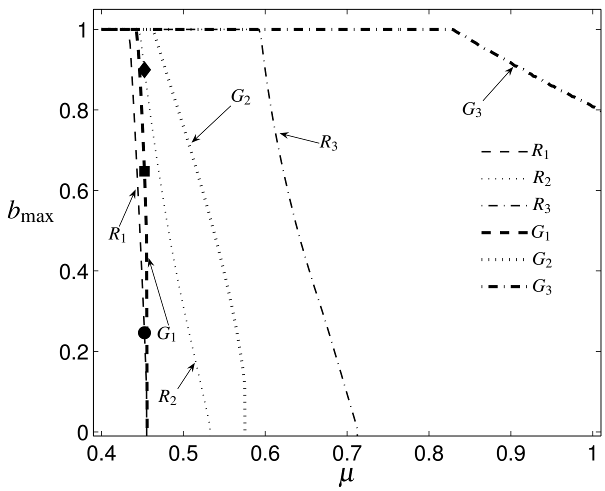


Fig. 1. Maximal bound $b_{\max}$ on the rate of parameter variation as a function of the scalar $\mu$.

For low values of $\mu$, all control designs result in controllers that allow the parameters to vary arbitrarily fast in the unit simplex since $b_{\max} = 1$. However, as $\mu$ increases, the maximal allowed bound $b_{\max}$ becomes smaller. Obviously, this occurs first for the robust case $R_1$, since it is the most restrictive control design. If more states can be measured, the curves, for the gain-scheduled and robust designs, move to the right, i.e., for the same $b_{\max}$, higher values of $\mu$ still give rise to feasible controllers. Note also that since the gain-scheduled controllers are less restrictive than the robust controllers, the curves associated with the gain-scheduled controllers are always on the right of the curves associated with the corresponding (in terms of output measurements) robust controllers.

To check the achieved performance, $\mu$ is now fixed to be $\mu = 0.4525$. Figure 2 shows the achieved upper bound $\eta$ on the closed-loop $\mathscr{H}_\infty$ performance as a function of the allowed bound $0 \leq b \leq 1$ on the rate of variation. For all control designs, it is clear from Figure 2, that as the bound $b$ increases, the performance becomes worse since the upper bound $\eta$ increases. In the robust case $R_1$ and $R_2$, and the

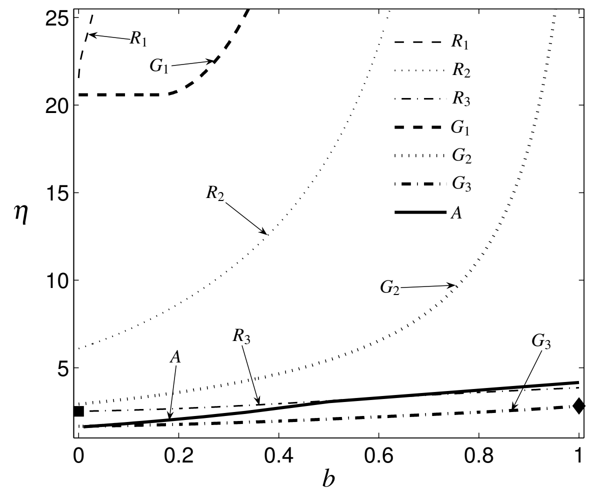


Fig. 2. Guaranteed upper bound $\eta$ on the $\mathscr{H}_\infty$ cost.

gain-scheduled case $G_1$, the upper bound $\eta$ increases very fast as the value of the bound $b$ increases. This can be expected since Figure 1 shows that for the robust case $R_1$ with $\mu = 0.4525$ the LMI conditions become infeasible for $b > 0.2465$ (circle) and for the the robust case $R_2$ the LMIs become infeasible for $b > 0.8997$ (diamond). In the gain-scheduled case $G_1$ the LMIs in Theorem 3 become infeasible for $b > 0.6475$ (square). In the other cases $R_3$, $G_2$ and $G_3$, the conditions are feasible for all values of $b$.

For the specific case $b = 1$, where the parameters can vary arbitrarily fast in the unit simplex $\Lambda_N$, the gain-scheduled case $G_3$ yields the performance $\eta = 2.8174$, which is exactly the same value as the one obtained using Theorem 2 from [20]. This is illustrated in Figure 2 by a diamond. As seen in Figure 2, the LMI conditions in Theorem 3, by explicitly considering the bound $b$ on the rate of variation, can provide a less conservative $\mathscr{H}_\infty$ bound $\eta$ for the gain-scheduled case $G_3$ as compared to the results in [20]. For the case $b = 0$, the robust case $R_3$ yields the performance $\eta = 2.5146$, which is the same value as the one obtained using Theorem 10 from [24]. This is illustrated in Figure 2 by a square.

Finally, the performance and numerical complexity of the conditions of Theorem 3 are compared to the conditions for gain scheduled state feedback controllers of [14, Theorem 6], which are based on a gridding of the parameter space in subintervals. In [14], the number of decision variables depends on the number of subintervals. To impose a bound $b$ on the rate of parameter variation, the number of subintervals needs to be at least $\nu = \lceil 1/b \rceil$. Thus, the number of decision variables grows rapidly for small values of $b$. Figure 2 shows that, compared to the results of [14, Theorem 6] (thick, solid line, denoted by $A$), the conditions of Theorem 3 result in smaller upper bounds $\eta$, for almost all values of $b$. Only for small values of $b$, the conditions of [14, Theorem 6] lead to a smaller upper bound. This, however, comes at the cost of a huge increase in the numerical complexity, which can be estimated by the number of scalar decision variables $V$ and number of LMIs rows $R$. Table I compares the numerical complexity of Theorem 3 and [14, Theorem 6] for 3 values of the bound $b$. The number of variables $V$ and LMI rows $R$ in the synthesis conditions of Theorem 3 can be calculated as follows (for the specific case of full state feedback)

$$V_{T3} = Nn(n^2+3n+2m)/2+1$$
$$R_{T3} = N^2(N+1)(2n+r+p)/2,$$

with $N$ the number of vertices, $n$ the number of states, $m$ the number of control inputs, $r$ the number of exogenous inputs and $p$ the number of outputs. Obviously, both $V_{T3}$ and $R_{T3}$ are independent of the bound $b$. However, for the conditions of [14, Theorem 6], both $V$ and $R$ increase fast, for decreasing values of $b$. Note that the results proposed in this paper can be further improved by using homogeneous polynomially parameter-dependent Lyapunov matrices and Pólya's relaxations, as currently investigated by the authors.

TABLE I

COMPARISON OF THE NUMERICAL COMPLEXITY.

| | | $V$ | | $R$ | |
|---|---|---|---|---|---|
| $b$ | $\nu$ | G3 | A | G3 | A |
| 1 | 1 | 48 | 16 | 37 | 10 |
| 0.1 | 10 | 48 | 448 | 37 | 91 |
| 0.01 | 100 | 48 | 4768 | 37 | 901 |

## VII. CONCLUSION

New LMI conditions are presented for the synthesis of gain-scheduled $\mathscr{H}_\infty$ static output feedback controllers for polytopic linear time-varying discrete-time systems, with a priori known bounds on the parameter variation.

It is worth to emphasize that previously published results in the literature can be recovered as special cases of the proposed results. In the case $b = 1$, where the parameter is allowed to vary arbitrarily fast, Theorem 3 provides equivalent results as the ones presented in [20]. Likewise, for $b = 0$, the conditions of Corollary 1 reduce to the robust $\mathscr{H}_\infty$ full state feedback conditions of [24].

Compared to the conditions of [14], the proposed approach yields similar results, with significantly less numerical complexity, estimated by the number of scalar variables and LMI rows.

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