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Asymptotic Stabilization using a Constructive Approach to Constrained Nonlinear Model Predictive Control

Juan S. Mejía and Dušan M. Stipanović

Abstract— **This paper presents a new constructive model predictive control approach to asymptotic stabilization of constrained, discrete time-invariant nonlinear dynamic systems. The constructive approach not only considers the traditional optimality problem on a finite horizon, but also considers a feasibility constraint imposed at the end of each finite horizon (prediction horizon). The feasibility constraint is included in the optimization formulation as a set of inequality constraints. Sufficient conditions for establishing asymptotic stability of discrete nonlinear systems are derived from the simultaneous solutions of the optimality and the feasibility problems on the finite horizon. The proposed approach is appealing in the sense that no necessary conditions regarding stabilizability of the linearization of the nonlinear dynamic system around an equilibrium, or the identification of an a priory stabilizing control law in a neighborhood of the equilibrium are needed; known as common requirements in many nonlinear model predictive control formulations. Simulation examples for the proposed approach are presented.**

I. INTRODUCTION

Since its origins in the late 70's and early 80's and until its blooming period in the 90's and thereafter, Model Predictive Control (MPC), an optimization based approach for stabilizing linear and nonlinear dynamic systems, has become an important control design methodology. The increase in popularity and therefore in applications is based on its ability to achieve desired system performance while at the same time being able to handle constraints on the controlled system [1]. Beside successful applications in industry [2], over the last two decades there has been a great increase in theoretical results and algorithms that focus on showing and enforcing properties such as stability and robustness ([3], [4], and [5]). These results aim to broaden the family of problems and systems where model predictive control has been considered, improving the way in which objectives promoted by higher standards are accomplished.

The main idea of constrained model predictive control (also known as constrained receding horizon control) is to solve a finite horizon optimal control problem for a system, starting from current states x_k over the time interval $[k\Delta t, (k+N)\Delta t]$, where Δt is a sampling time and N is the control horizon length, under a set of constraints on the system states and/or control inputs. After a solution from the optimization problem is obtained, a portion of the computed control actions is applied on the interval $[k\Delta t, (k+n)\Delta t]$, where n is the receding step, satisfying $n < N$. This process is then repeated as the finite horizon moves by *time steps* of $n\Delta t$ units of time, yielding a state feedback control scheme strategy.

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PhD student Juan S. Mejía and Assistant Professor Dušan M. Stipanović are both with the Industrial and Enterprise Systems Engineering Department and the Coordinated Science Laboratory, University of Illinois at Urbana-Champaign; {jsmejia,dusan}@illinois.edu

It has been expressed in [6] that even for the case where there is no model mismatch between the real system and the model, and no disturbances are present, the closed-loop inputs and state trajectories will differ from the ones obtained by the open-loop prediction in the optimization problem. Such observation motivates the fact that most implementation choose $n = 1$, holding a constant control input between the sampling times. By choosing $n = 1$, an optimization problem is solved at every sampling time, therefore minimizing the mismatch between the calculated and actual plant state response by considering the latest information used for the optimization process. Under assumptions that the nonlinear dynamic system would not deviate too far from the predicted trajectory, we have a freedom to choose $1 < n < N$, such that the on-line optimization is not performed at every sampling time, thus reducing the computational load.

In the case of constrained Nonlinear Model Predictive Control (NMPC), over the recent years different approaches to establish stability properties have been proposed, including stability constraints, contractive constraints, Control Lyapunov Functions (CLF), and inverse optimality based schemes, among others. Stability constraint approaches share two common ideas: 1) the use of the value function (optimal cost function over the finite horizon) as a valid Lyapunov function and 2) the definition of a terminal region of attraction $\mathbb{E}_F$ (non empty) in the state space, where the last state of the finite horizon at the first optimization iteration at time $k = 0$ must belong to $\mathbb{E}_F$. This terminal condition is then included in the optimization formulation in the form of equality or inequality constraints.

In [7], [8], and [9], a final equality constraint is used, defining $\mathbb{E}_F = \{0\}$, such that at time ΔtN, the last state in the horizon must reach the equilibrium. The problem with such approach is the high computational demand required for satisfying the equality constraint exactly, therefore not being efficient in practice. This issue motivated alternative formulations such as the ones presented in [10], [11], and [12], where instead of satisfying an equality constraint, an inequality constraint is used to define a terminal region $\mathbb{E}_F$ in a neighborhood of the origin. In this terminal region $\mathbb{E}_F$, a local linear control law K_L is assumed to be available and is obtained from the linearization of the nonlinear system at the equilibrium. Also in [11] and [12], a final penalty term or final cost $F(\cdot)$ is included, representing an upper bound on the optimal cost to go until reaching the equilibrium. Adequate selection of a terminal region $\mathbb{E}_F$, the local linear control law K_L and the cost $F(\cdot)$ in the mentioned approaches guarantee asymptotic stability of the considered nonlinear system. Another approach within the terminally constraint schemes is the one presented in [13], where no stabilizable linearization condition around the equilibrium is necessary, but a stabilizing control law needs to be identified within $\mathbb{E}_F$. The identification of such control law may be a challenging task for some systems. Resemblance to the CLF design methodology is clear in this setting.

Another approach is the contractive NMPC introduced in [14] and extended in [15]. This method, similarly to the the ones mentioned previously, also uses a Lyapunov-based approach. In [15] an a priori chosen Lyapunov function is decreasing at some discrete instances instead of being decreasing uniformly (at all sample times k). By imposing an extra inequality constraint in the optimization process, the a priori chosen Lyapunov function decreases at the desired discrete instances by a factor of $\alpha \in (0,1)$, guaranteeing that the system is exponentially stable. In [15] an assumption regarding the linearization of nonlinear systems around the origin being stabilizable is also used.

Other methods based on CLF and inverse optimality have also been considered in [16] and [17], respectively. For more detailed information on NMPC stability approaches, we refer to [4] and the excellent survey presented in [18].

The main idea of this paper is to present an NMPC scheme where asymptotic stability of the considered nonlinear system is guaranteed (under appropriate assumptions) without necessary conditions such as the nonlinear system being stabilizable once linearized around the desired equilibrium, or the identification of a stabilizing control law within a neighborhood of the origin; as assumed in many existing NMPC formulations. The exclusion of such conditions allows the proposed approach to broaden the types of nonlinear systems that can be stabilized using NMPC, including important systems such as certain nonholonomic systems, where some previously proposed approaches fail due to the associated assumptions.

The paper is structured as follows: In Section II we introduce some preliminaries on NMPC and the notation that is used throughout the paper. Section III presents the proposed NMPC approach, followed by the main results in Section IV. Simulation results based on the proposed NMPC approach are presented in Section V. Finally in Section VI we draw some conclusions and propose future research directions.

II. Preliminaries and Notation

Let us consider the following time-invariant nonlinear discrete model:

$$x_{k+1} = f(x_k, u_k), \tag{1}$$

where $x_k \in \mathbb{R}^p$ defines the set of states, $u_k \in \mathbb{R}^q$ defines the control inputs at time k and $f(\cdot,\cdot)$ is a continuous function, where $f(0,0) = 0$. All control inputs satisfy $u_k \in \mathbb{U}_N$, where $\mathbb{U}_N$ is a convex, compact set which defines the set of all admissible control inputs (for a horizon of length N). Having defined the system model in (1) we can state a formulation of the receding finite time horizon optimal control problem, denoted by $\mathscr{P}(k)$, starting from initial condition x_k at time k, as:

$$\mathscr{P}(k): \; V_N^\circ(x_k) \;=\; \min_{u_{k+i|k}} \{V_N(x_{k+i|k}, u_{k+i|k}) : u_{k+i|k} \in \mathbb{U}_N, \; i \in \{1,\ldots,N\}\}. \tag{2}$$

The formulation in (2) is subjected to a set of equality constraints $h(\cdot) = 0$ representing the system model in (1) over a finite horizon with N samples and a set of inequality constraints $g(\cdot) \leq 0$ that impose system's control input constraints. Future predicted states starting from initial conditions x_k are defined by $x_{k+i|k}, i \in \{1,\ldots,N\}$, and these are generated by predicted control inputs $u_{k+i|k}$, $i \in \{1,\ldots,N\}$. A sequence of control inputs on the finite horizon staring from initial conditions x_k will be defined as $U(x_k) = \left\{u_{k+1|k}, u_{k+2|k}, \ldots, u_{k+N|k}\right\}$. Optimal future predicted states and controls are represented by $x^\circ_{i+k|k}$ and $u^\circ_{k+i|k}, i \in \{1,\ldots,N\}$ respectively. The function to be minimized in $\mathscr{P}(k)$, $V_N(x_{k+i|k}, u_{k+i|k})$, penalizing future states and control inputs, is the following cost function:

$$V_N(x_k, U(x_k)) = \sum_{i=1}^{N} l(x_{k+i|k}, u_{k+i|k}) + F(x_{k+N|k}), \tag{3}$$

with two components, a running cost $\sum_{i=1}^{N} l(x_{k+i|k}, u_{k+i|k})$ and a final cost $F(x_{k+N|k})$. The running cost and the final cost are chosen (as in most NMPC formulations) as time-invariant, positive definite, radially unbounded functions. Advantages from such choices will become evident later in developing stability properties for the proposed control scheme.

The following notation will be used throughout the rest of the paper:

- $\mathbb{E}_F := \{x \in \mathbb{R}^p : 0 \leq H(x) \leq C\}$, is a closed invariant subset of $\mathbb{R}^p$ including the origin, for a given constant C and a positive semi-definite function $H(x)$. That is, after system states x_k enter $\mathbb{E}_F$, all future system states $x_{k+j},\; j \in \{1,\ldots,\infty\}$, must remain in that set.
- Instead of a time-invariant $\mathbb{E}_F \subset \mathbb{R}^p$, we define a time varying $\mathbb{E}_F(k) \subset \mathbb{R}^p$ such that as k increases, $\mathbb{E}_F(k+1) \subset \mathbb{E}_F(k)$.
- Predicted control inputs $u_{k+i|k}$ will also be denoted by $u_{k+i}(x_k)$, making more explicitly their dependence on the initial condition x_k.
- The sequence of N predicted optimal control actions that achieves V_N° starting from initial condition x_k is denoted as $U^\circ(x_k) = \{u^\circ_{k+1}(x_k), u^\circ_{k+2}(x_k), \ldots, u^\circ_{k+N}(x_k)\}$.
- The sequence of n predicted optimal control actions starting from initial condition x_k is denoted as $K_n^\circ(x_k) = \{u^\circ_{k+1}(x_k), u^\circ_{k+2}(x_k), \ldots, u^\circ_{k+n}(x_k)\}$, $K_n(x_k) \subset U^\circ(x_k)$.

- The sequence of n chosen control actions (not necessarily optimal) applied from initial condition $x^{\circ}_{k+N|k}$ such that the future states $x_{k+N+j|k}$, $j=\{1,\ldots,n\}$, stay within a closed invariant subset $\mathbb{E}_F(k)\subset\mathbb{R}^p$, $\mathbb{E}_F(k)=\left\{x\in\mathbb{R}^p : F(x)\le F(x^{\circ}_{k+N|k})\right\}$ is denoted as $K_F(x^{\circ}_{k+N|k}) = \{\hat{u}_{k+N+1}(x^{\circ}_{k+N|k}),\hat{u}_{k+N+2}(x^{\circ}_{k+N|k}),\ldots,\hat{u}_{k+N+n}(x^{\circ}_{k+N|k})\}$.
- The sequence of N optimal states generated by predicted optimal control inputs $U^{\circ}(x_k)$ starting from initial condition x_k is denoted as $X^{\circ}(x_k) = \{x^{\circ}_{k+1|k},x^{\circ}_{k+2|k},\ldots,x^{\circ}_{k+N|k}\}$.
- The sequence of n states generated by chosen control inputs $K_F(x^{\circ}_{k+N|k})$ starting from initial condition $x^{\circ}_{k+N|k}$ is denoted as $\hat{X}(x^{\circ}_{k+N|k}) = \{\hat{x}_{k+N+1|k},\hat{x}_{k+N+2|k},\ldots,\hat{x}_{k+N+n|k}\}$.
- The sequence of the control actions resulting from the union of $U^{\circ}(x_k)$ and $K_F(x^{\circ}_{N|k})$ is denoted as $\tilde{U}(x_k) = \{u^{\circ}_{k+1}(x_k),u^{\circ}_{k+2}(x_k),\ldots,u^{\circ}_{k+N}(x_k), \hat{u}_{k+N+1}(x^{\circ}_{N|k}),\hat{u}_{k+N+2}(x^{\circ}_{N|k}),\ldots,\hat{u}_{k+N+n}(x^{\circ}_{N|k})\}$.
- The sequence of the states generated by the control inputs $\tilde{U}(x_k)$ is denoted as $\tilde{X}(x_k) = \{x^{\circ}_{k+1|k},x^{\circ}_{k+2|k},\ldots,x^{\circ}_{k+N|k},\hat{x}_{k+N+1|k},\hat{x}_{k+N+2|k},\ldots,\hat{x}_{k+N+n|k}\}$.

Elements of a particular sequence $S(y)$ will be denoted by $S(i;y)$ where $i\in\{1,...,\text{length}(S(y))\}$ and y is an initial condition for the sequence.

III. Proposed NMPC approach

The main idea of the proposed NMPC approach is to solve, at each sampling time k, two problems simultaneously, an *optimality problem* $\mathscr{P}(k)$ and a *feasibility problem* $\mathscr{F}(k)$ where both problems depend on each other. In the proposed approach instead of considering the traditional *control horizon* of length N we will consider an extended horizon of length P called the *prediction horizon*, where $N<P$. The idea of a prediction horizon is not new [1], and it is used to obtain further insights about the future behavior of the plant based on current predictions. The prediction horizon is constructed by concatenating the usual control horizon of size N and a tail of size $P-N$ generated by applying the last control input $U^{\circ}(N;\cdot)$ from the sequence of optimal control actions as a constant input to obtain $P-N$ predicted states, starting from the last state in the control horizon. It is important to mention that in many approaches this tail is also penalized through the cost function. Instead of obtaining the tail of the predicted horizon in the traditional way (constant control input), we will allow the control inputs in the tail of the predicted horizon ($P=N+n$), defined by the control sequence $K_F(x^{\circ}_{k+N|k})$, to be variables which will not be incorporated (penalized) into the cost function, either directly or through the states they generate. The control sequence $K_F(x^{\circ}_{k+N|k})$ will be a solution to the feasibility problem ($\mathscr{F}(\cdot)$) defined by satisfying the following two conditions:

C1 The states $\hat{X}(x^{\circ}_{k+N|k})$ (predicted states) generated by $K_F(x^{\circ}_{k+N|k})$ belong to an invariant set $\mathbb{E}_F(k):=\{x\in\mathbb{R}^q|0\le F(x)\le F(x^{\circ}_{k+N|k})\}$

C2 The set $\mathbb{E}_F(k)$ contracts as k recedes in time.

The feasibility conditions introduced above can be included into the proposed NMPC formulation (optimization problem) by introducing two sets of inequality constraints that must be satisfied at each optimization horizon. The first set of inequalities ($g_{\mathbf{C1}}(\cdot)\le 0$),

$$F(\hat{X}(j;x^{\circ}_{k+N|k}))-F(x^{\circ}_{k+N|k})\le 0,\ \ \forall j\in\{1,...,n\}, \tag{4}$$

enforces condition **C1** such that $\hat{X}(x^{\circ}_{k+N|k})\in\mathbb{E}_F(k)$. On the other hand, the second set has just one inequality ($g_{\mathbf{C2}}\le 0$),

$$F(\hat{x}_{k+N+n|k})-F(x^{\circ}_{k+N|k})+\sum_{i=1}^{n} l(\hat{x}_{k+N+i|k},K_F(i;x^{\circ}_{k+N|k}))\le 0, \tag{5}$$

which enforces **C2** and implies the contraction of of the set $\mathbb{E}_F(k)$ in between finite horizons as k recedes.

A clear implication of this formulation is the requirement to use a final penalizing cost in the cost function. This final cost can be chosen as an appropriate scaled version of the last element (or associated last state) of the running cost that is being minimized over each horizon.

With all necessary elements being introduced we can redefine the optimization problem to be solved at each sampling time as,

$$\begin{aligned}\hat{\mathscr{P}}(k):\ V_N^{\circ}(x_k) \;\; = \;\; & \min_{u_{k+i|k}}\{V_N(x_{k+i|k},u_{k+i|k}): u_{k+i|k}\in\mathbb{U}_N,\\ & i\in\{1,\ldots,N\}\}, \qquad\qquad (6)\\ & \text{s.t.}\left\{\begin{array}{l} h(i)=0,\forall i\in\{1,\ldots,P\}\\ g(i)\le 0,\forall i\in\{1,\ldots,P\}\\ g_{\mathbf{C1}}(j)\le 0,\forall j\in\{1,\ldots,n\}\\ g_{\mathbf{C2}}\le 0,\end{array}\right.\end{aligned}$$

where again equality constraints $h(\cdot)=0$ represent the dynamic model in (1), inequality constraints $g(\cdot)\le 0$ represent constraints on the control input and inequality constraints $g_{\mathbf{C1}}(\cdot)\le 0$ and $g_{\mathbf{C2}}\le 0$ are the expressions that enforce conditions **C1** and **C2**. Notice that all inequality constraints are being expressed in the traditional negative null form.

From the formulation of $\hat{\mathscr{P}}(k)$ in (6) it can be observed that it includes elements from both the stability and the contractive constraint approach. Regarding relations with the stability constraint approach, instead of using a local control law (which may not be available) to enforce $\hat{X}(x^{\circ}_{k+N|k})\in\mathbb{E}_F(k)$, the proposed scheme uses $K_F(x^{\circ}_{k+N|k})$, a sequence of control action which is available provided there is a feasible solution for $\hat{\mathscr{P}}(k)$. The availability of $K_F(x^{\circ}_{k+N|k})$ by obtaining a solution to the feasibility problem (satisfaction of **C1-C2**) will allow us to later show stability properties in a constructive way. A difference with respect to the stability constraint approach is the fact that $\mathbb{E}_F(k)$ can change size and it is not static. This fact is what makes a direct resemblance to the contractive approach since the set $\mathbb{E}_F(k)$ contracts as the problem is iteratively solved and k recedes, guiding system states toward the desired equilibrium.

IV. Main Result

Before introducing our main result for establishing stability properties of the proposed NMPC approach defined in (6), let us state the following nonrestrictive assumptions:

- **A1** There exist at least one control sequence, $U_\infty \in \mathbb{U}_N$, such that as $k \to \infty$, $x_k \to 0$. This implies that at each optimization horizon there is at least one feasible solution $U^\circ(x_k)$ for $\hat{\mathscr{P}}(k)$, which implies satisfaction of the set of constraints $g_{\mathbf{C1}}(\cdot) \le 0$ and $g_{\mathbf{C2}} \le 0$ on the finite horizon of length P.
- **A2** $\mathbb{E}_F(k) \subset \mathbb{R}^p$, $\mathbb{E}_F(k)$ is closed and $\{0\} \in \mathbb{E}_F(k)$.
- **A3** $F(x^\circ_{N|0}) \le F(x_0)$ along the solutions of the first optimization problem. This condition can be enforced by choosing a suitable horizon length N.
- **A4** System state feedback is available at all sampling instances.
- **A5** No disturbances are considered in the system environment.

Using the proposed control scheme and the previous assumptions we introduce the following result.

Theorem 1. *Under the assumptions **A1-A5**, the proposed receding time horizon control scheme with $1 < n < N$ that iteratively solves the optimal control problem $\hat{\mathscr{P}}(\cdot)$ in (6), asymptotically stabilizes the system described in equation (1).*

Proof: Let us define the associated optimal cost starting from the initial conditions $x_{k+n|k}$ and apply control sequence $\tilde{U}(x_k)$ (well defined from **A1**), matching time accordingly by,

$$\begin{aligned} V_N(x_{k+n|k},\tilde{U}(x_k)) &= V_N^\circ(x_k,U^\circ(x_k)) - \sum_{i=1}^{n} l(x^\circ_{k+i|k},K_n^\circ(i;x_k)) \\ &- F(x^\circ_{k+N|k}) + \sum_{i=1}^{n} l(\hat{x}_{k+N+i|k},K_F(i;x^\circ_{k+N|k})) \\ &+ F(\hat{x}_{k+N+n|k}). \end{aligned} \quad (7)$$

The cost $V_N(x_{k+n|k},\tilde{U}(x_k))$ is an upper bound on the optimal cost $V_N^\circ(x_{k+n|k},U^\circ(x_{k+n|k}))$ and therefore the following is a consequence of (7):

$$V_N^\circ(x_{k+n|k},U^\circ(x_{k+n|k})) \le V_N^\circ(x_k,U^\circ(x_k)) - \sum_{i=1}^{n} l(x^\circ_{k+i|k},K_n^\circ(i;x_k)), \quad (8)$$

provided that,

$$F(\hat{x}_{k+N+n|k}) - F(x^\circ_{k+N|k}) + \sum_{i=1}^{n} l(\hat{x}_{k+N+i|k},K_F(i;x^\circ_{k+N|k})) \le 0.$$

The above inequality follows from the existence of a feasible solution (assumption **A1**) and therefore satisfaction of constraint $g_{\mathbf{C2}} \le 0$. Also, by satisfying $g_{\mathbf{C1}}(\cdot) \le 0$ (assumption **A1**), invariance of $x_{k+N+j|k}$, $j = \{1,\dots,n\}$ with respect to $\mathbb{E}_F(k)$ is guarantied.

Provided $\sum_{i=1}^{n} l(x^\circ_{k+i|k},K_n^\circ(i;x_k)) > 0$, $\forall x_k \neq \{0\}$, which is enforced by an appropriate selection of the cost function V_N, from (8) we obtain the relation,

$$\begin{aligned} \Delta V_N^\circ(k) &= V_N^\circ(x_{k+n|k},U^\circ(x_{k+n|k})) - V_N^\circ(x_k,U^\circ(x_k)) \\ &\le -\sum_{i=1}^{n} l(x^\circ_{k+i|k},K_n^\circ(i;x_k)) < 0, \forall x_k \neq \{0\}. \end{aligned} \quad (9)$$

The running cost and final cost are originally chosen as time-invariant, positive definite, radially unbounded functions. If we add such choice of the cost function to the condition in (9), the optimal cost can be identified as a valid discrete Lyapunov function. Therefore, from the condition in (9) we can conclude by Lyapunov second theorem for discrete systems ([19], [20]), that the proposed receding time horizon control scheme asymptotically stabilizes the constrained nonlinear system in (1). □

The following Lemma is used to reformulate the result of Theorem 1 for the case of $n = 1$ which is again almost always the exclusive choice in NMPC implementations.

Lemma 1. *In the case where the receding step $n = 1$, condition **C2** implies condition **C1***

Proof: Condition **C2** is captured by the feasibility of $g_{\mathbf{C2}} \le 0$. Then, suppose that the problem $\hat{\mathscr{P}}(k)$ at time k produces a feasible solution and therefore the inequality is satisfied. Now, let us rewrite the inequality $g_{\mathbf{C2}} \le 0$, for the case when $n = 1$ in the following form,

$$F(\hat{x}_{k+N+1|k}) - F(x^\circ_{k+N|k}) \le -l(\hat{x}_{k+N+1|k},K_F(1;x^\circ_{k+N|k})).$$

Since the portion of the cost $l(\hat{x}_{k+N+1|k},K_F(1;x^\circ_{k+N|k})) \ge 0$, given $l(\cdot,\cdot)$ is chosen as a positive definite function, then $F(\hat{x}_{k+N+1|k}) - F(x^\circ_{k+N|k}) \le 0$. This last inequality is the one that enforces condition **C1** for the case where $n = 1$. Thus the Lemma is proved. □

The following is a Corollary of Theorem 1 and Lemma 1.

Corollary 1. *Under the assumptions **A1-A5**, the receding time horizon control scheme with $n = 1$ that iteratively solves the optimal control problem $\hat{\mathscr{P}}(\cdot)$ in (6), asymptotically stabilizes the system described in (1) without explicitly including inequality $g_{\mathbf{C1}}(1) \le 0$ in $\hat{\mathscr{P}}(\cdot)$.*

V. Simulation Results

To illustrate the proposed NMPC scheme let us consider the following two simulation examples, where all considered systems when linearized around the origin are not stabilizable.

A. Example 1

Let us consider the following constrained continuous nonlinear dynamic system [13],

$$\dot{x} = xu, \quad x,u \in \mathbb{R}, \quad |u| \le 1,$$

which can be discretized to obtain,

$$x_{k+1} = x_k(1+\Delta t u), \quad |u| \leq 1, \tag{10}$$

where Δt is the sampling time.

Now, let us define the cost function by,

$$V_N(x_{k+i|k}, u_{k+i|k}) = \sum_{i=1}^{N} (\alpha(x_{k+i|k})^2 + \beta(u_{k+i|k})^2) + \gamma(x_{k+N|k})^2,$$

where $\alpha = 1$, $\beta = 0.001$, $\gamma = 20$, and choose parameters $N = 7$, $n = 1$ and $\Delta t = 0.1$[s].

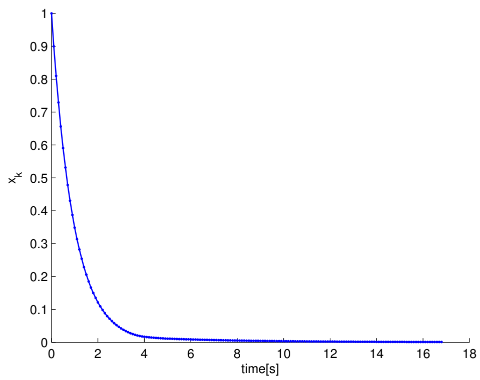

Fig. 1. System Response Example 1

Simulated state response obtained from implementing the proposed NMPC approach to the system in (10) starting from the initial condition $x_0 = 1$ is depicted in Fig. 1. From the figure it can be observed that the proposed NMPC approach asymptotically stabilizes the system in (10).

B. Example 2

Consider the so called *nonholonomic integrator* system,

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1 u_2 - x_2 u_1 \end{aligned} \tag{11}$$

introduced by Brockett in [21], where $x = [x_1, x_2, x_3]^T \in \mathbb{R}^3$ and $u = [u_1, u_2]^T \in \mathbb{R}^2$. This system has the particularity that it fails to meet Brockett's condition for smooth stabilizability [21], and as a consequence there is no time-invariant continuously differentiable control law that would make the origin asymptotically stable.

By discretizing the system in (11) the following discrete-time system is obtained:

$$\begin{aligned} x_{k+1}(1) &= x_k(1) + u_k(1)\Delta t \\ x_{k+1}(2) &= x_k(2) + u_k(2)\Delta t \\ x_{k+1}(3) &= x_k(3) + (x_k(1)u_k(2) - x_k(2)u_k(1))\Delta t. \end{aligned} \tag{12}$$

where the states are given by $x_k = [x_k(1), x_k(2), x_k(3)]^T \in \mathbb{R}^3$ and the control inputs by $u_k = [u_k(1), u_k(2)]^T \in \mathbb{R}^2$. The system considered in (12) is intended to minimize the following cost function,

$$\begin{aligned} V_N(x_{k+i|k}, u_{k+i|k}) &= \sum_{i=1}^{N} (x_{k+i|k}^T Q x_{k+i|k} + \beta u_{k+i|k}^T \, u_{k+i|k}) \\ &+ \gamma x_{k+N|k}^T \, x_{k+N|k}, \end{aligned} \tag{13}$$

where Q is a diagonal matrix with diagonal elements $\{1,1,2\}$, $\beta = 0$ and $\gamma = 100$, while subject to a constrained set of admissible control inputs $u_k(i) \in [-1,1]$, $\forall i \in \{1,2\}, \forall k \geq 0$, and additional constraints $|u_{k+1}(i) - u_k(i)| \leq 0.5$, $\forall i \in \{1,2\}, \forall k \geq 0$. Horizon and receding parameters are $N = 4$ and $n = 2$, and sampling time $\Delta t = 0.2$[s]. Superscript T in equation (13) denotes vector transpose.

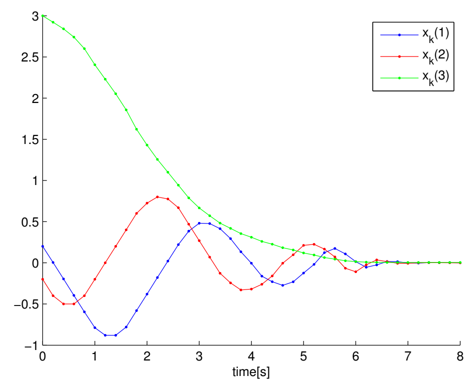

Fig. 2. System Response Example 2

The system response starting from initial conditions $x_0 = [0.2, -0.2, 3]^T$ can be observed in Fig. 2. From the figure it can be noticed that the system converges asymptotically to the origin.

VI. Conclusions

The proposed constructive NMPC approach is an alternative scheme for stabilizing discrete nonlinear dynamic systems that are not stabilizable when linearized around an equilibrium, or discrete nonlinear dynamic systems for which finding an a priory stabilizing control law in a neighborhood of the origin is a hard task. Sufficient conditions for guaranteeing asymptotic stability were presented for receding parameter $1 < n < N$. These conditions were later specialized for the case when $n = 1$, showing that a simplified version of the proposed optimization problem could also guarantee system's asymptotic stability. Two simulation examples including systems that are not stabilizable when linearized around an equilibrium were provided. In both examples the proposed approach proved to stabilize the considered systems. Our future work will focus on the consideration of disturbances and also in determining more insightful directions in the choice of the final cost function.

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