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Robust LMIs with parameters in multi-simplex: Existence of solutions and applications

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Abstract— **This paper presents new results concerning the existence of solutions for robust (parameter-dependent) LMIs with parameters lying in a Cartesian product of simplexes, called *multi-simplex*. These results allow to derive convergent procedures based on LMI relaxations to check the positivity of polynomial matrices with parameters in multi-simplexes. As an application, the robust stability analysis of uncertain linear systems is investigated. As an immediate advantage of this flexible representation, polynomially parameter-dependent Lyapunov functions can be constructed to handle simultaneously time-invariant, arbitrarily time-varying and bounded time-varying parameters in an appropriate way. Numerical experiments illustrate the advantages of the method.**

I. Introduction

Linear Matrix Inequalities (LMIs) subject to uncertain data are known as robust (or parameter-dependent) LMIs. Optimization procedures based on robust LMIs are convex but of infinite dimension [1] and several efforts, coming from different fronts [2–6], have been devoted in the last few years to provide LMI relaxations that completely characterize the solution of robust LMIs. Notoriously, control problem as stability analysis, stabilizability, filtering, $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performance analysis, and other related design issues, cast straightforwardly in the form of robust LMIs. Particularly in the case of robust stability analysis of uncertain linear systems with parameters lying in compact sets, several contributions to solve the associated robust LMIs have appeared in the literature. In general, the solutions are expressed in terms of a hierarchy of LMI relaxations which provides better and better approximations, some with guaranteed convergence. Basically, three major classes of uncertain parameters can be distinguished: time-invariant parameters, time-varying parameters with bounded rates of variation and arbitrarily fast time-varying parameters.

In the case of linear time-invariant uncertain systems, the robust stability analysis methods available nowadays have reached a high level of maturity, allowing to treat the problem in terms of convergent relaxations [4, 7–11]. Most of the strategies rely on the use of parameter-dependent Lyapunov functions whose existence is verified by LMI relaxations. It is worth mentioning in this context the methods that use region-dividing techniques and can be conclusive about the solution under a given precision [12, 13].

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The problem of robust stability analysis of linear time-varying uncertain systems where the parameters can vary arbitrarily fast, has been tackled, in general, by approaches that use Lyapunov functions independent of the parameters. Among others, one can mention the quadratic stability method, results based on piecewise quadratic Lyapunov functions [14, 15] and strategies based on Lyapunov functions with homogeneous polynomial dependence of arbitrary degree in the state [16, 17].

In the context of time-varying uncertain systems where the parameters have bounded rates of variation, several contributions using affine [18–20], quadratic [21], and polynomially [22–25] parameter-dependent Lyapunov functions can be found. It is also worthy to mention [26, 27] that use the IQC (Integral Quadratic Constraint) approach.

As a general observation, the aforementioned methods are not flexible in the sense of being easily adapted to cope with the other classes of parameters. The reason is that methods are highly dependent on the characteristics of the space where the parameters can assume values. In general, the uncertain parameters are assumed either to lie in a polytope, or to be individually bounded — thus resulting in a hypercube uncertainty set. Hypercubes are special polytopes, and reciprocally polytopes can be parametrized as hypercubes, at the expense of a possible overparametrization. However, according to the nature of the parameters, the corresponding change of variables can be rather unnatural, and computationally cumbersome. Moreover, this rewriting can lead to conservative considerations when time-invariant and time-varying parameters are merged in the same polytope. (As an example, one can see easily that the time variation of *every* parameters should be considered unbounded in the new parametrization as soon as *one* parameter has unbounded variations in the initial setting). The main application of the results of this paper is to provide a unified and direct approach to investigate the problem of robust stability of continuous-time systems with parameters in a Cartesian product of simplexes, (called *multi-simplex* in the sequel). In this setting, the Lyapunov functions assessing robust stability can be appropriately constructed accordingly to the uncertain parameters class: invariant, arbitrarily time-varying or time-varying with bounded rates of variation. The proposed approach produces better results when compared to others from the literature, even in the case where all the uncertain parameters belong to the same class, as illustrated by numerical experiments.

Notation: $\mathbb{N}$ denotes the natural numbers and $\mathbb{R}$ the real numbers; The space of symmetric matrices in $\mathbb{R}^{p\times p}$ is denoted by $\mathbb{S}^p$; The symbol $(')$ indicates transpose; $P > 0$ (≥ 0) means that P is symmetric positive (semi)definite; 0_p

is the zero matrix of dimension $p \times p$; $\otimes$ stands for the Kronecker product.

II. MULTI-SIMPLEX AND CORRESPONDING HOMOGENEOUS POLYNOMIAL

The unit simplex Λ_r of dimension $r \geq 2$ is given by

$$\Lambda_r = \left\{ \alpha = (\alpha_1 \cdots \alpha_r)' \in \mathbb{R}^r : \sum_{i=1}^{r} \alpha_i = 1, \ \alpha_i \geq 0, \ i = 1, \ldots, r \right\}.$$

Definition 1 (Multi-simplex): A *multi-simplex* Λ is the Cartesian product $\Lambda_{N_1} \times \cdots \times \Lambda_{N_m}$ of a finite number of simplexes $\Lambda_{N_1}, \ldots, \Lambda_{N_m}$, $i = 1, \ldots, m$. The *dimension* of Λ is defined as the index $N = (N_1, \ldots, N_m)$. For ease of notation, $\mathbb{R}^N$ denotes the space $\mathbb{R}^{N_1 + \cdots + N_m}$. A given element α of Λ is decomposed as $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ according to the structure of Λ and, subsequently, each α_i (being in Λ_i), is decomposed in the form $(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{iN_i})$. ■

As an example, let $\Lambda = \Lambda_2 \times \Lambda_3 \times \Lambda_2$. Then a generic element of Λ writes as $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_1 = (\alpha_{11}, \alpha_{12}) \in \Lambda_2$, $\alpha_2 = (\alpha_{21}, \alpha_{22}, \alpha_{23}) \in \Lambda_3$ and $\alpha_3 = (\alpha_{31}, \alpha_{32}) \in \Lambda_2$.

Definition 2 (Λ-homogeneous polynomial): Given a multi-simplex Λ of dimension N, a polynomial $P(\alpha)$ defined on $\mathbb{R}^N$ and taking values in a finite dimensional vector space is said Λ-*homogeneous* if, for any $i_0 \in \{1, \ldots, m\}$, and for any given $\alpha_{N_i} \in \mathbb{R}^{N_i}$, $i \in \{1, \ldots, N\} \setminus \{i_0\}$, the partial application $\alpha_{N_{i_0}} \in \mathbb{R}^{N_{i_0}} \mapsto P(\alpha)$ is a *homogeneous* polynomial. ■

As an illustration, considering the previous example for Λ, $P(\alpha) = 3\alpha_{11}(\alpha_{21}^2 + \alpha_{22}\alpha_{23}) + \alpha_{12}\alpha_{23}^2$ is Λ-homogeneous (of degree 1 in the components of $\alpha_1 \in \Lambda_2$ and of degree 2 in the components of $\alpha_2 \in \Lambda_3$).

Definition 3 (Λ-completion of a polynomial): Given a multi-simplex Λ of dimension N and a polynomial $P(\alpha)$ defined on $\mathbb{R}^N$ taking values in a finite dimensional vector space, the Λ-*completion of* $P(\alpha)$, denoted $\mathrm{comp}_\Lambda(P(\alpha))$, is the (unique) polynomial Λ-homogeneous of minimal degree equal to $P(\alpha)$ on Λ. ■

The Λ-completion of $P(\alpha)$ is easily constructed by introducing, in each term of the sum of factors defining $P(\alpha)$, factors of the form $(\alpha_{i1} + \cdots + \alpha_{iN_i})^{\beta_i}$ with minimal degree β_i. In this notation, $(\alpha_{i1}, \ldots, \alpha_{iN_i})$ corresponds to the component of α lying into the unit simplex Λ_{N_i}. For example, with $\Lambda = \Lambda_2 \times \Lambda_3$, the Λ-completion of $P(\alpha) = 3\alpha_{11}(\alpha_{22} + \alpha_{23}^2) + 2$ is given by

$$\begin{aligned}\mathrm{comp}_\Lambda(P(\alpha)) = 3\alpha_{11}(\alpha_{22}(\alpha_{21} + \alpha_{22} + \alpha_{23}) + \alpha_{23}^2) \\ + 2(\alpha_{11} + \alpha_{12})(\alpha_{21} + \alpha_{22} + \alpha_{23})^2\end{aligned}$$

which is a homogeneous polynomial of degree 1 in Λ_2 and homogeneous of degree 2 in Λ_3, equal to $P(\alpha)$ on Λ, and it is clearly the polynomial of minimal degree having these properties.

III. A KEY EXISTENCE RESULT FOR LMIS WITH PARAMETERS IN MULTI-SIMPLEX

In general, robust LMI with parameters belonging to Λ can be written as $F(x, \alpha) > 0_p$ where the map F is *affine in* x and *polynomial in* α. The next theorem establishes an existence result for the solution of this robust LMI.

Theorem 1: Let Λ be a multi-simplex of dimension N and $F : \mathbb{R}^\ell \times \mathbb{R}^N \to \mathbb{S}^p$ a map that defines a feasibility problem based on LMIs with parameters in Λ. The following properties are equivalent.

(a) For all $\alpha \in \Lambda$, there exists $x(\alpha) \in \mathbb{R}^\ell$ such that $F(x(\alpha), \alpha) > 0_p$.
(b) There exists a Λ-homogeneous polynomial x^* taking values in $\mathbb{R}^\ell$, such that all the coefficients of $\mathrm{comp}_\Lambda(F(x^*(\alpha), \alpha))$ are positive definite. ■

Proof: The demonstration, based on Pólya's theorem, is made by recursion on the number m of terms in the Cartesian product defining Λ. Firstly, the property is shown for the case $m = 1$. Consider a multi-simplex formed by only one simplex, i.e. $\Lambda = \Lambda_{N_1}$, and a problem based on robust LMIs with parameters belonging to Λ. As proved in [28], there exists a solution $x(\alpha)$ solving the LMI for any α belonging to Λ if and only if there exists a homogeneous polynomial solution $x^{**}(\alpha)$. For such a solution, the validity of the positivity constraint on the coefficients is equivalent to the existence of a nonnegative (large enough) integer β such that every coefficients of $(\alpha_{11} + \cdots + \alpha_{1N_1})^\beta \mathrm{comp}_\Lambda(F(x^{**}(\alpha), \alpha)$ are positive definite. This claim is nothing but an application of the extension of Pólya's Theorem to polynomials with matrix-valued coefficients [10, 29]. It is immediate to verify that $(\alpha_{11} + \cdots + \alpha_{1N_1})^\beta \mathrm{comp}_\Lambda(F(x^{**}(\alpha), \alpha) = \mathrm{comp}_\Lambda(F((\alpha_{11} + \cdots + \alpha_{1N_1})^\beta x^{**}(\alpha), \alpha)$, since F is affine in x. As a consequence, $x^*(\alpha) = (\alpha_{11} + \cdots + \alpha_{1N_1})^\beta x^{**}(\alpha)$ solves the problem.

Assume now that the property is valid for any product of up to m simplexes: our aim is to deduce it for any Λ with $m+1$ components. Let Λ be such multi-simplex, i.e. $\Lambda = \Lambda_1 \times \cdots \times \Lambda_{m+1}$. For any fixed $\alpha_{m+1} \in \Lambda_{m+1}$, to check the existence, for any $\hat{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \hat{\Lambda} \doteq \Lambda_1 \times \cdots \times \Lambda_m$ of an $x(\hat{\alpha}, \alpha_{m+1}) \in \mathbb{R}^\ell$ such that $F(x(\hat{\alpha}, \alpha_{m+1}), (\hat{\alpha}, \alpha_{m+1})) > 0_p$ is an LMI problem with parameters in the simplex $\hat{\Lambda}$ of dimension m. By the recursion assumption, the latter problem is equivalent to the existence of a $\hat{\Lambda}$-homogeneous polynomial $\hat{x}^*_{\alpha_{m+1}}(\hat{\alpha})$ such that every coefficients of $\mathrm{comp}_{\hat{\Lambda}}(F(\hat{x}^*_{\alpha_{m+1}}(\hat{\alpha}), (\hat{\alpha}, \alpha_{m+1}))) > 0_p$ are positive definite.

The latter problem can be considered as an LMI with the coefficients of $\hat{x}^*_{\alpha_{m+1}}(\hat{\alpha})$ being the new variables (intervening in an affine way), and with parameters α_{m+1} belonging to Λ_{m+1} (entering polynomially): applying the recursion hypothesis to the case of a unique simplex shows that the LMI solvability is equivalent to the existence of $x^*(\alpha)$ such that $\mathrm{comp}_{\Lambda_{m+1}}(\mathrm{comp}_{\hat{\Lambda}}(F(x^*(\alpha), \alpha))) > 0_p$. Note that polynomials in the variables $\alpha_1, \ldots, \alpha_{m+1}$ can be viewed as polynomials in the variables $\alpha_1, \ldots, \alpha_m$, whose coefficients are polynomials in α_{m+1}. Thus, it is immediate to observe that this quantity is equal to $\mathrm{comp}_\Lambda(F(x^*(\alpha), \alpha))$. This establishes the property for $m+1$. In conclusion, the result has been proved by induction. ■

The result of Theorem 1 guarantees that an LMI with parameters lying in multi-simplexes can be completely characterized by Λ-homogeneous polynomials of arbitrary degrees. Note that the degrees do not need to be necessarily equal and can be chosen independently.

IV. Representation of Λ-Homogeneous Polynomial Matrices

Some definitions and notations to handle Λ-homogeneous polynomials are necessary.

For $N, g \in \mathbb{N}$, let $\mathscr{K}_N(g)$ be the set of N-tuples obtained from all possible combinations of N nonnegative integers with sum g. The number of elements in $\mathscr{K}_N(g)$ is thus

$$J_N(g) = \text{ card } \mathscr{K}_N(g) = \frac{(N+g-1)!}{g!(N-1)!} .$$

Let now $N, g \in \mathbb{N}^m$. The set $\mathscr{K}_N(g)$ is defined as the Cartesian product $\mathscr{K}_N(g) = \mathscr{K}_{N_1}(g_1) \times \ldots \times \mathscr{K}_{N_m}(g_m)$.

One is now in position to represent Λ-homogeneous polynomials. Any Λ-homogeneous polynomial matrix $P(\alpha)$ of partial degrees $g = (g_1, \ldots, g_m)$ can be generically represented by

$$P(\alpha) = \sum_{k \in \mathscr{K}(g)} \alpha^k P_k, \tag{1}$$

where the α^k are monomials which are homogeneous of degree g_i in each variable α_i:

$$\alpha^k = \alpha_1^{k_1} \alpha_2^{k_2} \cdots \alpha_m^{k_m}, \quad \alpha_i^{k_i} = \alpha_{i1}^{k_{i1}} \alpha_{i2}^{k_{i2}} \cdots \alpha_{iN_i}^{k_{iN_i}}$$

where $k_i = (k_{i1}, k_{i2}, \ldots, k_{iN_i})$ is such that $k_{i1} + k_{i2} + \cdots + k_{iN_i} = g_i$; and $P_k \in \mathbb{R}^{n \times n}$ are the corresponding matrix-valued coefficients.

For instance, a Λ-homogeneous polynomial with dimensions: $m = 2$, $g = (1,2)$, $N = (2,2)$ yields $\mathscr{K}_N(g) = \mathscr{K}_2(1) \times \mathscr{K}_2(2)$=$\{(0,1),(1,0)\} \times \{(0,2),(1,1),(2,0)\}$, with $J_{N_1}(1) = 2$ and $J_{N_2}(2) = 3$, corresponding to the following matrix-valued polynomial

$$\begin{aligned} P(\alpha) = \alpha_{11}\big(&\alpha_{21}^2 P_{((1,0),(2,0))} + \alpha_{21}\alpha_{22} P_{((1,0),(1,1))} \\ &+ \alpha_{22}^2 P_{((1,0),(0,2))}\big) + \alpha_{12}\big(\alpha_{21}^2 P_{((0,1),(2,0))} \\ &\quad + \alpha_{21}\alpha_{22} P_{((0,1),(1,1))} + \alpha_{22}^2 P_{((0,1),(0,2))}\big). \end{aligned}$$

Finally, note that the indices $k = (k_1, k_2, \ldots, k_m)$ are obtained by combining all the N-tuples of the sets $\mathscr{K}_{N_i}(g_i)$, $i = 1, \ldots, m$, yielding a total of $J_N(g)$ monomials equal to

$$J_N(g) = \prod_{i=1}^{m} J_{N_i}(g_i).$$

In the previous example, $J_N(g) = 6$.

By definition, for N-tuples $k, \tilde{k}$ one writes $k \preceq \tilde{k}$ if $k_{ij} \leq \tilde{k}_{ij}$, $i = 1, \ldots, m$, $j = 1, \ldots, N_i$. Operations of sum $k + \tilde{k}$ and subtraction $k - \tilde{k}$ (whenever $\tilde{k} \preceq k$) are defined componentwise. In the case $m = 1$, i.e. multi-simplex formed by only one simplex, the definitions and notations presented are similar to the ones used in [10].

V. Systems with Time-Invariant Parameters

Now, let $A(\alpha)$ be a Λ-homogeneous polynomial matrix of partial degrees $r = (r_1, \ldots, r_m)$ with time-invariant parameters $\alpha \in \Lambda$. Through the Lyapunov stability theory, Hurwitz robust stability of $A(\alpha)$ can be investigated as follows:

Lemma 1: Matrix $A(\alpha)$ is Hurwitz robustly stable if and only if there exists a symmetric parameter-dependent matrix $P(\alpha) \in \mathbb{R}^{n \times n}$ such that, for all $\alpha \in \Lambda$,

$$P(\alpha) > 0_n, \quad A(\alpha)' P(\alpha) + P(\alpha) A(\alpha) < 0_n$$

■

Clearly, the inequalities of Lemma 1 are robust LMIs depending upon scalar parameters (entries of matrix $P(\alpha)$). Applying Theorem 1 yields to the following LMI relaxations.

Theorem 2: Let Λ be a multi-simplex of dimension $N = (N_1, \ldots, N_m)$. The Λ-homogeneous polynomial matrix $A(\alpha)$ of partial degrees $r = (r_1, \ldots, r_m)$ is Hurwitz robustly stable $\forall \alpha \in \Lambda$ if and only if there exist $g = (g_1, \ldots, g_m)$, $k \in \mathscr{K}_N(g)$ and matrices $P_k \in \mathbb{S}^n$ such that the following LMIs are verified

$$P_k > 0_n, \quad \forall k \in \mathscr{K}_N(g) \tag{2}$$

$$\sum_{\substack{\tilde{k} \in \mathscr{K}_N(r) \\ \tilde{k} \preceq k}} A_{\tilde{k}}' P_{k-\tilde{k}} + P_{k-\tilde{k}} A_{\tilde{k}} < 0_n, \quad \forall k \in \mathscr{K}_N(g+r). \tag{3}$$

■

Proof: Necessity is demonstrated using condition (b) of Theorem 1, which guarantees that the desired Λ-homogeneous solution can be constrained to the class of Λ-homogeneous polynomials with positive definite matrix-valued coefficients. For the sufficiency, note that

$$A(\alpha)' P(\alpha) + P(\alpha) A(\alpha) = \sum_{k \in \mathscr{K}_N(g+r)} \alpha^k \Big(\sum_{\substack{\tilde{k} \in \mathscr{K}_N(r) \\ \tilde{k} \preceq k}} A_{\tilde{k}}' P_{k-\tilde{k}} + P_{k-\tilde{k}} A_{\tilde{k}} \Big)$$

whose right hand-side is negative definite whenever the LMIs (3) are fulfilled. To conclude, note that the LMIs (2) assure that the Λ-homogeneous matrix $P(\alpha)$ is positive definite. ■

VI. Systems with Time-varying Parameters

It is assumed now that the parameters $\alpha_i(t)$, $i = 1, \ldots, m$ are time-varying with bounded rates of variation in the form

$$\underline{b}_{ij} \leq \dot{\alpha}_{ij}(t) \leq \overline{b}_{ij}, \quad \underline{b}_{ij}, \overline{b}_{ij} \in \mathbb{R}, \tag{4}$$

with $\underline{b}_{ij}$, $\overline{b}_{ij}$ given. This situation adds a supplemental term concerning the derivative of the Λ-homogeneous Lyapunov matrix with respect to time, i.e. the following inequality must be tested

$$A(\alpha(t))' P(\alpha(t)) + P(\alpha(t)) A(\alpha(t)) + \sum_{i=1}^{m} \sum_{j=1}^{N_i} \frac{\partial P(\alpha(t))}{\partial \alpha_{ij}(t)} \dot{\alpha}_{ij}(t) < 0 \tag{5}$$

for all $\alpha(t) \in \Lambda$ and $\dot{\alpha}(t) \in \Omega = \Omega_1 \times \cdots \times \Omega_m$. As a first observation, note that the parameters $\alpha_i(t)$ are independent from each other, $i = 1, \ldots, m$ and so do their time-derivatives. Thus, the sets Ω_i, where the parameters $\dot{\alpha}_i(t)$ can assume values, are built independently. For each $\alpha_i(t)$, the construction of Ω_i follows from (4), known by the user, and

$$\dot{\alpha}_{i1}(t) + \dot{\alpha}_{i2}(t) + \cdots + \dot{\alpha}_{iN_i}(t) = 0, \tag{6}$$

since $\alpha_i(t) \in \Lambda_{Ni}$. For any i, the vector $(\dot{\alpha}_{i1}(t), \ldots, \dot{\alpha}_{iN_i}(t))$ thus lies in a polytope, which is constructed from the constraints (4) and (6).

Let $G^{(i)}$ denotes the i-th column of matrix G. The sets Ω_i, $i = 1, \ldots, m$ are defined as

$$\Omega_i = \Big\{ \delta \in \mathbb{R}^{N_i} : \delta = \sum_{\ell=1}^{M_i} \eta_{i\ell} H_i^{(\ell)}, \\ \sum_{\ell=1}^{N_i} H_i(\ell, j) = 0,\ j = 1, \ldots, N_i, \quad \eta_i \in \Lambda_{M_i} \Big\}. \quad (7)$$

For instance, let

$$-1 \le \dot{\alpha}_{11}(t) \le 1, \quad -1 \le \dot{\alpha}_{12}(t) \le 1, \quad -2 \le \dot{\alpha}_{13}(t) \le 2 \quad (8)$$

The extremal solutions of (6) under (8) are $\{(1,1,-2), (-1,-1,2), (1,-1,0), (-1,1,0)\}$. Taking the convex combination of these solutions, one has

$$\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \eta_{11} + \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \eta_{12} + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \eta_{13} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \eta_{14}, \quad \eta_i \in \Lambda_4$$

or

$$\underbrace{\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ -2 & 2 & 0 & 0 \end{bmatrix}}_{H_1} \begin{bmatrix} \eta_{11} \\ \eta_{12} \\ \eta_{13} \\ \eta_{14} \end{bmatrix}, \quad H_1 \in \mathbb{R}^{N_1 \times M_1}$$

with $N_1 = 3$ (number of parameters in Λ_1) and $M_1 = 4$ (number of solutions of (6) under (8)). Note the null sum of the rows of any column, as defined in (7). The number M_1 is not known *a priori*, being determined by the number of extremal solutions. In this example $M_1 = 4$, but it could be different if distinct bounds were considered in (8). In fact, if one consider $-1 \le \dot{\alpha}_{13}(t) \le 1$ in (8), the number of extremal solutions would be $M_1 = 6$. For more details see [20, 24], where this model has also been used.

Since $\alpha(t) \in \Lambda$ implies $\dot{\alpha}(t) \in \Omega$ for all $t \ge 0$ and using the definition of a generic $\dot{\alpha}_{ij}(t)$ belonging to Ω_i, inequality (5) can be rewritten as

$$A(\alpha)'P(\alpha) + P(\alpha)A(\alpha) \\ + \sum_{i=1}^{m} \sum_{j=1}^{N_i} \frac{\partial P(\alpha)}{\partial \alpha_{ij}} \sum_{\ell=1}^{M_i} \eta_{i\ell} H_i(j, \ell) < 0. \quad (9)$$

For Λ-homogeneous functions $P(\alpha)$ and $A(\alpha)$ of partial degrees $g = (g_1, \ldots, g_m)$ and $r = (r_1, \ldots, r_m)$ respectively, the total degree of the first two terms $A(\alpha)'P(\alpha) + P(\alpha)A(\alpha)$ is of course $\bar{g} = (g_1 + r_1, g_2 + r_2, \ldots, g_m + r_m)$. Thus, the first task is to homogenize accordingly the third term in α.

The general expression for the derivative of the Lyapunov matrix $P(\alpha)$ with respect to the i-th component of the multi-simplex, $i = 1, \ldots, m$ and then with respect to its j-th component, $j = 1, \ldots, N_i$ is given by

$$\frac{\partial P(\alpha)}{\partial \alpha_{ij}} = \sum_{k \in \mathscr{K}_N(g)} k_{ij} \alpha_1^{k_1} \cdots \alpha_{i1}^{k_{i1}} \cdots \alpha_{ij}^{k_{ij}-1} \cdots \alpha_{iN_i}^{k_{iN_i}} \cdots \alpha_m^{k_m} P_k \\ = \sum_{k \in \mathscr{K}_N(g - e_{i|m})} \alpha^k \Big((k + e_{i|m} \otimes e_{j|N_i})_{ij} P_{k + e_{i|m} \otimes e_{j|N_i}} \Big)$$

where by definition $e_{i|m}$ is the vector of dimension m with zero components, except 1 in the i-th position. To fit (on α) with the partial degrees $\bar{g}$, the following homogenization is necessary:

$$\sum_{i=1}^{m} (\alpha_{i1} + \cdots + \alpha_{iN_i})^{r_i+1} \sum_{j=1}^{N_i} \frac{\partial P(\alpha)}{\partial \alpha_{ij}} = \\ \sum_{i=1}^{m} \sum_{j=1}^{N_i} \sum_{k \in \mathscr{K}_N(g+r)} \alpha^k \Bigg(\sum_{\substack{\hat{k} \in \mathscr{K}_N(r + e_{i|m}) \\ \hat{k} \preceq k}} \times \\ \frac{(r_i+1)!}{\pi(\hat{k}_i)} \Big((k - \hat{k} + e_{i|m} \otimes e_{j|N_i})_{ij} P_{k - \hat{k} + e_{i|m} \otimes e_{j|N_i}} \Big) \Bigg) \quad (10)$$

where $\pi(k_i) = (k_{i1}!)(k_{i2}!) \cdots (k_{iN_i}!)$. Now, the third term of (9) must be homogenized to become multi-affine on η. This is done as follows

$$\prod_{\substack{p=1 \\ p \neq i}}^{m} \big(\eta_{p1} + \cdots + \eta_{pM_p} \big) \sum_{\ell=1}^{M_i} \eta_{i\ell} H_i(j, \ell) = \\ \sum_{p_1=1}^{M_1} \cdots \sum_{p_i=1}^{M_i} \cdots \sum_{p_m=1}^{M_m} \eta_{1p_1} \cdots \eta_{ip_i} \cdots \eta_{mp_m} H_i(j, p_i). \quad (11)$$

Taking into account (10) and (11), the third term in the left-hand side of (9) can be equivalently written as

$$\sum_{i=1}^{m} \sum_{j=1}^{N_i} \frac{\partial P(\alpha)}{\partial \alpha_{ij}} \sum_{\ell=1}^{M_i} \eta_{i\ell} H_i(j, \ell) = \\ \sum_{p_1=1}^{M_1} \cdots \sum_{p_m=1}^{M_m} \eta_{1p_1} \cdots \eta_{mp_m} \Bigg(\sum_{k \in \mathscr{K}_N(g+r)} \alpha^k \\ \sum_{i=1}^{m} \sum_{j=1}^{N_i} \sum_{\substack{\hat{k} \in \mathscr{K}_N(r + e_{i|m}) \\ \hat{k} \preceq k}} \frac{(r_i+1)!}{\pi(\hat{k}_i)} \times \\ \Big((k - \hat{k} + e_{i|m} \otimes e_{j|N_i})_{ij} P_{k - \hat{k} + e_{i|m} \otimes e_{j|N_i}} \Big) H_i(j, p_i) \Bigg). \quad (12)$$

Now, observe that

$$\prod_{p=1}^{m} \big(\eta_{p1} + \cdots + \eta_{pM_p} \big) \big(A(\alpha)'P(\alpha) + P(\alpha)A(\alpha) \big) = \\ \sum_{p_1=1}^{M_1} \cdots \sum_{p_i=1}^{M_i} \cdots \sum_{p_m=1}^{M_m} \eta_{1p_1} \cdots \eta_{ip_i} \cdots \eta_{mp_m} \times \\ \big(A(\alpha)'P(\alpha) + P(\alpha)A(\alpha) \big) \quad (13)$$

and finally, (9) can be tested since all terms have the same partial degrees on both α and η. Next theorem presents LMI relaxations of increasing precision for the problem of robust stability analysis of matrix $A(\alpha)$ with parameters $\alpha \in \Lambda$, $\dot{\alpha} \in \Omega$.

Theorem 3: Let Λ be a multi-simplex of dimension $N = (N_1, \ldots, N_m)$. The Λ-homogeneous polynomial matrix $A(\alpha)$ of partial degrees $r = (r_1, \ldots, r_m)$ is robustly stable $\forall \alpha \in \Lambda$, $\dot{\alpha} \in \Omega$ if there exists $g = (g_1, \ldots, g_m)$, $k \in \mathscr{K}_N(g)$ and matrices $P_k \in \mathbb{S}^n$ such that (2) and for all $(i_1, \ldots, i_m) \in$

$\{1,\ldots,M_1\}\times\cdots\times\{1,\ldots,M_m\}$ the following LMIs are verified

$$T_k = \sum_{\substack{\tilde{k}\in\mathscr{K}_N(r)\\ \tilde{k}\preceq k}} \left(A'_{\tilde{k}}P_{k-\tilde{k}} + P_{k-\tilde{k}}A_{\tilde{k}}\right) + \Xi_k < 0_n, \quad \forall k \in \mathscr{K}_N(g+r) \tag{14}$$

where

$$\Xi_k = \sum_{i=1}^{m}\sum_{j=1}^{N_i}\sum_{\substack{\hat{k}\in\mathscr{K}_N(r+e_{i|m})\\ \hat{k}\preceq k}} \frac{(r_i+1)!}{\pi(\hat{k}_i)}\times \Big((k-\hat{k}+e_{i|m}\otimes e_{j|N_i})_{ij}P_{k-\hat{k}+e_{i|m}\otimes e_{j|N_i}}\Big)H_i(j,p_i). \tag{15}$$

■

Proof: First note that the left-hand side of (9) can be written as

$$\sum_{i_1=1}^{M_1}\cdots\sum_{i_m=1}^{M_m}\eta_{1i_1}\cdots\eta_{mi_m}\left(\sum_{k\in\mathscr{K}_N(g+r)}\alpha^k T_k\right)$$

which is negative definite when the constraints (14) are fulfilled. To conclude the proof note that (2) guarantees that the Lyapunov matrix $P(\alpha)$ is positive definite. ■

Remark 1: Since the existence result of Theorem 1 is only applicable to robust LMIs with time-invariant parameters, the relaxations of Theorem 3 are only sufficient. However, less and less conservative evaluations can be obtained as the degrees of the Λ-homogeneous Lyapunov matrix increase. If Theorem 3 is used in the case of only time-invariant parameters, the same results of Theorem 2 are retrieved. In this case, it is always recommended to use the conditions of Theorem 2 since the conditions of Theorem 3 must be tested for all $(i_1,\ldots,i_m)\in\{1,\ldots,M_1\}\times\cdots\times\{1,\ldots,M_m\}$ that, in the case of frozen parameters, always produce the same (redundant) LMIs.

Remark 2: If the conditions of Theorem 3 are used to analyze the robust stability of an uncertain system where nothing is known about the time-variation of its i-th parameter, the degree of the Λ-homogeneous Lyapunov matrix $P(\alpha)$ associated to this parameter (one of the simplex inside the multi-simplex) is chosen to be zero. In this case, matrices H_i can be anything since they are not used in the algorithm during the construction of the LMIs. On the other hand, the degrees associated to the other parameters can be freely chosen. Such "decoupling" property between time-varying and time-invariant parameters of the system is one of the main advantages of the applications of the results of Theorem 1.

Remark 3: Both Theorems 2 and 3 require that the coefficients of matrix $A(\alpha)$ must be given in the multi-simplex representation. These coefficients can be easily obtained from affine or multi-affine models using the Λ-completion operation. Moreover, in the time-varying case, the conversion of the time-derivative bounds can be done without introducing conservativeness.

VII. Numerical Experiments

The numerical complexity associated to optimizations problems involving LMIs can be estimated as a function of the number V of scalar variables and of the number L of LMI rows. The experiments presented were performed in a PC equipped with: Athlon 64 X2 6000+ (3.0 GHz), 2GB RAM (800 MHz), using SeDuMi [30] and Yalmip [31]. All the information necessary to implement the conditions of Theorems 2 and 3 were given in the paper. However, specially concerning the conditions of Theorem 3, the interested reader may find difficulties during the implementation, because the number of combinations $(i_1,\ldots,i_m)$ is not known *a priori* since m (number of simplexes inside the multi-simplex) is given by the system under consideration. To facilitate the task, a generic code has been implemented and is available for download at `http://www.dt.fee.unicamp.br/~ricfow/robust.htm`. A routine to convert a system originally in the affine form to the multi-simplex representation is available as well.

Example 1: Consider the time-invariant system

$$A(\theta) = A_0 + \theta_1 A_1 + \theta_2 A_2, \quad -1 \le \theta_i \le 1, \quad i = 1,2$$

$$\left[\,A_0 \,\vdots\, A_1 \,\vdots\, A_2\,\right] = \left[\begin{array}{ccc:ccc:ccc} -4 & 2 & -2 & -5 & -3 & -13 & 0 & 2 & 2\\ 5 & -6 & 1 & -5 & 0 & 0 & 0 & 0 & 0\\ -2 & 2 & -7 & 10 & 13 & 16 & 0 & -1 & 0 \end{array}\right].$$

The aim is to test the robust stability of this system using the results of [7], that handles directly the affine model; [10] (OP07), that tests the equivalent polytopic model (2^m vertices); and Theorem 2 that deals with the multi-simplex representation of the system. The minimal degrees necessary to test positively robust stability, the number V (L) of scalar variables (LMI rows) and computational times (in seconds) are ($[7]_{k=3}$, V = 1422, $L = 75$, Time = 31.14); ($[10]_{g=4}$, $V = 210$, $L = 273$, Time = 0.21); (Theorem $2_{(4,2)}$, $V = 90$, $L = 117$, Time = 0.09). As it can be seen, the conditions of Theorem 2 provide the best results in terms of the numerical complexity, achieving a positive evaluation of robust stability with the partial degrees $g = (4,2)$.

Example 2: Consider the system $\dot{x}(t) = A(\theta)x(t)$ with $A(\theta) = A_0 + \theta_1(t)A_1 + \theta_2(t)A_2 + \theta_3 A_3$ and

$$\left[\,A_0 \,\vdots\, A_1 \,\vdots\, A_2 \;\; A_3\,\right] = \left[\begin{array}{cc:cc:cc:cc} -2 & 1 & 2 & 1 & 1 & 1 & 0 & 0\\ -1 & -3 & 0 & -1 & -1 & 1 & 0 & -1 \end{array}\right],$$

where the parameter $\theta_1(t)$ is time-varying with bounded rate of variation, $\theta_2(t)$ can vary arbitrarily fast (unknown time-variation) and θ_3 is time-invariant. The aim here is to determine the maximum variation rate γ of the parameter $\theta_1(t)$, i.e.$|\dot{\theta}_1| \le \gamma_{max}$ such that the system is robustly stable. Table I shown the robust stability analysis results provided by Theorem 3 using different values for the partial degrees of the Λ-homogeneous Lyapunov matrix associated to the parameters $\theta_1(t)$ and θ_3.

It is important to mention that the conditions of Theorem 3 are of increasing precision, yielding less and less conservative results as the degrees increase. But, undoubtedly, the most important trend associated to Theorem 3 is its ability to handle independently the parameters of the system in terms of allowing different degrees for the Λ-homogeneous Lyapunov matrix. The last two rows of Table I indicate that the estimation of the maximum variation rate of parameter $\theta_1(t)$ is insensitive to the degree of the Λ-homogeneous Lyapunov matrix associated to the parameter θ_3. If $g_3 = 1$, the same results are obtained when g_1 increases, demanding less computational effort. To the authors knowledge, this

TABLE I

ROBUST STABILITY ANALYSIS RESULTS OF EXAMPLE 3 USING THEOREM 2 WITH DIFFERENT PARTIAL DEGREES g_1 AND g_3 FOR $g_2 = 0$; V SCALAR VARIABLES; L LMI ROWS.

T3 (g_1, g_3)	γ_{max}	V	L	Time
$(1,1)$	2.277	12	80	0.06 s
$(2,2)$	3.326	27	146	0.06 s
$(3,3)$	3.970	48	232	0.08 s
$(4,4)$	4.267	75	338	0.14 s
$(5,5)$	4.507	108	464	0.20 s
$(6,6)$	4.650	147	610	0.26 s
$(7,7)$	4.737	192	776	0.41 s
$(8,8)$	4.810	243	962	0.52 s
$(8,1)$	4.810	54	276	0.15 s
$(1,8)$	2.277	54	276	0.10 s

is the first method to allow such degree of flexibility for the problem of robust stability analysis of uncertain linear systems.

VIII. CONCLUSION

Existence results for robust LMIs with parameters lying into a multi-simplex were presented. As illustrated for simplicity with robust stability analysis of uncertain linear systems, this setting allows to take into account in a unified and flexible way time-varying parameters and time-invariant parameters, without adding in itself supplementary conservatism.

Future investigations on this topic include the design of controllers that can take advantage of the "decoupling" effect between time-invariant and time-varying parameters.

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