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A dynamic estimator on $SE(3)$ using Range-Only measurements

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Abstract— **This paper addresses the problem of estimating the attitude and position of a rigid body when the available measurements consist only of the relative distances between a set of body fixed beacons and a set of Earth fixed landmarks. The proposed solution is given in terms of a dynamical system evolving on the Special Euclidean group $SE(3)$, the trajectories of which are shown to locally converge to the actual attitude and position of the rigid body. Local asymptotic stability of the dynamical system is proven by using a suitable Lyapunov function, under the assumption that there is a set of noncoplanar landmarks and beacons. Simulation results are shown to illustrate the behaviour of the proposed estimator.**

I. Introduction

The problem of estimating the attitude and the position of a rigid body by using only relative distance measurements between a set of body fixed beacons and Earth fixed landmarks arises often in practice. Examples of applications include indoor localization systems based on wireless networks, determination of the attitude/position of a body underwater resorting to acoustics, and computation of the forward kinematics of Stewart-Gough platforms [1]. In some cases, range-only attitude and positioning systems represent an alternative, or a complement, to more expensive and sophisticated inertial navigation systems with the clear advantage of being robust to magnetic disturbances and temporal drifts. Relative distance measurements are usually obtained by measuring the time it takes an electromagnetic or acoustic signal to travel between an emitter and a receiver given that the speed of propagation of the signals is known.

The attitude estimation problem has attracted the attention of the scientific community for a long time. The most usual setup is that in which vector observations are considered [2] [3] [4]. In some applications, vector observations can be obtained from range observations by making the planar waveform approximation [5] [6]. Despite the fact that the attitude and positioning problems are strongly coupled, a simultaneous treatment of the problem is seldom encountered in the literature (see [7] for an example with landmark observations, and [8] for an example using line of sight observations).

There are two major issues that make the range-only attitude and positioning problem quite challenging. First, the highly nonlinear nature of the range observations and second, the non-Euclidean geometry of the Special Euclidean group $SE(3)$. The problem of simultaneous range-only attitude and position estimation was addressed in a Maximum likelihood framework in [9] by formulating a constrained optimization problem on the Special Euclidean group $SE(3)$ and resorting to generalized intrinsic gradient and Newton algorithms. The performance of the derived estimator is very close to the theoretical bounds provided by the Intrinsic Variance Lower Bound IVLB [10]. However, no convergence warranties were given and simulations revealed the existence of local minima. Some of the difficulties encountered may originate in the complex structure of the maximum likelihood cost function which contained squared roots, cross-term products, and was not even differentiable at some points.

The present paper presents a novel approach to the range-only attitude and positioning problem that aims at overcoming some of the difficulties encountered. A modified cost function is considered based on the square of the range observations that exhibits good properties and helps decouple the attitude and position estimation errors. The presented solution is inspired by the ideas in [11] [12] [7] [4] where, in some cases, parameter estimation and signal processing problems are solved by resorting to special classes of dynamical systems, the properties of which can be analyzed from a system theoretic point of view. It is hoped that the dynamic formulation given here will give some insight to the solution of more complex problems that arise when the rigid body undergoes motion in space.

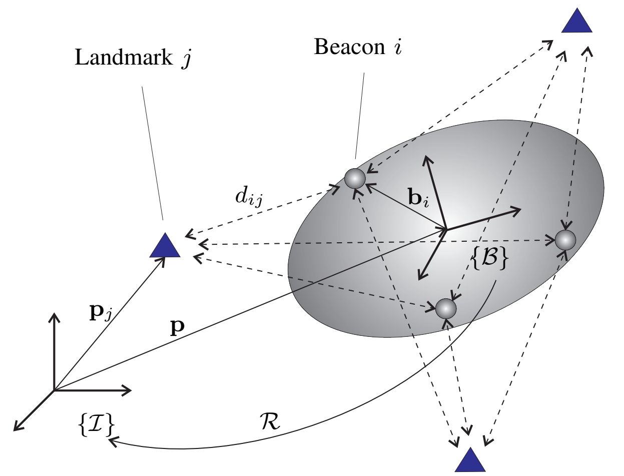

Fig. 1. Estimation of the attitude and position $(\mathcal{R}, \mathbf{p}) \in SE(3)$ of a rigid body using only relative distance measurements d_{ij}.

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Using a suitable Lyapunov candidate function based on the range observations, and under certain conditions, local asymptotical convergence of the position and attitude estimates to the true values is proven. The conditions required to obtain this result consist in having at least a set of noncoplanar landmarks and placing the origin of the body reference frame at the centroid of the beacons. In spite of the result obtained being local, close examination of the literature indicates that it is not trivial in that it addresses explicitly the fact that only range measurements are available. Furthermore, and because the problem is directly formulated in SE(3), a non global result is expected if one draws an analogy with control problems. In fact, the results in [13] [14], [15] show that due to the non-Euclidean nature of SE(3) it is not possible to render a system evolving on this manifold globally asymptotically stable (GAS) by resorting to continuous feedback control laws. This is only possible when the state space is diffeomorphic to $\mathbb{R}^n$, which is not the case for $SE(3)$. At most, one may expect to obtain an Almost Global Asymptotic Stability (AGAS) type of result as in [7] and show that the region of attraction is the entire state space except for a nowhere dense set of zero measure [13] [14]. Although not proven analytically, simulation results obtained with the estimator here proposed did not reveal the existence of local minima and suggest that, under the noncoplanarity conditions, the presented estimator may exhibit AGAS, a conjecture that warrants further research.

II. Problem Formulation

Suppose that one is interested in estimating the configuration (that is, position and attitude) of a rigid body in space. To this effect, define a Inertial reference frame $\{\mathcal{I}\}$ and a reference frame $\{\mathcal{B}\}$ attached to the rigid body. Let $\mathbf{p} \in \mathbb{R}^3$ denote the position of the origin of $\{\mathcal{B}\}$ with respect to $\{\mathcal{I}\}$ expressed in $\{\mathcal{I}\}$ and let $\mathcal{R} \in SO(3)$ denote the rotation matrix from $\{\mathcal{B}\}$ to $\{\mathcal{I}\}$, where

$$SO(3) = \left\{\mathcal{R} \in \mathbb{R}^{3\times 3} \,:\, \mathcal{R}^T\mathcal{R} = \mathbf{I}_3,\ \det(\mathcal{R}) = 1\right\} \quad (1)$$

is the Special Orthogonal group [16]. In the above expression, $\mathbf{I}_3$ stands for the 3×3 identity matrix and $\det(\cdot)$ is the matrix determinant operator. The attitude and position of the rigid body $(\mathcal{R}, \mathbf{p})$ can then be identified with an element of the Special Euclidean group $SE(3) = SO(3)\times\mathbb{R}^3$. There is a vast and rich literature on $SE(3)$, also referred as the group of rigid body motions, a smooth manifold that is not globally diffeomorphic to $\mathbb{R}^n$ and that can be given the structure of a Lie group [17] [16] [15] [18].

Let $\mathfrak{so}(3)$ denote the set of 3×3 skew-symmetric matrices with real entries, i.e. $\mathfrak{so}(3) = \left\{K \in \mathbb{R}^{3\times 3} : K + K^T = 0\right\}$. Define the map $\mathcal{S} : \mathbb{R}^3 \to \mathfrak{so}(3)$ and its inverse $\mathcal{S}^{-1} : \mathfrak{so}(3) \to \mathbb{R}^3$ by

$$\mathcal{S}\left(\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}\right) = \begin{bmatrix}0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{bmatrix}, \quad (2)$$

$$\mathcal{S}^{-1}\left(\begin{bmatrix}0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{bmatrix}\right) = \begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}. \quad (3)$$

Suppose the rigid body has attached a set of solidary beacons with known positions $\mathbf{b}_i \in \mathbb{R}^3, i \in \{1,\ldots,p\}$ expressed in $\{\mathcal{B}\}$. Let us further consider that there is a set of fixed landmarks distributed in the ambient space with known positions $\mathbf{p}_j \in \mathbb{R}^3, j \in \{1,\ldots,m\}$ expressed in $\{\mathcal{I}\}$ (see Fig. 1). Suppose one measures d_{ij}, the square of the distance between the i'th beacon and the j'th landmark, defined as

$$d_{ij} = \|\mathcal{R}\mathbf{b}_i + \mathbf{p} - \mathbf{p}_j\|^2. \quad (4)$$

Inspired by the work in [11] [12] [7] [4], one possible solution to the range-only attitude and positioning problem is to use a dynamical system, or adaptive identifier, the trajectories of which converge asymptotically to the actual attitude and position. This paper will focus on this class of solutions. We are now ready to formulate the problem rigorously.

Problem statement : Consider a static rigid body with attitude and position represented by $(\mathcal{R}, \mathbf{p}) \in SE(3)$. Suppose one measures the squared distances d_{ij} between a set of body-fixed beacons with positions $\mathbf{b}_i;\ i \in \{1,\ldots,p\}$ expressed in $\{\mathcal{B}\}$ and a set of Earth-fixed landmarks with positions $\mathbf{p}_j;\ j \in \{1,\ldots,m\}$ expressed in $\{\mathcal{I}\}$ (see Fig.1). Consider the dynamical system

$$\begin{cases}\dot{\widehat{\mathbf{p}}} = \widehat{\mathbf{v}}\\ \dot{\widehat{\mathcal{R}}} = \widehat{\mathcal{R}}\mathcal{S}(\widehat{\boldsymbol{\omega}})\end{cases} \quad (5)$$

with initial conditions $(\widehat{\mathcal{R}}(0), \widehat{\mathbf{p}}(0)) \in SE(3)$. Compute the functions $\widehat{\mathbf{v}} = \widehat{\mathbf{v}}(\widehat{\mathcal{R}}, \widehat{\mathbf{p}}, d_{11}, ..., d_{pm}) \in \mathbb{R}^3$ and $\widehat{\boldsymbol{\omega}} = \widehat{\boldsymbol{\omega}}(\widehat{\mathcal{R}}, \widehat{\mathbf{p}}, d_{11}, ..., d_{pm}) \in \mathbb{R}^3$ such that the estimated attitude and position $(\widehat{\mathcal{R}}(t), \widehat{\mathbf{p}}(t))$ generated by (5) converge to the actual rigid body attitude and position $(\mathcal{R}, \mathbf{p}) \in SE(3)$ as t tends to infinity.

Note that the pair $(\widehat{\mathcal{R}}\mathcal{S}(\widehat{\boldsymbol{\omega}}), \widehat{\mathbf{v}})$ is a valid tangent vector of $SE(3)$ at $(\widehat{\mathcal{R}}, \widehat{\mathbf{p}})$, which means that (5) defines a flow on the Special Euclidean group [16] [18] [12]. Hence, ideal integration of the system equations will produce an estimate that evolves naturally on $SE(3)$ without the need to chose a particular parametrization or to resort to normalization schemes.

Define the error rotation $\mathcal{R}_e = \widehat{\mathcal{R}}^T\mathcal{R} \in SO(3)$ and the estimation errors

$$\begin{cases}\widetilde{\mathbf{p}} = \mathbf{p} - \widehat{\mathbf{p}}\\ \widetilde{\mathcal{R}} = \mathcal{R}_e - \mathbf{I}_3 = \widehat{\mathcal{R}}^T\mathcal{R} - \mathbf{I}_3.\end{cases} \quad (6)$$

Note that $\widetilde{\mathbf{p}} \in \mathbb{R}^3$ but, in general, $\widetilde{\mathcal{R}} \notin SO(3)$. Assuming a static rigid body ($\dot{\mathcal{R}} = 0$ and $\dot{\mathbf{p}} = 0$), the error dynamics can be written from (5)-(6) as

$$\begin{cases}\dot{\widetilde{\mathbf{p}}} = -\widehat{\mathbf{v}} \\ \dot{\widetilde{\mathcal{R}}} = -\mathcal{S}(\widehat{\boldsymbol{\omega}})\widehat{\mathcal{R}}^T\mathcal{R} = -\mathcal{S}(\widehat{\boldsymbol{\omega}})(\widetilde{\mathcal{R}}+\mathbf{I}_3).\end{cases} \tag{7}$$

An equivalent formulation of the problem statement can be made using the error variables $(\widetilde{\mathcal{R}}, \widetilde{\mathbf{p}})$ as follows. Determine $\widehat{\mathbf{v}}$ and $\widehat{\boldsymbol{\omega}}$ as functions of the current estimates $(\widehat{\mathcal{R}}, \widehat{\mathbf{p}})$ and the measurements d_{ij} such that $\lim_{t\to\infty}(\widetilde{\mathcal{R}}(t), \widetilde{\mathbf{p}}(t)) = (0,0)$.

III. Statement of Main Result

Let $\mathbf{P} = [\mathbf{p_1} \ \dots \ \mathbf{p}_m] \in \mathbb{R}^{3\times m}$, and $\mathbf{B} = [\mathbf{b_1} \ \dots \ \mathbf{b}_p] \in \mathbb{R}^{3\times p}$ be matrices containing the landmark and beacon coordinates, respectively. Define the centering matrices

$$\mathbf{M}_m = \mathbf{I}_m - \frac{1}{m}\mathbf{1}_m\mathbf{1}_m^T \in \mathbb{R}^{m\times m}, \tag{8}$$

$$\mathbf{M}_p = \mathbf{I}_p - \frac{1}{p}\mathbf{1}_p\mathbf{1}_p^T \in \mathbb{R}^{p\times p}, \tag{9}$$

where $\mathbf{1}_k \in \mathbb{R}^k$, $k \in \{m,p\}$ denotes a vector of ones. The centering matrices are projection operators and satisfy $\mathbf{M}_k = \mathbf{M}_k^T$, $\mathbf{M}_k\mathbf{M}_k = \mathbf{M}_k$, and $\mathbf{M}_k\mathbf{1}_k = 0$. Let $\mathbf{P}_c = \mathbf{P}\mathbf{M}_m$ be a matrix containing the landmark coordinates expressed with respect to the landmark centroid. Similarly, define $\mathbf{B}_c = \mathbf{B}\mathbf{M}_p$ as the matrix containing the centered beacon coordinates. Further define the matrices

$$\mathbf{D} := \begin{bmatrix} d_{11} & \dots & d_{p1} \\ \vdots & \ddots & \vdots \\ d_{1m} & \dots & d_{pm}\end{bmatrix} \in \mathbb{R}^{m\times p},$$

$$\widehat{\mathbf{D}} := \begin{bmatrix} \hat{d}_{11} & \dots & \hat{d}_{p1} \\ \vdots & \ddots & \vdots \\ \hat{d}_{1m} & \dots & \hat{d}_{pm}\end{bmatrix} \in \mathbb{R}^{m\times p} \tag{10}$$

containing the actual and estimated square range measurements, respectively, between landmarks and beacons, i.e., with entries $d_{ij} = \|\mathcal{R}\mathbf{b}_i + \mathbf{p} - \mathbf{p}_j\|^2$ and $\widehat{d}_{ij} = \|\widehat{\mathcal{R}}\mathbf{b}_i + \widehat{\mathbf{p}} - \mathbf{p}_j\|^2$.

Assume the following assumptions hold:

Assumption 1: The body reference frame $\{\mathcal{B}\}$ has its origin at the centroid of the beacons.

Note that if this is true, then $\mathbf{B}_c = \mathbf{B}$ and $\mathbf{B}\mathbf{1}_p = \mathbf{B}_c\mathbf{1}_p = \mathbf{B}\mathbf{M}_p\mathbf{1}_p = 0$. This assumption is not restrictive and will help greatly to simplify the development.

Assumption 2: There is a set of noncoplanar beacons and landmarks.

Since three noncollinear points in $\mathbb{R}^3$ always define a plane, this is equivalent to requiring that at least four beacons and landmarks be noncoplanar. Note that if this assumption is satisfied then it is easy to show that matrices $\mathbf{P}_c$ and $\mathbf{B}_c$ have full column rank.

Consider the following simple Lyapunov function candidate

$$V = \frac{1}{8}\|\mathbf{M}_m(\mathbf{D} - \widehat{\mathbf{D}})\|_F^2 \tag{11}$$

where, given a matrix $\mathbf{A}$, $\|\mathbf{A}\|_F^2 = \operatorname{tr}\left(\mathbf{A}^T\mathbf{A}\right)$ is the matrix Frobenius norm. Note that V depends only on the available range measurements $\mathbf{D}$ and the estimated rigid body attitude and position, since $\widehat{\mathbf{D}} = \widehat{\mathbf{D}}(\widehat{\mathcal{R}}, \widehat{\mathbf{p}})$. This Lyapunov function candidate has an interesting property:

Lemma 1: The function V in (11) can be decomposed in two quadratic terms, one depending only on the actual and estimated position and the other on the attitude estimation error, i.e.,

$$V = \frac{1}{2}\widetilde{\mathbf{p}}^T\boldsymbol{\Theta}_1\widetilde{\mathbf{p}} + \frac{1}{2}\operatorname{vec}(\widetilde{\mathcal{R}})^T\boldsymbol{\Theta}_2\operatorname{vec}(\widetilde{\mathcal{R}}) \tag{12}$$

where $\operatorname{vec}()$ is the operator that stacks the columns of a matrix from left to right,

$$\boldsymbol{\Theta}_1 = m\mathbf{P}_c\mathbf{P}_c^T \in \mathbb{R}^{3\times 3}, \tag{13}$$

$$\boldsymbol{\Theta}_2 = \mathbf{B}\mathbf{B}^T \otimes \widehat{\mathcal{R}}^T\mathbf{P}_c\mathbf{P}_c^T\widehat{\mathcal{R}} \in \mathbb{R}^{9\times 9}, \tag{14}$$

and $\otimes$ denotes the Kronecker product of matrices.

Proof: see Appendix ■

Corollary 1: Suppose assumptions 1-2 are satisfied. Then, V is a positive definite function of the estimation errors.

Proof: According to Lemma 1, it is enough to show that, when 1-2 are satisfied, $\boldsymbol{\Theta}_1$ and $\boldsymbol{\Theta}_2$ are positive definite matrices. If $\mathbf{P}_c, \mathbf{B} \in \mathbb{R}^{3\times m}$ are full column rank, then $\mathbf{P}_c\mathbf{P}_c^T \in \mathbb{R}^{3\times 3}$ and $\mathbf{B}\mathbf{B}^T \in \mathbb{R}^{3\times 3}$ are positive definite. If $\widehat{\mathcal{R}}$ is a rotation matrix it is nonsingular, and therefore $\widehat{\mathcal{R}}^T\mathbf{P}_c\mathbf{P}_c^T\widehat{\mathcal{R}}$ is also positive definite. Since the Kronecker product of two positive definite matrices is also positive definite, the result follows. ■

We now state the main result of the paper:

Theorem 1: Suppose that assumptions 1-2 are fulfilled. Consider the adaptive estimator in (5) with

$$\begin{cases}\widehat{\mathbf{v}} &= -\mathbf{K}_v\mathbf{P}_c(\mathbf{D}-\widehat{\mathbf{D}})\mathbf{1}_p \\ \widehat{\boldsymbol{\omega}} &= -\mathbf{K}_\omega\mathcal{S}^{-1}\left(\boldsymbol{\Psi} - \boldsymbol{\Psi}^T\right),\end{cases} \tag{15}$$

where $\boldsymbol{\Psi} = \frac{1}{2}\mathbf{B}(\mathbf{D}-\widehat{\mathbf{D}})^T\mathbf{P}_c^T\widehat{\mathcal{R}} \in \mathbb{R}^{3\times 3}$ and $\mathbf{K}_v, \mathbf{K}_\omega \in \mathbb{R}^{3\times 3}$ are positive definite matrix gains. Then, the error system (7) has an asymptotically stable equilibrium point at the origin $(\widetilde{\mathcal{R}}, \widetilde{\mathbf{p}}) = (0,0)$.

Proof: Consider the estimation error variables $(\widetilde{\mathcal{R}}, \widetilde{\mathbf{p}})$ defined in (6). The time derivative of the Lyapunov function along the trajectories of the system can be computed as

$$\dot{V} = m\widetilde{\mathbf{p}}^T\mathbf{P}_c\mathbf{P}_c^T\dot{\widetilde{\mathbf{p}}} + \operatorname{tr}\left(\mathbf{P}_c^T(\dot{\widehat{\mathcal{R}}}\widetilde{\mathcal{R}} + \widehat{\mathcal{R}}\dot{\widetilde{\mathcal{R}}})\mathbf{B}\mathbf{B}^T\widetilde{\mathcal{R}}^T\widehat{\mathcal{R}}^T\mathbf{P}_c\right).$$

From the error dynamics in (7),

$$\dot{\widehat{\mathcal{R}}}\widetilde{\mathcal{R}} + \widehat{\mathcal{R}}\dot{\widetilde{\mathcal{R}}} = \widehat{\mathcal{R}}\mathcal{S}(\widehat{\boldsymbol{\omega}})\widetilde{\mathcal{R}} - \widehat{\mathcal{R}}\mathcal{S}(\widehat{\boldsymbol{\omega}})\widehat{\mathcal{R}}^T\mathcal{R} \tag{16}$$

$$= -\widehat{\mathcal{R}}\mathcal{S}(\widehat{\boldsymbol{\omega}}) \tag{17}$$

and

$$\begin{aligned}\dot{V} &= -m\widetilde{\mathbf{p}}^T\mathbf{P}_c\mathbf{P}_c^T\widehat{\mathbf{v}} - \operatorname{tr}\left(\mathbf{P}_c^T\widehat{\mathcal{R}}\mathcal{S}(\widehat{\boldsymbol{\omega}})\mathbf{B}\mathbf{B}^T\widetilde{\mathcal{R}}^T\widehat{\mathcal{R}}^T\mathbf{P}_c\right)\\ &= -m\widetilde{\mathbf{p}}^T\mathbf{P}_c\mathbf{P}_c^T\widehat{\mathbf{v}} - \operatorname{tr}\left(\mathcal{S}(\widehat{\boldsymbol{\omega}})\mathbf{B}\mathbf{B}^T\widetilde{\mathcal{R}}^T\widehat{\mathcal{R}}^T\mathbf{P}_c\mathbf{P}_c^T\widehat{\mathcal{R}}\right)\\ &= -m\widetilde{\mathbf{p}}^T\mathbf{P}_c\mathbf{P}_c^T\widehat{\mathbf{v}} - \boldsymbol{\xi}^T\widehat{\boldsymbol{\omega}} \qquad (18)\end{aligned}$$

where, according to property 1 in the appendix, $\boldsymbol{\xi} := \mathcal{S}^{-1}(\boldsymbol{\Sigma} - \boldsymbol{\Sigma}^T) \in \mathbb{R}^3$ and $\boldsymbol{\Sigma} = \mathbf{B}\mathbf{B}^T\widetilde{\mathcal{R}}^T\widehat{\mathcal{R}}^T\mathbf{P}_c\mathbf{P}_c^T\widehat{\mathcal{R}}$. A natural choice for $\widehat{\mathbf{v}}$ and $\widehat{\boldsymbol{\omega}}$ is

$$\begin{cases}\widehat{\mathbf{v}} &= \mathbf{K}_v\mathbf{P}_c\mathbf{P}_c^T\widetilde{\mathbf{p}},\\ \widehat{\boldsymbol{\omega}} &= \mathbf{K}_\omega\boldsymbol{\xi},\end{cases} \qquad (19)$$

where $\mathbf{K}_v$ and $\mathbf{K}_\omega$ are positive definite matrix gains that make the derivative of V, computed as

$$\dot{V} = -m\widetilde{\mathbf{p}}^T\mathbf{P}_c\mathbf{P}_c^T\mathbf{K}_v\mathbf{P}_c\mathbf{P}_c^T\widetilde{\mathbf{p}} - \boldsymbol{\xi}^T\mathbf{K}_\omega\boldsymbol{\xi}, \qquad (20)$$

negative semidefinite. Moreover, using Lemma 3 from the appendix it can be shown that $\boldsymbol{\xi}$ is different from zero in a neighborhood of $\widetilde{\mathcal{R}} = 0$ and therefore $\dot{V}$ is a negative definite function in a neighborhood of the equilibrium $(\widetilde{\mathcal{R}}, \widetilde{\mathbf{p}}) = (0, 0)$.

It is still necessary to show that the derived estimation laws can actually be computed using only the available information such that (15) and (19) are equivalent. This can be seen by using Lemma 2 in the appendix, thus concluding the proof. ■

IV. Simulation Results

Simulation results using MATLAB are presented to illustrate the behaviour of the adaptive range-only attitude and position estimator. The simulation setup is shown in Fig. 2, where four ($m = 4$) Earth fixed landmarks and a rigid body with four ($p = 3$) beacons were considered. The initial attitude and position estimate $(\widehat{\mathcal{R}}(0), \widehat{\mathbf{p}}(0))$ were set as a random rotation and a random position, respectively as depicted in Fig. 2 and Fig. 5. The estimator differential equations (5) were integrated until $t = 10\ s$ using the gains $\mathbf{K}_v = \mathbf{K}_\omega = \gamma\mathbf{I}_3$ with $\gamma = 10^{-3}$. The actual and estimated attitude and position are plotted in Fig.3. The attitude and position estimation errors are depicted in Fig.4. In order to plot the attitude estimation error, exponential coordinates of the error rotation $\mathcal{R}_e = \widehat{\mathcal{R}}^T\mathcal{R}$ are used, i.e., the error rotation is parameterized by vector $\boldsymbol{\theta} = [\theta_1\ \theta_2\ \theta_3]^T$ where $\mathcal{R}_e = \exp(\mathcal{S}(\boldsymbol{\theta}))$. Note that this is done only for visualization purposes and that no particular parametrization of the Special Orthogonal group $SO(3)$ is used elsewhere in the paper. The residuals $\epsilon_{ij} := |d_{ij} - \hat{d}_{ij}|^{1/2}$ are shown in Fig.6.

V. Conclusions

This paper presented a novel approach to the range-only attitude and position estimation problem. The solution derived is based on a dynamic estimator that defines a flow in the Special Euclidean group $SE(3)$. Since no particular parametrization of the rotation matrices is used, common problems such as singularities and normalization schemes are avoided. Under the assumptions that there is a set of at least 4 noncoplanar beacons and landmarks, local asymptotic stability to the actual attitude and position was proven. Simulations results illustrated the behaviour of the dynamic estimator. Although the present paper considers the 3-dimensional problem, all the derived results can be easily applied with minor modifications to the 2 dimensional attitude and position problem. Future work will include the extension of the estimator derived to deal with rigid bodies undergoing motion in space and the inclusion of bounded disturbances in the range measurements.

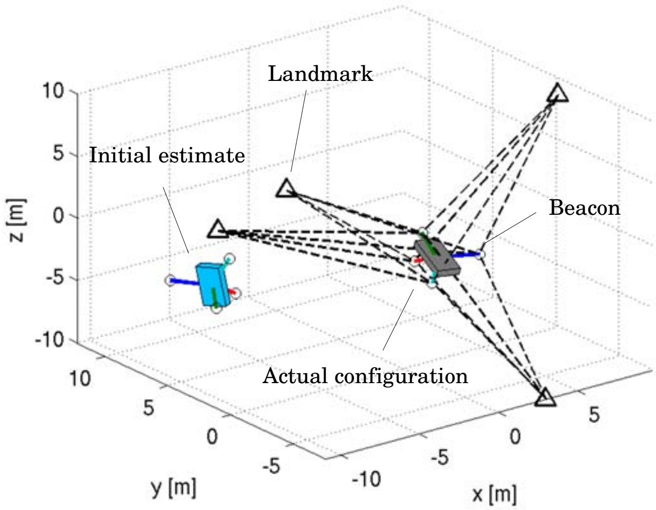

Fig. 2. Simulation setup. Actual and initial estimated rigid body configurations together with beacon and landmark locations.

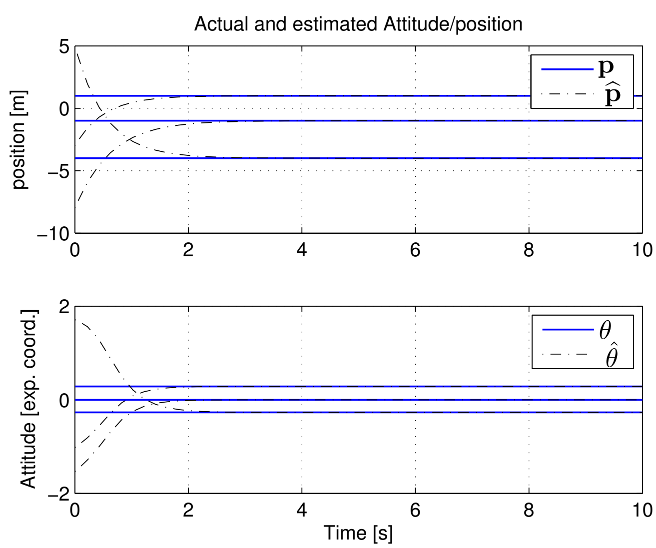

Fig. 3. Actual and estimated position / attitude. (Top) Entries of vectors $\mathbf{p}$, and $\hat{\mathbf{p}}$. (Bottom) Entries of vectors $\boldsymbol{\theta}$ and $\hat{\boldsymbol{\theta}}$, the exponential coordinates of $\mathcal{R}$ and $\hat{\mathcal{R}}$, respectively.

Appendix

Lemma 2: Let $\mathbf{D}$ and $\widehat{\mathbf{D}}$ be the matrices containing the actual and estimated squared ranges, as defined in (10). The

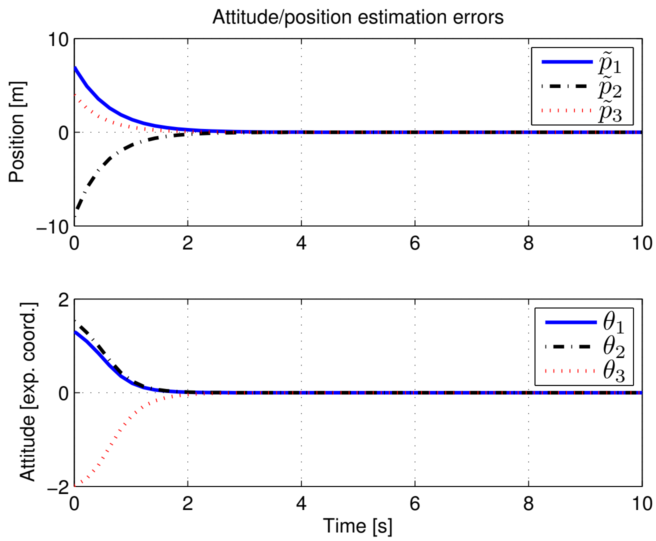

Fig. 4. Position / attitude estimation errors. (Top) Entries of vector $\tilde{\mathbf{p}} = \mathbf{p} - \widehat{\mathbf{p}} = [\tilde{p}_1\ \tilde{p}_2\ \tilde{p}_3]^T$. (Bot.) Entries of vector $\boldsymbol{\theta} = [\theta_1\ \theta_2\ \theta_3]^T$, the exponential coordinates of $\mathcal{R}_e = \mathcal{R}^T\widehat{\mathcal{R}}$, i.e., $\mathcal{R}_e = \exp(\mathcal{S}(\boldsymbol{\theta}))$.

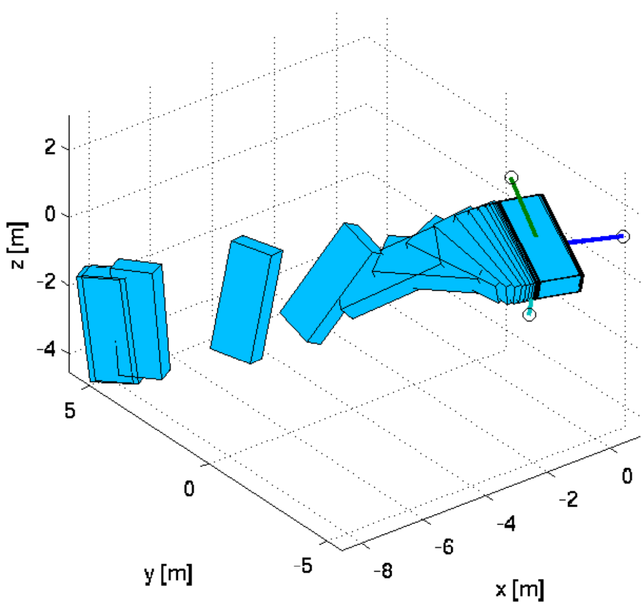

Fig. 5. Time evolution of the estimated rigid body configuration.

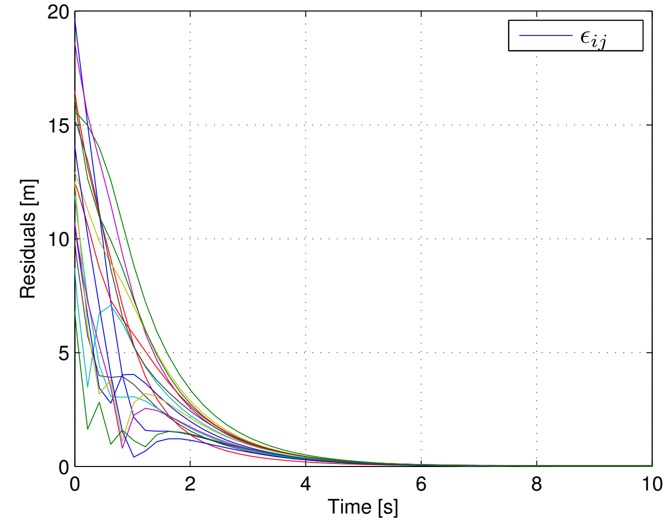

Fig. 6. Time evolution of the square root of the absolute value of the residuals $\epsilon_{ij} := |d_{ij} - \hat{d}_{ij}|^{1/2}$ (in meters).

following equalities hold:

$$\mathbf{M}_m(\mathbf{D} - \widehat{\mathbf{D}}) = -2\mathbf{P}_c^T\widetilde{\mathbf{p}}\mathbf{1}_p^T + 2\mathbf{P}_c^T\widehat{\mathcal{R}}\widetilde{\mathcal{R}}\mathbf{B}, \tag{21}$$

$$\mathbf{M}_m(\mathbf{D} - \widehat{\mathbf{D}})\mathbf{1}_p = -2p\mathbf{P}_c^T\widetilde{\mathbf{p}}, \tag{22}$$

$$\mathbf{M}_m(\mathbf{D} - \widehat{\mathbf{D}})\mathbf{M}_p = -2\mathbf{P}_c^T\widehat{\mathcal{R}}\widetilde{\mathcal{R}}\mathbf{B}. \tag{23}$$

Proof: The entries of $\mathbf{D}$ and $\widehat{\mathbf{D}}$ have the form

$$d_{ij} = (\mathcal{R}\mathbf{b}_i + \mathbf{p} - \mathbf{p}_j)^T(\mathcal{R}\mathbf{b}_i + \mathbf{p} - \mathbf{p}_j),$$
$$\hat{d}_{ij} = (\widehat{\mathcal{R}}\mathbf{b}_i + \widehat{\mathbf{p}} - \mathbf{p}_j)^T(\widehat{\mathcal{R}}\mathbf{b}_i + \widehat{\mathbf{p}} - \mathbf{p}_j).$$

Simple algebraic manipulations show that the ij entry of $\mathbf{D} - \widehat{\mathbf{D}}$ can be written as

$$d_{ij} - \hat{d}_{ij} = (\mathbf{p} + \widehat{\mathbf{p}} - 2\mathbf{p}_j + 2\widehat{\mathcal{R}}\mathbf{b}_i)^T\widetilde{\mathbf{p}} + 2(\mathbf{p} - \mathbf{p}_j)^T\widehat{\mathcal{R}}\widetilde{\mathcal{R}}\mathbf{b}_i.$$

Defining the vector variables

$$\mathbf{d}_i := \begin{bmatrix} d_{i1} \\ \vdots \\ d_{im} \end{bmatrix} \in \mathbb{R}^m, \quad \hat{\mathbf{d}}_i := \begin{bmatrix} \hat{d}_{i1} \\ \vdots \\ \hat{d}_{im} \end{bmatrix} \in \mathbb{R}^m$$

yields

$$\mathbf{d}_i - \hat{\mathbf{d}}_i = \left((\mathbf{p} + \widehat{\mathbf{p}})\mathbf{1}_m^T - 2\mathbf{P}\right)^T\widetilde{\mathbf{p}} + 2\mathbf{1}_m\widetilde{\mathbf{p}}^T\widehat{\mathcal{R}}\mathbf{b}_i + 2(\mathbf{p}\mathbf{1}_m^T - \mathbf{P})^T\widehat{\mathcal{R}}\widetilde{\mathcal{R}}\mathbf{b}_i.$$

As a consequence,

$$\mathbf{D} - \widehat{\mathbf{D}} := \left[\mathbf{d}_1 - \hat{\mathbf{d}}_1 \quad \ldots \quad \mathbf{d}_p - \hat{\mathbf{d}}_p\right]$$

can be written as

$$\mathbf{D} - \widehat{\mathbf{D}} = \left((\mathbf{p} + \widehat{\mathbf{p}})\mathbf{1}_m^T - 2\mathbf{P}\right)^T\widetilde{\mathbf{p}}\mathbf{1}_p^T + 2\mathbf{1}_m\widetilde{\mathbf{p}}^T\widehat{\mathcal{R}}\mathbf{B} + 2(\mathbf{p}\mathbf{1}_m^T - \mathbf{P})^T\widehat{\mathcal{R}}\widetilde{\mathcal{R}}\mathbf{B}.$$

The result follows by using the facts that $\mathbf{M}_m\mathbf{1}_m = 0$, $\mathbf{P}\mathbf{M}_m = \mathbf{P}_c$, $\mathbf{B}\mathbf{1}_p = 0$, $\mathbf{1}_p^T\mathbf{1}_p = p$, and $\mathbf{M}_p\mathbf{1}_p = 0$. ■

Property 1: Let $\mathbf{a} \in \mathbb{R}^3$ and $\mathbf{B} \in \mathbb{R}^{3\times 3}$. Then

$$\mathrm{tr}\,(\mathcal{S}(\mathbf{a})\mathbf{B}) = -\mathbf{a}^T\mathbf{b}, \quad \mathbf{b} = \mathcal{S}^{-1}(\mathbf{B} - \mathbf{B}^T). \tag{24}$$

Proof: [Proof of Lemma 1] Using Lemma 2 and the fact that $\mathbf{B}\mathbf{1}_p = 0$ yields

$$\begin{aligned} V &= \frac{1}{8}\|\mathbf{M}_m(\mathbf{D}-\widehat{\mathbf{D}})\|_F^2 \\ &= \frac{1}{8}\mathrm{tr}\left(\mathbf{M}_m(\mathbf{D}-\widehat{\mathbf{D}})(\mathbf{D}-\widehat{\mathbf{D}})^T\mathbf{M}_m^T\right) \\ &= \frac{1}{2}\mathrm{tr}\left(m\mathbf{P}_c^T\widetilde{\mathbf{p}}\widetilde{\mathbf{p}}^T\mathbf{P}_c + \mathbf{P}_c^T\widehat{\mathcal{R}}\widetilde{\mathcal{R}}\mathbf{B}\mathbf{B}^T\widetilde{\mathcal{R}}^T\widehat{\mathcal{R}}^T\mathbf{P}_c\right) \\ &= \frac{1}{2}m\widetilde{\mathbf{p}}^T\mathbf{P}_c\mathbf{P}_c^T\widetilde{\mathbf{p}} + \frac{1}{2}\mathrm{tr}\left(\mathbf{P}_c^T\widehat{\mathcal{R}}\widetilde{\mathcal{R}}\mathbf{B}\mathbf{B}^T\widetilde{\mathcal{R}}^T\widehat{\mathcal{R}}^T\mathbf{P}_c\right). \end{aligned}$$

The first term is clearly a quadratic function of the position estimation error $\widetilde{\mathbf{p}}$. The second term can also be shown to be a quadratic function of the attitude estimation error. Since for conformable matrices $\mathrm{vec}(\mathbf{ABC}) = (\mathbf{C}^T\otimes\mathbf{A})\mathrm{vec}(\mathbf{B})$, we have

$$\mathrm{vec}(\mathbf{P}_c^T\widehat{\mathcal{R}}\widetilde{\mathcal{R}}\mathbf{B}) = (\mathbf{B}^T\otimes\mathbf{P}_c^T\widehat{\mathcal{R}})\mathrm{vec}(\widetilde{\mathcal{R}}). \quad (25)$$

Moreover, using the identity $\mathrm{tr}\left(\mathbf{A}\mathbf{A}^T\right) = \mathrm{vec}(\mathbf{A})^T\mathrm{vec}(\mathbf{A})$ we obtain

$$\begin{aligned} &\mathrm{tr}\left(\mathbf{P}_c^T\widehat{\mathcal{R}}\widetilde{\mathcal{R}}\mathbf{B}\mathbf{B}^T\widetilde{\mathcal{R}}^T\widehat{\mathcal{R}}^T\mathbf{P}_c\right) = \mathrm{tr}\left(\mathbf{B}^T\widetilde{\mathcal{R}}^T\widehat{\mathcal{R}}^T\mathbf{P}_c\mathbf{P}_c^T\widehat{\mathcal{R}}\widetilde{\mathcal{R}}\mathbf{B}\right) \\ &= \mathrm{tr}\left((\mathbf{P}_c^T\widehat{\mathcal{R}}\widetilde{\mathcal{R}}\mathbf{B})^T\mathbf{P}_c^T\widehat{\mathcal{R}}\widetilde{\mathcal{R}}\mathbf{B}\right) \\ &= \mathrm{vec}(\mathbf{P}_c^T\widehat{\mathcal{R}}\widetilde{\mathcal{R}}\mathbf{B})^T\mathrm{vec}(\mathbf{P}_c^T\widehat{\mathcal{R}}\widetilde{\mathcal{R}}\mathbf{B}) \\ &= \mathrm{vec}(\widetilde{\mathcal{R}})^T(\mathbf{B}^T\otimes\mathbf{P}_c^T\widehat{\mathcal{R}})^T(\mathbf{B}^T\otimes\mathbf{P}_c^T\widehat{\mathcal{R}})\mathrm{vec}(\widetilde{\mathcal{R}}) \\ &= \mathrm{vec}(\widetilde{\mathcal{R}})^T(\mathbf{B}\mathbf{B}^T\otimes\widehat{\mathcal{R}}^T\mathbf{P}_c\mathbf{P}_c^T\widehat{\mathcal{R}})\mathrm{vec}(\widetilde{\mathcal{R}}) \end{aligned}$$

and the result follows. ■

Lemma 3: Suppose that conditions 1-2 are satisfied. Then there exist a neighborhood of $\widetilde{\mathcal{R}} = 0$ in which $\boldsymbol{\xi} \neq 0$ for all $\widetilde{\mathcal{R}} \neq 0$.

Proof: Let $\mathcal{R}_e = \widehat{\mathcal{R}}^T\mathcal{R} \in SO(3)$. Note that since $\widetilde{\mathcal{R}} = \mathcal{R}_e - \mathbf{I}_3$ we can write

$$\begin{aligned} \boldsymbol{\Sigma} &= \mathbf{B}\mathbf{B}^T\widetilde{\mathcal{R}}^T\widehat{\mathcal{R}}^T\mathbf{P}_c\mathbf{P}_c^T\widehat{\mathcal{R}} \\ &= \mathbf{B}\mathbf{B}^T(\mathbf{I}_3-\mathcal{R}_e)\mathcal{R}^T\mathbf{P}_c\mathbf{P}_c^T\mathcal{R}\mathcal{R}_e^T \\ &= \mathbf{B}\mathbf{B}^T(\mathbf{I}_3-\mathcal{R}_e)\mathbf{C}\mathbf{C}^T\mathcal{R}_e^T, \end{aligned}$$

where we defined $\mathbf{C} = \mathcal{R}^T\mathbf{P}_c$. Regarding $\mathcal{R}, \mathbf{B}$, and $\mathbf{P}_c$ as fixed parameters, matrix $\boldsymbol{\Sigma}$ can be seen as a function of the error rotation $\mathcal{R}_e$. Define a map $\phi : SO(3) \to \mathfrak{so}(3)$ as $\phi(\mathcal{R}_e) = \boldsymbol{\Sigma} - \boldsymbol{\Sigma}^T$ and note that $\boldsymbol{\xi} = \mathcal{S}^{-1}(\boldsymbol{\Sigma} - \boldsymbol{\Sigma}^T)$ can be seen as the composition map $\boldsymbol{\xi} = \mathcal{S}^{-1}\circ\phi$. Moreover, note that $\boldsymbol{\xi}(\mathcal{R}_e) = 0$ if and only if $\phi(\mathcal{R}_e) = 0$. The tangent space of $SO(3)$ at the identity matrix $\mathbf{I}_3$ can be identified with the set of skew symmetric matrices $\mathfrak{so}(3)$. The *push forward* of ϕ at $\mathcal{R}_e = \mathbf{I}_3$ is a map $\phi_* : \mathfrak{so}(3) \to \mathfrak{so}(3)$ [17]. After some simplifications this map can be written as

$$\phi_*(\Delta) = -\mathbf{B}\mathbf{B}^T\Delta\mathbf{C}\mathbf{C}^T - \mathbf{C}\mathbf{C}^T\Delta\mathbf{B}\mathbf{B}^T$$

since $\Delta^T = -\Delta$. With some abuse of notation we can write the vectorized version of the *push forward* as $\mathrm{vec}(\phi_*(\Delta)) = \boldsymbol{\Phi}\mathrm{vec}(\Delta)$, where

$$\boldsymbol{\Phi} = -\mathbf{C}\mathbf{C}^T\otimes\mathbf{B}\mathbf{B}^T - \mathbf{B}\mathbf{B}^T\otimes\mathbf{C}\mathbf{C}^T.$$

It is easy to show that $\boldsymbol{\Phi} \in \mathbb{R}^{9\times9}$ is negative definite and therefore invertible. Under assumptions 1-2, matrices $\mathbf{B}\mathbf{B}^T \in \mathbb{R}^{3\times3}$ and $\mathbf{C}\mathbf{C}^T \in \mathbb{R}^{3\times3}$ are positive definite. The Kronecker product of two positive definite matrices is also positive definite, and so is the sum of two positive definite matrices. Hence, $\boldsymbol{\Phi}$ is nonsingular, and we have that $\phi_*(\Delta) = 0$ if and only if $\Delta = 0$. This shows that the push forward of ϕ at $\mathcal{R}_e = \mathbf{I}_3$ is an isomorphism and therefore, using the inverse function theorem [17, p.105], ϕ is a local dipheomorphism. This implies that there is a neighborhood $\Omega \subset SO(3)$ of $\mathcal{R}_e = \mathbf{I}_3$, such that $\phi(\mathcal{R}_e) \neq 0$ for all $\mathcal{R}_e \in \Omega/\mathbf{I}_3$. This in turn implies that $\boldsymbol{\xi}(\mathcal{R}_e) \neq 0$ for all $\mathcal{R}_e \in \Omega/\mathbf{I}_3$. Now since $\widetilde{\mathcal{R}} = \mathcal{R}_e - \mathbf{I}_3$, the result follows. ■

REFERENCES

[1] J. Merlet, "Direct kinematics of parallel manipulators," *IEEE Trans. on Robotics and Automation, 9(6):842–846*, December 1993.

[2] G. Wahba, "A Least-Squares Estimate of Spacecraft Attitude," *SIAM Review, Vol. 7, No.3*, p. 409, July 1965.

[3] A. Nadler, Y. Bar-Itzhack, and H. Weiss, "On Algorithms for Attitude Estimation using GPS," in *Proceedings of the 2000 IEEE Conference on Decision and Control, Sidney, Australia*, December 2000, pp. 3321–3326.

[4] J. Kinsey and L. Whitcomb, "Adaptive Identification on the Group of Rigid-Body Rotations and its Application to Underwater Vehicle Navigation," *IEEE Trans. on Robotics , 23(1):124–136*, Feb. 2007.

[5] C. E. Cohen, "Attitude Determination Using GPS," Ph.D. dissertation, Dept. of Aeronautics and Astronautics, Stanford Univ., Stanford, CA., December 1992.

[6] J. L. Crassidis and F. L. Markley, "New Algorithm for Attitude Determination Using Global Positioning System Signals," *AIAA Journal of Guidance, Control, and Dynamics, Vol. 20, No. 5*, pp. 891–896, Sept.-Oct. 1997.

[7] R. Cunha, C. Silvestre, and J. Hespanha, "Output feedback control for stabilization on SE(3)," in *Proceedings of the 45th IEEE Conference on Decision and Control, San Diego, USA*, December 2006, pp. 3825–3830.

[8] J. L. Crassidis, R. Alonso, and J. L. Junkins, "Optimal Attitude and Position Determination from Line-of-Sight Measurements," *The Journal of the Astronautical Sciences, Vol. 48, Nos. 2-3*, pp. 391–408, April-Sept. 2000.

[9] A. Alcocer, P. Oliveira, A. Pascoal, and J. Xavier, "Estimation of Attitude and Position from Range only Measurements using Geometric Descent Optimization on the Special Euclidean Group," in *International Conference on Information Fusion, FUSION'06, Florence, Italy.*, July 2006.

[10] J. Xavier and V. Barroso, "Intrinsic Variance Lower Bound (IVLB): an extension of the Cramér-Rao bound to Riemannian manifolds," in *IEEE Int. Conf. on Acoust., Sp. and Sig. Proc. (ICASSP)*, March 2005.

[11] R. W. Brockett, "Dynamical systems that sort lists, diagonalize matrices and solve linear programming problems," in *Proceedings of the 27th IEEE Conference on Decision and Control, Austin, TX, USA*, December 1988, pp. 799–803.

[12] U. Helmke and J. Moore, *Optimization and Dynamical Systems*. London: Springer-Verlag, 1994, ISBN: 4-540-19857-1.

[13] M. Malisoff, M. Krichman, and E. Sontag, "Global stabilization for systems evolving on manifolds," *Journal of Dynamical and Control Systems , 12(2):161–184*, April. 2006.

[14] D. Angeli, "Almost global stabilization of the inverted pendulum via continuous state feedback," *Automatica, 12(2):161–184*, vol. 37, pp. 1103–1108, July 2001.

[15] F. Bullo and A. Lewis, *Geometric Control of Mechanical Systems*. Springer Verlag, 2004.

[16] R. M. Murray, Z. Li, and S. S. Sastry, *A Mathematical Introduction to Robotic Manipulation*. Boca Raton, FL, USA: CRC Press, 1994.

[17] J. M. Lee, *Introduction to Smooth Manifolds*. Springer Verlag, New York, 2003.

[18] F. C. Park, "Distance Metrics on the Rigid Body Motions with Applications to Mechanism Design," *Transactions of the ASME Journal of Mechanical Design, Vol.117(1), pp:48–54.*, March 1995.