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Stability analysis of linear hyperbolic systems with switching parameters and boundary conditions

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***Abstract*— We study asymptotic stability of an infinite dimensional system that switches between a finite set of modes. Each mode is governed by a system of one-dimensional, linear, hyperbolic partial differential equations on a bounded space interval. The switching system is fairly general in that the space dependent system matrix functions as well as the boundary conditions may switch in time. For the case in which the switching occurs between subsystems in canonical diagonal form, we provide two sets of sufficient conditions for asymptotic stability under arbitrary switching signals. These results are direct generalizations of the corresponding results for the unswitched case. Furthermore, we provide an explicit dwell-time bound on the switching signals that guarantee asymptotic stability of the switched system under the assumption that each of the subsystems are stable. Our results of stability under arbitrary switching generalize to the case where switching occurs between non-diagonal hyperbolic systems that are diagonalizable using a common transformation. For the case where no such transformation exists, we prove existence of a dwell-time bound on the switching signals such that asymptotic stability is guaranteed. To motivate our study, we discuss a potential application to stability of water flow in one-dimensional open channels governed by linearized Saint-Venant equations.**

I. INTRODUCTION

Flows in physical infrastructure networks such as transportation systems [1], irrigation canal systems [2], [3], and gas distribution systems [4] can be modeled by systems of hyperbolic conservation laws in one spatial dimension. These physical networked systems can be monitored and controlled at nodes by *supervisory control and data acquisition* (SCADA) systems [5]. A common control problem studied in the context of these conservation laws is the problem of stability and stabilization under boundary control actions. Recent years have witnessed a significant amount of research activity on this topic [6], [7], [8], [2], [3], [9], [10].

From a practical point-of-view, it is of interest to consider situations in which during the period of operation, the parameters of the system exhibit switching in time triggered by external factors [11]. In addition, a controller based on externally specified logical rules may switch between one of the several possible control actions. The present article focuses on stability properties of hyperbolic conservation laws in bounded domains, where the system's parameters and the boundary conditions may (autonomously) switch in time.

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This switched initial boundary value problem is posed as a hybrid system problem on an infinite dimensional state space. While hybrid systems in which modes are governed by *ordinary differential equations* (ODEs) and *differential algebraic equations* (DAEs) in $\mathbb{R}^n$ are extensively considered in the literature [11], [12], hybrid systems in which modes are governed by *partial differential equations* (PDEs) represent a relatively unexplored and potentially rich field of study [13]. In general, systems modeled by PDEs may exhibit hybrid behavior in a variety of ways: switching sequentially in time or sequentially in space or distributed in the space/space-time domain. Here, we focus on the case in which switching occurs in time as the system we consider allows an abstract ODE treatment using semigroup theory. Our work is different from earlier results on stability of infinite-dimensional switching systems [14], [15] in that our analysis accounts for boundary conditions. We consider the main driving problems:

(A) Find conditions that guarantee asymptotic stability of the switched PDE for arbitrary switching signals.
(B) Alternatively, characterize a (preferably large) class of switching signals such that the switched PDE is asymptotically stable.

These problems are relevant when the switching mechanism is either unknown or too complicated for a more careful stability analysis, in particular when the switching happens autonomously as for instance in networked transport systems [16]. For the PDE under consideration here, the switching may either affect the advective velocities or the boundary conditions or both. We can thus expect that a potentially de-stabilizing switching of the advective velocities can be compensated by stabilizing boundary conditions and vice-versa.

The article is organized as follows. In Section II we consider switching the hyperbolic system in canonical diagonal form and derive a joint spectral radius sufficient condition in view of problem (A). In Section III, we switch non-diagonal systems and show that the joint spectral radius condition from the former section is no longer sufficient. However, we obtain existence of a dwell-time such that the system is asymptotically stable for slow enough switching in view of problem (B). A potential application to the linearized Saint Venant equation is discussed in Section IV. Some final remarks are given in Section V.

II. STABILITY OF SWITCHED HYPERBOLIC SYSTEM IN DIAGONAL FORM

A. Switched hyperbolic system in diagonal form

We consider a switched system in which the dynamics in each mode are governed by a system of linear hyperbolic PDEs in diagonal form and the mode switches in time $t \geq 0$ occurs according to a switching signal $\sigma(\cdot)$ with $\sigma(t) \in \mathcal{Q} \simeq \{1, \ldots, N\}$:

$$\begin{cases} \dfrac{\partial}{\partial t}\begin{pmatrix}\xi_I(t,s)\\ \xi_{II}(t,s)\end{pmatrix} + \Lambda^{\sigma(t)}(s)\dfrac{\partial}{\partial s}\begin{pmatrix}\xi_I(t,s)\\ \xi_{II}(t,s)\end{pmatrix} = 0 \\ \xi_{II}(t,a) = G_L^{\sigma(t)}\xi_I(t,a),\ \xi_I(t,b) = G_R^{\sigma(t)}\xi_{II}(t,b) \\ \xi(0,s) = \bar{\boldsymbol{\xi}}(s) \end{cases} \quad (1)$$

for the unknown function $\xi(t,s) = (\xi^1(t,s), \ldots, \xi^n(t,s))^\top$ partitioned as $(\xi_I^\top(t,s), \xi_{II}^\top(t,s))^\top$ with $\xi_I \in \mathbb{R}^m$, $\xi_{II} \in \mathbb{R}^{n-m}$ on the space-time strip $\mathcal{T} \times [a,b]$, $\mathcal{T} = \{t \geq 0\}$, where for all *modes* $j \in \mathcal{Q}$

- (H_1) $\Lambda^j(s) = \mathrm{diag}(\Lambda_I^j(s), \Lambda_{II}^j(s)) \in \mathbb{R}^{n\times n}$ with $\Lambda_I^j(s) = \mathrm{diag}(\lambda_1^j(s), \ldots, \lambda_{m_j}^j(s)) < 0$, $\Lambda_{II}^j(s) = \mathrm{diag}(\lambda_{m_j+1}^j(s), \ldots, \lambda_n^j(s)) > 0$ and $\lambda_i^j(\cdot) \in C^1([a,b])$ for $i = 1, \ldots, n$ and $1 < m_j < n$ specify the *advective velocities*;
- (H_2) $G_L^j \in \mathbb{R}^{(n-m_j)\times m_j}$ and $G_R^j \in \mathbb{R}^{m_j \times (n-m_j)}$ specify the *boundary data*;

and where

- (H_3) $\sigma(\cdot)$: $\mathcal{T} \longrightarrow \mathcal{Q}$ is a piecewise constant *switching signal* with switching times $\tau_k \in \mathcal{T}$ ($k \in \mathbb{N}$) at which $\sigma(\cdot)$ discontinuously switches from one mode $j \in \mathcal{Q}$ to another mode $j' \in \mathcal{Q}$, denoted as $j \curvearrowright j'$ (piecewise constant meaning that there are only finitely many switches $j \curvearrowright j'$ in each finite time interval of $\mathcal{T}$).

Subsequently, we make use of the following hypothesis

- (H_4) Let $\dim(\Lambda_I^j) = \dim(\Lambda_I^{j'})$, i.e. $m_j = m_{j'}$, for all $j, j' \in \mathcal{Q}$.

In each mode $j \in \mathcal{Q}$, along the C^1-curve defined by the equation

$$\frac{ds}{dt} = \lambda_i^j(s) \quad (2)$$

the component ξ^i of the solution (ξ_I, ξ_{II}) remains constant; see Figure 1 (a). The above equations have for given initial conditions a unique solution because (H_1) implies a Lipschitz-bound of λ_i^j on $[a,b]$. For the switched system, these characteristic curves become *characteristic paths*, given by solutions of the switched ODE (2) with $j = \sigma(t)$. The form of boundary conditions in system (1) arises in many applications and is called reflecting boundary conditions; see Figure 1 (b).

We consider the dynamics of (1) in the Hilbert space $\mathcal{H} = (L^2[a,b])^n$ with norm $\|\cdot\|_2$ and define for a fixed $j \in \mathcal{Q}$ the following unbounded operator $\mathcal{A}^j$: $D^j(\mathcal{A}^j)(\subset \mathcal{H}) \longrightarrow \mathcal{H}$ by

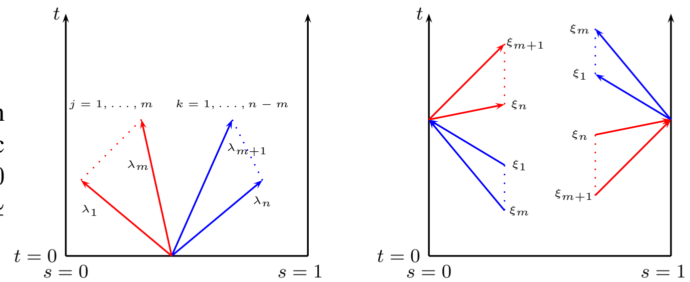

Fig. 1. (a) Characteristic lines. (b) Reflecting boundary conditions.

$$\begin{cases} \mathcal{A}^j\begin{pmatrix}\xi_I(s)\\ \xi_{II}(s)\end{pmatrix} = -\Lambda^j(s)\dfrac{\partial}{\partial s}\begin{pmatrix}\xi_I(s)\\ \xi_{II}(s)\end{pmatrix}, \\ D^j(\mathcal{A}^j) = \left\{\begin{pmatrix}\xi_I\\ \xi_{II}\end{pmatrix} \in (H^1[a,b])^{m_j} \times (H^1[a,b])^{n-m_j} \mid \right. \\ \left. \xi_{II}(a) = G_L^j\xi_I(a),\ \xi_I(b) = G_R^j\xi_{II}(b)\right\}. \end{cases} \quad (3)$$

With this operator, the system (1) can be written as a switched evolution equation on $\mathcal{H}$:

$$\frac{d\boldsymbol{\xi}(t)}{dt} = \mathcal{A}^{\sigma(t)}\boldsymbol{\xi}(t), \quad t > 0 \quad (4)$$

with $\boldsymbol{\xi}(t) = (\xi_I(t,\cdot)^\top, \xi_{II}(t,\cdot)^\top)^\top$.

The following result is well-known [6]:

Lemma 1: For a fixed $j \in \mathcal{Q}$, the operator (3) generates a C_0-semigroup $\{T^j(t)\}_{t\geq 0}$ on $\mathcal{H}$. □

Thus for a given initial condition $\bar{\boldsymbol{\xi}} \in \mathcal{H}$, the solution $\boldsymbol{\xi}(\cdot) \in C([0,\infty), \mathcal{H})$ of the switched system (1) can be represented as

$$\boldsymbol{\xi}(t) = T^{\sigma(\tau_K)}(t-\tau_K)\cdots T^{\sigma(\tau_1)}(\tau_2-\tau_1)T^{\sigma(0)}(\tau_1)\bar{\boldsymbol{\xi}} \quad (5)$$

with $\tau_K = \max_{k\in\mathbb{N}}\{\tau_k \mid \tau_k < t\}$.

Remark 1: If we assume the initial condition to be piecewise continuously differentiable, $\bar{\boldsymbol{\xi}} \in \mathcal{PC}^1([a,b], \mathbb{R}^n)$, then a solution of switched system (1) also inherits the same property [17]. □

B. Stability of switched hyperbolic system in diagonal form

We consider stability and stabilizability of the switching system (1), motivated by a simple PDE counterpart to the classical ODE observation [11] that asymptotic stability of all subsystems is *not* sufficient for the asymptotic stability of the switched system:

Example 1: Consider system (1) with $\mathcal{Q} = \{1,2\}$, $\Lambda^j = \mathrm{diag}(-1,+1)$, $[a,b] = [0,1]$, $G_L^j = 1.5(j-1)$, $G_R^j = 1.5(2-j)$ and $\bar{\boldsymbol{\xi}}(\cdot) \equiv 1$. For the case of no switching, the solution of the system $\boldsymbol{\xi}(\cdot)$ for $j = 1$ and $j = 2$ is zero for all $t > 2$, but the solution of the system with switching times $\tau_k = 0.5, 1.5, 2.5, \ldots$, blows up (i.e. $\lim_{t\to\infty}\|\boldsymbol{\xi}(t)\|_\infty = \infty$), because the values on the right-going characteristic ξ^2 emerging from all $s \in (0, 0.5)$ always increase by reflection of the characteristics along the boundary. □

Note that starting from an initial condition $\bar{\boldsymbol{\xi}} \equiv 0$ the solution $\boldsymbol{\xi}(\cdot)$ of the switched system (1) satisfies $\boldsymbol{\xi}(t) = 0$ for all $t \geq 0$. Without loss of generality, we consider the problem of asymptotic stability only for this *equilibrium state*. For a given switching sequence $\sigma(\cdot)$, we say that the system is *stable*, if for all $\varepsilon > 0$ sufficiently small, there exists a $\delta(\varepsilon) > 0$ such that if $\|\boldsymbol{\xi}(0)\|_2 \leq \delta$, then $\|\boldsymbol{\xi}(t)\|_2 \leq \varepsilon$ for all $t \geq 0$. We say that the system is *asymptotically stable*, if it is stable and $\lim_{t\to\infty} \|\boldsymbol{\xi}(\cdot)\|_2 = 0$. In view of main problem (A), we then say that the switched system is *absolutely asymptotically stable* if it is asymptotically stable for all switching sequences $\sigma(\cdot)$ satisfying assumption (H_3). Finally, in view of problem (B), we say that a value $\tau > 0$ is a *dwell-time* of a switching signal $\sigma(\cdot)$, if the intervals between consecutive switches are no shorter than τ, that is, $\tau_{k+1} - \tau_k \geq \tau$ for all $k > 0$.

It was shown in [15], that infinite dimensional switched systems like (4) are exponentially stable for arbitrary switching if all subsystems are exponentially stable and the operators commute pairwise. However, due to the presence of boundary conditions in (1), the operators $\mathcal{A}^j$ defined in (3) do not commute pairwise in general. Thus we will focus on conditions for the boundary data under which the switched system is absolutely asymptotically stable. We begin with a very strong sufficient condition where we ask all boundary data to be *strictly dissipative*, compare [7].

Assumption 1: (Strict dissipativeness) For all $j \in \mathcal{Q}$ and for all $v_I \in \mathbb{R}^{m_j}$ and $v_{II} \in \mathbb{R}^{n-m_j}$ let the following conditions hold

$$\begin{pmatrix} v_I \\ v_{II} \end{pmatrix}^\top \frac{\partial \Lambda^j(s)}{\partial s} \begin{pmatrix} v_I \\ v_{II} \end{pmatrix} \leq 0 \quad \text{for all } s \in [a,b] \tag{6}$$

$$v_I^\top (\Lambda_I^j(a) + (G_L^j)^\top \Lambda_{II}^j(a) G_L^j) v_I \leq -r_I \|v_I\|^2 \tag{7}$$

$$v_{II}^\top (\Lambda_{II}^j(b) + (G_R^j)^\top \Lambda_I^j(b) G_R^j) v_{II} \geq +r_{II} \|v_{II}\|^2 \tag{8}$$

where $r_I, r_{II} \geq 0$ are constants such that $r_I + r_{II} > 0$.

Theorem 1: Let the switching system (1) under hypotheses (H_1)-(H_4) satisfy Assumption 1. Then the system is absolutely asymptotically stable.

Proof:

We suppose $r_I > 0$ (the case $r_{II} > 0$ is analogous). We have

$$\begin{aligned}
\frac{d}{dt}\|\boldsymbol{\xi}(t)\|_2^2 &= 2\int_a^b \xi^\top(t,s)\frac{d}{dt}\xi(t,s)ds \\
&= -2\int_a^b \xi^\top(t,s)\Lambda^j(s)\frac{\partial}{\partial s}\xi(t,s)ds \\
&= -\xi(t,b)^\top\Lambda^j(b)\xi(t,b) + \xi(t,a)^\top\Lambda^j(a)\xi(t,a) \\
&\quad + \int_a^b \xi(t,s)^\top\left(\frac{\partial\Lambda^j(s)}{\partial s}\right)\xi(t,s)ds \\
&\leq -\xi(t,b)^\top\Lambda^j(b)\xi(t,b) + \xi(t,a)^\top\Lambda^j(a)\xi(t,a) \\
&= -\xi_{II}^\top(t,b)\left(\Lambda_{II}^j(b) + (G_R^j)^\top\Lambda_I^j(b)G_R^j\right)\xi_{II}(t,b) \\
&\quad + \xi_I^\top(t,a)\left(\Lambda_I^j(a) + (G_L^j)^\top\Lambda_{II}^j(a)G_L^j\right)\xi_I(t,a) \\
&\leq -r_I\|\xi_I(t,a)\|^2
\end{aligned}$$

where the inequalities follow from conditions (6)–(8). Thus, we have

$$\|\boldsymbol{\xi}(t)\|_2^2 \leq \|\boldsymbol{\xi}(0)\|_2^2 - r_I\int_0^t \|\xi_I(\vartheta,a)\|^2 d\vartheta. \tag{9}$$

So we see that $\|\boldsymbol{\xi}(t)\|_2$ is a non-increasing function of t for all $j \in \mathcal{Q}$. Consider $t > \bar{\tau}$ where $\bar{\tau}$ is given by (12). Let t_u (resp. t_l) be the time taken by the slowest left-going (resp. right-going) characteristic path passing through (t,a) (resp. $(0,a)$) to hit the boundary $s = b$. By standard energy estimates for the equations $\partial_s\xi = -(A^j(s))^{-1}\partial_t\xi$, we have that

$$\int_{t_l}^{t_u} \|\boldsymbol{\xi}(y)\|_2^2 dy \leq K^j\int_0^t \|\xi_I(\vartheta,a)\|^2 d\vartheta,$$

where $K^j > 0$ are constants. Using that $\|\boldsymbol{\xi}(\cdot)\|_2$ is decreasing for all $j \in \mathcal{Q}$, we have that

$$(t_u - t_l)\|\boldsymbol{\xi}(t)\|_2^2 \leq \int_{t_l}^{t_u} \|\boldsymbol{\xi}(y)\|_2^2 dy,$$

so using (9) with a constant $0 < \gamma = \frac{(t_u - t_l)}{(r_I \max_{j\in\mathcal{Q}} K^j)}$ between t and $t+\bar{\tau}$, we have $\|\boldsymbol{\xi}(t+\bar{\tau})\|_2^2 \leq \|\boldsymbol{\xi}(t)\|_2^2 - \gamma\|\boldsymbol{\xi}(t+\bar{\tau})\|_2^2$. With the constant $K = \sqrt{1/(1+\gamma)} < 1$, this implies

$$\|\boldsymbol{\xi}(t+\bar{\tau})\|_2 \leq K\|\boldsymbol{\xi}(t)\|_2$$

for all switching signals $\sigma(\cdot)$ satisfying (H_3). Thus, by induction, we have

$$\|\boldsymbol{\xi}(t+i\bar{\tau})\|_2 \leq K^i\|\boldsymbol{\xi}(0)\|_2 = \exp(-i|\ln(K)|)\|\boldsymbol{\xi}(0)\|_2$$

and finally, for a suitable positive constant $c > 0$,

$$\|\boldsymbol{\xi}(t)\|_2 \leq c\exp(-t|\ln(K)|)\|\boldsymbol{\xi}(0)\|_2.$$

■

Remark 2: Theorem 1 also holds if the assumption $r_I + r_{II} \geq 0$ is dropped but the inequality (6) for all $j \in \mathcal{Q}$ is strict for some $s \in [a,b]$. □

We will later consider a weaker sufficient condition similar on the following *spectral radius condition* also known for quasi-linear hyperbolic systems [8].

Assumption 2: For all $j \in \mathcal{Q}$, let the following hold:

$$\inf_{\substack{\gamma=\text{diag}\{\gamma_i\},\gamma_i>0 \\ (i=1,\ldots,n)}} \left\|\gamma\begin{pmatrix} 0 & G_R^j \\ G_L^j & 0 \end{pmatrix}\gamma^{-1}\right\|_\infty < 1, \tag{10}$$

where $\|M\|_\infty := \max\{\sum_{j=1}^n |M|_{ij}; i \in \{1,\ldots,n\}\}$ for $M \in \mathbb{R}^{n\times n}$. □

Note that condition (10) is a spectral radius condition because is the same as saying that the maximum eigenvalue of the characterizing matrix

$$G^j := \begin{pmatrix} 0 & |G_R^j| \\ |G_L^j| & 0 \end{pmatrix}$$

is less than one. Under assumption 2, the subsystems for fixed $j \in \mathcal{Q}$ are known to be exponentially stable [8], i.e. it is known that there exists constants $M, \beta > 0$ such that the semigroups in Lemma 1 satisfy

$$\|T^j(t)\| \leq M\exp(-\beta t), \tag{11}$$

where $\|\cdot\|$ denotes the induced operator norm. However, assumption 2 is no longer sufficient for the switched system to be asymptotically stable, noting that G_L^j, G_R^j in Example 1 satisfy (10). Nevertheless, as common for switched ODE systems, the switched system (1) satisfying assumption (2) can be stabilized by switching slow enough, i. e.

Corollary 1: (Dwell-Time) Consider (1) under assumption 2 and define the following values

$$\bar\tau_j := (b-a)\left\{\Big(\min_{\substack{s\in[a,b]\\ i=1,\ldots,m_j}} |\lambda_i^j(s)|\Big)^{-1} + \Big(\min_{\substack{s\in[a,b]\\ i=m_j+1,\ldots,n}} |\lambda_i^j(s)|\Big)^{-1}\right\}$$
$$\bar\tau := \max_{j\in\mathcal{Q}} \bar\tau_j. \tag{12}$$

Then, for any $\tau \geq \bar\tau$ assumed as dwell-time for the switching signal $\sigma(\cdot)$, the system (1) under hypotheses (H_1)-(H_4) is asymptotically stable.

Proof: Observe that in mode j, the first summand in the definition of $\bar\tau_j$ denotes the time taken by the slowest left moving characteristic curve starting from $s=b$ to cross $(b-a)$; similarly the second summand for the slowest right moving characteristic curve. So $\bar\tau$ denotes the time in which all characteristic curves starting at any point in $(b-a)$ will have hit both left and right boundary at least once independent of the mode $j\in\mathcal{Q}$. The dwell time result then follows directly by induction in time over $\bar\tau$-steps. ■

Assumption 2 can be generalized to the following *joint spectral radius condition*.

Assumption 3: For all $j, j' \in \mathcal{Q}$, let the following hold:

$$\inf_{\substack{\gamma=\mathrm{diag}\{\gamma_i\},\gamma_i>0\\ (i=1,\ldots,n)}} \left\| \gamma \begin{pmatrix} 0 & G_R^{j'} \\ G_L^j & 0 \end{pmatrix} \gamma^{-1} \right\|_\infty < 1. \tag{13}$$

□

Under the above condition one can show absolute asymptotic stability.

Theorem 2: Let the switching system (1) under hypotheses (H_1)–(H_4) satisfy Assumption 3. Then the system is absolutely asymptotically stable (indeed, L^∞-exponentially stable).

Proof: The proof given in [17] can be easily extended to the more general situation considered here. ■

III. Stability of Hyperbolic Switching Systems in Non-Diagonal Form

A. Switched system in non-diagonal form

In this section, we draw attention to systems such as the ones considered in Section II, but where the advective velocity matrices $A^j(s)$ are only supposed to be equivalent to diagonal matrices $\Lambda^j(s)$ for all $j\in\mathcal{Q}$ via transformations

$$S_j(s)A^j(s)S_j^{-1}(s) = \Lambda^j(s)$$

with $S_j(\cdot)$ and $S_j^{-1}(\cdot)$ in $(C^1[a,b])^{n\times n}$. Such switching systems result for instance from sequential linearization in time along equilibrium states of non-linear hyperbolic systems. The switched system in non-diagonal form in which the switches in time are again for all $t\geq 0$ governed by the switching signal $\sigma(\cdot)$ with $\sigma(t)\in\mathcal{Q}\simeq\{1,\ldots,N\}$ is given by

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,s) + A^{\sigma(t)}(s)\dfrac{\partial}{\partial s}u(t,s) = 0 \\ D_L^{\sigma(t)}u(t,a) = 0, \quad D_R^{\sigma(t)}u(t,b) = 0 \\ u(0,s) = \bar{\mathbf{u}}(s) \end{cases} \tag{14}$$

where for all $j\in\mathcal{Q}$

(H_1') $A^j(s)\in\mathbb{R}^{n\times n}$ is strictly hyperbolic, i. e. A^j has m_j negative and $(n-m_j)$ positive eigenvalues $\lambda_i^j(s)$ with n corresponding linearly independent left (resp. right) eigenvectors $l_i^j(s)$ ($r_i^j(s)$), $s\in[a,b]$; $S_j(s) = [l_1^j(s)|\ldots|l_n^j(s)]^\top$, $s\in[a,b]$.

Under the transformation $u(t,s) = S_{\sigma(t)}^{-1}(s)\xi(t,s)$, the PDE in each mode becomes

$$\frac{\partial}{\partial t}\xi(t,s) + \Lambda^{\sigma(t)}(s)\frac{\partial}{\partial s}\xi(t,s) = 0, \tag{15}$$

with initial and the boundary conditions $\bar{\boldsymbol{\xi}}(s) = S_{\sigma(0)}(s)\bar{\mathbf{u}}(s)$, $\tilde D_L^{\sigma(t)}\xi(t,a) = 0$, $\tilde D_R^{\sigma(t)}\xi(t,b) = 0$, using $\tilde D_L^{\sigma(t)} = D_L^{\sigma(t)}S_{\sigma(t)}^{-1}(a)$, $\tilde D_R^{\sigma(t)} = D_R^{\sigma(t)}S_{\sigma(t)}^{-1}(b)$. For all $j\in\mathcal{Q}$, we use the representation

$$\xi(t,s) = \begin{pmatrix} \xi_I(t,s) \\ \xi_{II}(t,s)\end{pmatrix}, \quad \xi_I(t,s)\in\mathbb{R}^{m_j}, \xi_{II}(t,s)\in\mathbb{R}^{n-m_j}$$

and $\tilde D_L^j = [\tilde D_{L,I}^j|\tilde D_{L,II}^j]$, $\tilde D_R^j = [\tilde D_{R,I}^j|\tilde D_{R,II}^j]$.

We introduce the following hypothesis:

(H_2') For all $j\in\mathcal{Q}$, $D_L^j\in\mathbb{R}^{(n-m_j)\times n}$ and $D_R^j\in\mathbb{R}^{m_j\times n}$ are such that $\tilde D_{L,II}^j\in\mathbb{R}^{(n-m_j)\times(n-m_j)}$ and $\tilde D_{R,I}^j\in\mathbb{R}^{m_j\times m_j}$ are both non-singular.

and recall the following result for a fixed $j\in\mathcal{Q}$ [18]:

Lemma 2: Under hypothesis (H_1'), (H_2') and (H_3), the subsystems for a fixed $j\in\mathcal{Q}$ in (14) are well-posed. □

We can see that the boundary conditions for the switched system (14) can be written as

$$\begin{aligned} \xi_{II}^j(t,a) &= -(\tilde D_{L,II}^j)^{-1}\tilde D_{L,I}^j\xi_I^j(t,a) \\ \xi_I^j(t,b) &= -(\tilde D_{R,I}^j)^{-1}\tilde D_{R,II}^j\xi_I^j(t,b). \end{aligned} \tag{16}$$

Thus for a given initial condition $\bar{\mathbf{u}}\in\mathcal{H}$, the solution $\mathbf{u}(\cdot)\in C([0,\infty),\mathcal{H})$ of the switched system (1) can be represented as

$$\begin{aligned}\mathbf{u}(t) = S_{\sigma(t)}^{-1}T^{\sigma(\tau_K)}(t-\tau_K)\tilde T^{\sigma(\tau_{K-1})}(\tau_K-\tau_{K-1}) \\ \cdots\tilde T^{\sigma(\tau_1)}(\tau_2-\tau_1)\tilde T^{\sigma(0)}(\tau_1)S_{\sigma(0)}\bar{\mathbf{u}}\end{aligned} \tag{17}$$

with

$$\tilde T^{\sigma(\tau_k)}(\tau_{k+1}-\tau_k) := S_{\sigma(\tau_{k+1}+)}S_{\sigma(\tau_{k+1}-)}^{-1}T^{\sigma(\tau_k)}(\tau_{k+1}-\tau_k)$$

and $\tau_K = \max_{k\in\mathbb{N}}\{\tau_k\,|\,\tau_k < t\}$ and $\tau_0 = 0$, where $\{T^j(t)\}_{t\geq 0}$ is the semigroup generated for the system (15) with (16), cf. Lemma 1.

B. Stability of switched hyperbolic system in non-diagonal form

Again, we begin with an example, showing that the joint spectral radius condition (13) for the boundaries is no longer sufficient for the switched system to be absolutely asymptotically stable.

Example 2: Consider the system (14) on $\mathcal{T} \times [0,1]$ with two modes ($\mathcal{Q} = \{1,2\}$) and

$$A^1 = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}, \; A^2 = \begin{pmatrix} 2.6 & 14.4 \\ -0.4 & -2.6 \end{pmatrix},$$

$$D_L^1 = \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix}^\top, D_L^2 = \begin{pmatrix} -\frac{1}{11} \\ 1 \end{pmatrix}^\top, D_R^1 = \begin{pmatrix} 1 \\ -\frac{1}{4} \end{pmatrix}^\top, D_R^2 = \begin{pmatrix} 1 \\ 14 \end{pmatrix}^\top$$

for an alternating switching signal $\sigma(\cdot)$ with switching times $\mathcal{G} = \{0.5, 1, 1.5, \dots\}$ and let the initial condition be

$$\bar{u}(s) = \begin{cases} (\bar{v}_I^1, \bar{v}_I^2)^\top & s \in [0, 0.5] \\ (\bar{v}_{II}^1, \bar{v}_{II}^2)^\top & s \in (0.5, 1], \end{cases} \quad \sigma(0) = 1. \tag{18}$$

It can be easily seen that this example satisfies (H_1') and (H_2') and that

$$S_1 A^1 S_1^{-1} = S_2 A^2 S_2^{-1} = \mathrm{diag}(-1, 1) := \Lambda, \tag{19}$$

where $S_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $S_2 = \begin{pmatrix} 0.1 & 0.9 \\ 0.2 & 0.8 \end{pmatrix}$. Defining the characteristic variables $\xi(t,s) = (\xi^1(t,s), \xi^2(t,s))^\top$ as $\xi = S_j u$ for $j \in \{1,2\}$, the switched system (2) in characteristic variables becomes:

$$\begin{aligned} \partial_t \xi + \Lambda \partial_s \xi = 0, \\ \xi^2(t,0) = \frac{3}{2}\xi^1(t,0), \quad \xi^1(t,1) = \frac{1}{4}\xi^2(t,1), \end{aligned} \tag{20}$$

and the initial condition (18) becomes

$$\bar{\xi}(s) = \begin{cases} (\bar{v}_I^1, \bar{v}_I^2)^\top & s \in [0, 0.5] \\ (\bar{v}_{II}^1, \bar{v}_{II}^2)^\top & s \in (0.5, 1], \end{cases} \quad \sigma(0) = 1. \tag{21}$$

The characterizing matrix for $j \in \{1,2\}$ is both

$$G^j = \begin{pmatrix} 0 & \frac{1}{4} \\ \frac{3}{2} & 0 \end{pmatrix}$$

which has a (joint) spectral radius 0.6124 which is less than 1.
It is easy to observe that for the system (2),(18), the solution at all times that take values in $\mathcal{G} := \{0, 0.5, 1.0, 1.5, \dots\}$ is constant in $s \in [0, 0.5]$ and $s \in (0.5, 1]$. So consider the system at times $\tau_k \in \mathcal{G}$, $k \in \mathbb{N}$ and let the value of solution be

$$u(\tau_k, s) = \begin{cases} (v_I^1(\tau_k), v_I^2(\tau_k))^\top & s \in [0, 0.5] \\ (v_{II}^1(\tau_k), v_{II}^2(\tau_k))^\top & s \in (0.5, 1]. \end{cases} \tag{22}$$

The values of the solution at τ_{k+2} is then

$$u(\tau_{k+2}, s) = \begin{cases} (v_I^1(\tau_{k+2}), v_I^2(\tau_{k+2}))^\top & s \in [0, 0.5] \\ (v_{II}^1(\tau_{k+2}), v_{II}^2(\tau_{k+2}))^\top & s \in (0.5, 1]. \end{cases} \tag{23}$$

The quantities in equations (22) and (23) are related as

$$\begin{pmatrix} v_I^1(\tau_{k+2}) \\ v_I^2(\tau_{k+2}) \\ v_{II}^1(\tau_{k+2}) \\ v_{II}^2(\tau_{k+2}) \end{pmatrix} = M \begin{pmatrix} v_I^1(\tau_k) \\ v_I^2(\tau_k) \\ v_{II}^1(\tau_k) \\ v_{II}^2(\tau_k) \end{pmatrix}, \tag{24}$$

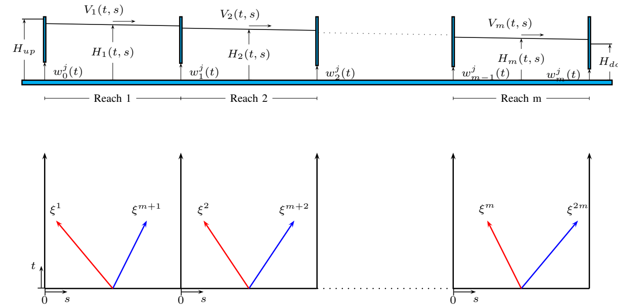

Fig. 2. (a) Cascade of canals operated by multi-mode underflow sluice gates. (b) Characteristic variables characterizing each reach.

where

$$M = \begin{pmatrix} 18.225 & -7.200 & 1.350 & -0.200 \\ -2.025 & 1.800 & -0.150 & 0.050 \\ 10.800 & -1.600 & 1.800 & -0.100 \\ -1.200 & -0.400 & -0.200 & -0.025 \end{pmatrix}.$$

The eigenvalues of M are $19.9, 1.08, 0.84, -0.02$. Thus, the matrix M is unstable and this implies that $\|u(\tau_k)\|_\infty$ tends to infinity as $k \to \infty$. □

As a direct consequence of symmetric diagonalizability of all subsystems (14) to (1), we have the following.

Corollary 2: All stability results from the former section hold for pairwise commuting matrices $A^j A^{j'} = A^{j'} A^j$ ($j, j' \in \mathcal{Q}$). □

For general A^j, our main concern is that the discrete time system

$$\begin{cases} u_{k+1} = B_k u_k \\ u_0 = S_{\sigma(0)} \bar{u} \end{cases} \tag{25}$$

with $B_k = S_{\sigma(\tau_{k+1}+)} S^{-1}_{\sigma(\tau_{k+1}-)} T^{\sigma(\tau_k)}(\tau_{k+1} - \tau_k)$ may be unstable. However, it should be clear that a dwell-time $\tau \geq \tau_{k+1} - \tau_k$ has a stabilizing role.

Proposition 1: For any switching system (14) under hypotheses (H_1')–(H_2') and (H_3) satisfying Assumption 2 with

$$G_L^j = -(\tilde{D}_{L,II}^j)^{-1} \tilde{D}_{L,I}^j, \quad G_R^j = -(\tilde{D}_{R,I}^j)^{-1} \tilde{D}_{R,II}^j,$$

there exists a value $\bar{\tau}$ such that for any $\tau \geq \bar{\tau}$ assumed as dwell-time for the switching signal $\sigma(\cdot)$, the switched system is asymptotically stable.

Proof: We have that

$$\begin{aligned} \rho(B_k) &\leq \|S_{\sigma(\tau_{k+1}+)} S^{-1}_{\sigma(\tau_{k+1}-)}\| \|T^{\sigma(\tau_k)}(\tau_{k+1} - \tau_k)\| \\ &\leq \max_{j,j' \in \mathcal{Q}} \big(\max_{s \in [a,b]} \|S_j S_{j'}^{-1}\| \big) \|T^{\sigma(\tau_k)}(\tau_{k+1} - \tau_k)\| \\ &=: K \|T^{\sigma(\tau_k)}(\tau_{k+1} - \tau_k)\| \end{aligned}$$

where $\rho(\cdot)$ denotes the spectral radius. Here, $K\|T^{\sigma(\tau_k)}(\tau_{k+1} - \tau_k)\|$ can always be made smaller than one under Assumption 2 using (11). ■

IV. Application for linearized Saint-Venant equations

We motivate our study by an example of a cascade of m canal reaches as depicted in Figure 2 (a). Consider a supervisory controller orchestrating a finite set of boundary feedback controlled underflow sluice gates with corresponding gate openings w_i^j for reach i in mode j. The flow of water in reach i is characterized by velocity $V_i(t,s)$ and elevation $H_i(t,s)$. For horizontal, prismatic canals with rectangular cross-section, frictionless walls and normalized length, the flow, under gravity g, satisfies the Saint-Venant equations [2]

$$\frac{\partial}{\partial t}\begin{pmatrix} H_i \\ V_i \end{pmatrix} + \begin{pmatrix} V_i & H_i \\ g & V_i \end{pmatrix}\frac{\partial}{\partial s}\begin{pmatrix} H_i \\ V_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (26)$$

for $i = 1,\ldots,m$, each defined on the domain $\{(t,s) : 0 \le t < \infty, 0 \le s \le 1\}$. Let the initial data be given by $H_i(0,s)$, $V_i(0,s)$ and the boundary conditions modeling decentralized feedback control actions in mode j together with flow conservation for each reach i be given by

$$\begin{aligned}
&f_1^j(w_0^j(t), H_1(t,0), V_1(t,0), H_{\text{up}}) = 0\\
&f_i^j(w_i^j(t), H_i(t,1), H_{i+1}(t,0), V_i(t,1), V_{i+1}(t,0)) = 0\\
&f_m^j(w_m^j(t), H_m(t,1), V_m(t,1), H_{\text{do}}) = 0\\
&H_i(t,1)V_i(t,1) - H_{i+1}(t,0)V_{i+1}(t,0) = 0
\end{aligned}$$

where H_{up}, H_{do} are the (known) up and down stream water levels.

Assume that under constant gate openings $\bar{w}_i$ and constant H_{up}, H_{do}, each reach attains a uniform steady state $(\bar{H}_i, \bar{V}_i)$ such that $H_{\text{do}} < \bar{H}_m < \ldots < \bar{H}_1 < H_{\text{up}}$ and $\bar{H}_1\bar{V}_1 > 0$. Using $v_i(x,t) = V_i(x,t) - \bar{V}_i$ and $h_i(x,t) = H_i(x,t) - \bar{H}_i$, the linearized model can be written as

$$\frac{\partial}{\partial t}\begin{pmatrix} h_i \\ v_i \end{pmatrix} + \begin{pmatrix} \bar{V}_i & \bar{H}_i \\ g & \bar{V}_i \end{pmatrix}\frac{\partial}{\partial s}\begin{pmatrix} h_i \\ v_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (27)$$

with initial conditions $h_i(0,\cdot)$, $v_i(0,\cdot)$ for $i = 1,\ldots,m$. With a change of coordinates $\xi_i(t,s) = h_i(t,s) + v_i\sqrt{\bar{H}_i/g}$, $\xi_{m+i}(t,s) = h_i(t,s) - v_i\sqrt{\bar{H}_i/g}$ the system becomes

$$\frac{\partial}{\partial t}\begin{pmatrix} \xi_i \\ \xi_{m+i} \end{pmatrix} + \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_{m+i} \end{pmatrix}\frac{\partial}{\partial s}\begin{pmatrix} \xi_i \\ \xi_{m+i} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (28)$$

with $\lambda_i = (\sqrt{g\bar{H}_i} - \bar{V}_i)$ and $\lambda_{m+i} = (\sqrt{g\bar{H}_i} + \bar{V}_i)$.

Under sub-critical flow, the eigenvalues satisfy $\lambda_i < 0 < \lambda_{m+i}$. For the system of $m-$canal reaches, equation (28) can be written in the form

$$\partial_t\xi + \Lambda\partial_s\xi = 0, \quad (29)$$

where $\xi = (\xi_1,\ldots,\xi_m,\xi_{m+1},\ldots,\xi_{2m})^\top$ and $\Lambda = \text{diag}(\lambda_1,\ldots,\lambda_{2m})$ (see Figure 2 (b)). Moreover, setting $\xi_I = (\xi_1,\ldots,\xi_m)$, $\xi_{II} = (\xi_{m+1},\ldots,\xi_{2m})$ and taking into account the coordinate transformation while assuming sufficient regularity of f_i^j, the boundary conditions in linearized form for each j can be rewritten as

$$\xi_{II}(t,0) = G_L^j\xi_I(t,0) \quad \xi_I(t,1) = G_R^j\xi_{II}(t,1) \quad (30)$$

with appropriately defined jacobians G_L^j, G_R^j (for details on the derivation for an explicit control law f_i^j see [3]).

Our results from Section II provide a set of sufficient conditions for solutions of (29)-(30) to decay for any admissible supervisory control action, e. g. as to persue superior objectives. In this context, the dwell-time results appear to be conventional, taking into account the multiscale peculiarity of the modeling.

V. Final remarks

We presented first results on stability of switching among systems of linear hyperbolic PDEs involving boundary data. It should have become clear that, although the switching signal was taken as global, all results apply for switching the boundary conditions or system matrices individually by introducing appropriate auxiliary modes, this is just a matter of notational convenience. Eventually, our results motivate further study of stability of PDE system that undergo switching in time, in particular, future direction of work should include extension of a Lyapunov theory for switched PDE systems.

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