Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Modeling for control of an inflatable space reflector, the nonlinear 1-D case

T. Voß
University of Groningen
Fac. of Math. & Natural Sciences
DTPA group
Nijenborgh 4
9747 AG Groningen, NL
t.voss@rug.nl

J.M.A. Scherpen
University of Groningen
Fac. of Math. & Natural Sciences
DTPA group
j.m.a.scherpen@rug.nl

P.R. Onck
University of Groningen
Fac. of Math. & Natural Sciences
Micromechanics group
p.r.onck@rug.nl

Abstract— **In this paper we develop a mathematical model of the dynamics for an inflatable space reflector, which can be used to design a controller for the shape of the inflatable structure. Inflatable structures have very nice properties, suitable for aerospace applications. We can construct e.g. a huge light weight reflector for a satellite which consumes very little space in the rocket because it can be inflated when the satellite is in the orbit. So with this technology we can build inflatable reflectors which are about 100 times bigger than solid ones. But to be useful for telescopes we have to actively control the surface of the inflatable to achieve the desired surface accuracy. The starting point of the control design is modeling for control, in the case port-Hamiltonian (pH) modeling. We will show how to derive a nonlinear infinite dimensional pH model of a 1-D Euler-Bernoulli beam with piezo actuation. In the future we will also focus on 2-D models.**

I. INTRODUCTION

Inflatable structures are a very promising technology for space applications [3]. With this emerging technology we are able to build bigger space crafts, which are cheaper in terms of costs but still use the same space in the orbiting device.

As a consequence, the developments may enable us to build huge solar panels, reflectors, solar panels, or even human habitats, which, with the state-of-the-art technology, are not possible to build.

Due to the fact that any inflatable structure is build of a polymer casing which is folded on earth and then inflated with a gas in space, it is clear that an inflatable structure cannot have the same surface accuracy as a rigid body. This disadvantage is the reason why inflatable structures currently are not the best option for high accuracy situations, i.e. a visible light reflector.

However, this problem may be solved by using smart materials which have the possibility to change their properties on demand, e.g. piezoelectric polymers [7]. This means that with smart materials it is actually possible to change the shape of an element by means of an applied voltage. Since these materials are made of polymers it is possible to build extremely thin actuators which can then be bonded to the casing of the inflatable structure.

In this paper we show how to develop a model for a 1-D flexible structure with a piezoelectric element as actuator in the port-Hamiltonian (pH) modeling framework [1]. The approach we propose differs from [4], [9], because we derive a model which can represent nonlinear deformations of the beam. Additionally we derive the equations of motion by the generalized Hamiltonian's principle, see [8]. We approach the problem with the purpose to extend it to the 2-D case in the future.

The paper is organized as follows. In Section II we introduce the basic physical relations which we use to formulate the model. A distributed pH model for a nonlinear piezoelectric Euler-Bernoulli beam is defined in Section III. In Section IV we show how the proposed model can be used to define a model for a piezoelectric composite, which will be a possible actuator for the shape control of an inflatable structure. Some simulation results of the final model are shown in Section V. Finally in Section VI conclusions are drawn and possible future directions are presented

The proposed model can also be used for modeling other structure, namely any flexible structure with a piezo actuation e.g. for vibration control in civil engineering.

II. BACKGROUND ON CONTINUUM DYNAMICS AND THE PIEZOELECTRIC EFFECT

In this section we briefly introduce the physics we use in the following sections. In this paper we focus on linear materials and large deformations [2].

We first introduce the constitutive equations of the model. If we consider a beam without a piezo actuation we know from Hooke's Law that the stress-strain relations can be described as

$$\sigma = C^E \varepsilon,$$

where we used the common matrix notation instead of the tensor notation. Here σ is the stress, ε the related strain and C^E a matrix which relates the stress and the strain. In general σ and ε are second order tensors of dimension 3, e.g. σ_{ij} describes the stress in the ij direction ($i, j \in \{1,2,3\}$), and C^E is a fourth order tensor. The subscripts correspond to directions in the coordinate system, 1 corresponds to x, 2 to y and 3 to the z direction.

For piezoelectric material we have that the piezo effect induces an additional strain in the material which is caused by an electrical field (actuation property). Similarly the deformation of the piezoelectric element also changes the electrical field in the element (sensing property). So the coupled constitutive relations for piezo electric material [6] can be described as

$$\begin{bmatrix} \sigma \\ D \end{bmatrix} = \begin{bmatrix} C^E & -e^T \\ e & \varepsilon^e \end{bmatrix} \begin{bmatrix} \varepsilon \\ E \end{bmatrix}. \tag{1}$$

Here D is the electrical displacement and E is the electrical field in the piezo element, ε^e is the electrical permittivity and e is the piezoelectric constant of the material.

The strain ε in the beam is related to the deformation $\mathbf{u}$ of the beam. The electrical field and the electrical displacement can be described by Maxwell's equations. So we can state the kinematic equations as

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_{k=1}^{3} \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right), \tag{2}$$

$$E_i = -\frac{\partial \varphi}{\partial x_i}, \rho_e = \sum_i \frac{\partial D_i}{\partial x_i}, \tag{3}$$

here φ is the electrical potential and ρ_e is the electrical charge density. Note: $x_1 = x$, $x_2 = y$ and $x_3 = z$.

The equations of motion for the model we derive via the generalized Hamilton's principle [8]. This principle states that for a piezoelectric material it must hold that

$$\delta \int_{t_0}^{t_1} (K - P + W)dt = 0,$$

where $K = \frac{1}{2}\int_V \rho ||\dot{\mathbf{u}}||^2 dV$ is the kinetic energy of the beam (ρ being the mass of the beam material), $P = \frac{1}{2}\int_V \sigma^T \varepsilon + DEdV$ is the potential energy and $W = \int_V \mathbf{f}_V^T \mathbf{u} dV + \oint_B \mathbf{f}_B^T \mathbf{u} dB$ are the external forces caused by body forces $\mathbf{f}_V$ and surface tractions $\mathbf{f}_B$. Here V stands for the volume of the element and B for its surface. This expression can then be reformulated to the following equations of motion in integral form

$$\int_V -\rho \ddot{\mathbf{u}}^T \delta \mathbf{u} - \sigma^T \delta \varepsilon + \mathbf{f}_V^T \delta \mathbf{u} dV + \oint_B \mathbf{f}_B^T \delta \mathbf{u} dB = 0. \tag{4}$$

In order to derive the equations of motion in differential form we will use (4).

III. PORT-HAMILTONIAN MODELING OF AN PIEZOELECTRIC EULER-BERNOULLI BEAM

In this section we want to introduce a pH model, see [1], [5], [4], [9], for a flexible piezoelectric beam, described in the nonlinear Euler-Bernoulli framework. We assume that a surface force is acting on the beam, e.g. a pressure. We first derive an infinite dimensional pH model. We start with the modeling of a simple piezoelectric Euler-Bernoulli beam to make the steps of the modeling process clear. In the next section we derive a pH model for a piezoelectric composite beam.

The derivation of the pH model can be subdivided in four parts. First we define the strain and the electrical field in the beam. We also need to define the geometry of the beam. The second step is then to derive the Hamiltonian of the beam that describes the energy stored in the structure. Thirdly we calculate the equations of motion with the generalized Hamiltonian's principle. The last step is then to define an interconnection structure which represents the physics of the system and gives us the final pH model.

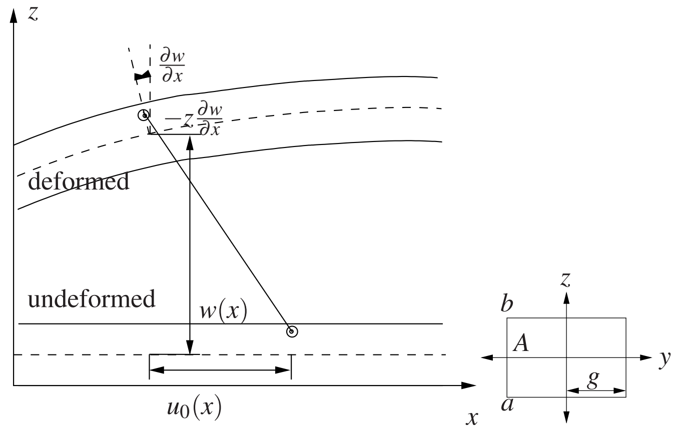

Fig. 1. Deformation of a beam under external influences (left), Cross sectional area of the beam (right)

A. Strain and electrical field of the beam

To derive the distributed pH model for the beam, we first have to define the strain which is caused by its deformation. For an Euler-Bernoulli beam it is in general assumed that the displacement takes place in the x and z direction only, so

$$\mathbf{u} = [u_0(x) - z\phi(x), 0, w(x)]^T.$$

That means we assume we have pure bending. This yields the following strain in the x direction,

$$\varepsilon_{11}(x,z) = u_0'(x) - z\phi(x) + \frac{1}{2}\left(u_0'(x) - z\phi'(x)\right)^2 + \frac{1}{2}\phi^2(x) \tag{5}$$

where $u_0(x)$ is the displacement of a material point at the neutral line of the beam and $w(x)$ describes the deflection of the beam from the undeformed configuration. $\phi(x)$ is defined to be the slope of the beam, hence $\phi(x) = \frac{\partial w}{\partial x}$, see Figure 1. In the Euler-Bernoulli beam framework it is assumed that all other strains and shears in the beam are zero, therefore from now on we neglect the subscripts for the strain. We also define the prime operator for a coordinate as the spatial derivative, e.g. $u_0' = \frac{\partial}{\partial x}u_0$.

Before we define the energies stored in the beam due to bending we have to define the geometry of the beam. The beam will have the length L ($x \in [0,\ldots,L]$) a height of $b - a$ ($z \in [a,b]$, $a < b$), and a width which is symmetric ($y \in [-g,g]$). The cross sectional area (in the yz-plane) of the beam is denoted as A, see Figure 1.

Now we also state some assumptions for the electrical field E of the piezoelectric beam. To be able to connect the beam to an electrical power source we assume that the upper and the lower side of the beam are covered with an electrode. Due to the applied potential an electrical field will be created. The width of the electrode $g_e(x)$ is depending on the position along the beam so that we can tune the electrical field locally. The structure of the electrodes and the piezoelectric material is similar to a parallel plate capacitor. So we assume, similar to a plate capacitor, that electrical displacement in the beam is given as $D = \frac{Q}{A_Q}$, where Q is the total charge of the electrodes and $A_Q = \int_0^L g_e(x)dx$ is the area of the electrodes.

B. Hamiltonian of the piezoelectric beam.

So as to derive the Hamiltonian of the system we first define the energy stored in the system. The Hamiltonian has the following general form

$$H(\mathbf{u},\varepsilon,E) = \frac{1}{2}\int_V \rho\,\|\dot{\mathbf{u}}\|^2 + \sigma\varepsilon + DEdV.$$

Let us now consider the kinetic energy of the beam. If we define the moment of a specific atom in the beam $\mathbf{p}_a = \rho\dot{\mathbf{u}}$ we can express the kinetic energy in the beam as

$$K = \frac{1}{2}\int_0^L \int_A \frac{\|\mathbf{p}_a\|^2}{\rho} dAdx.$$

It is easy to see that the kinetic energy is defined as a volume integral, but $\mathbf{p}_a$ depends on x and z. In consequence, we can express the kinetic energy of the beam as a line integral if we calculate the moment of a slice of the beam at position x,

$$\begin{aligned}\int_A \frac{\|\mathbf{p}_a\|^2}{\rho} dA &= \int_A \rho\left(\left(\dot{u}_0 - z\dot{\phi}\right)^2 + \dot{w}^2\right) dA \\ &= \rho\left(A\dot{u}_0^2 - 2I_0\dot{u}\dot{\phi} + I\dot{\phi}^2 + A\dot{w}^2\right)\end{aligned}$$

Then we can rewrite the kinetic energy as

$$K(\mathbf{p}) = \frac{1}{2}\int_0^L \mathbf{p}^T\mathbf{M}^{-1}\mathbf{p}dx,$$

where

$$\mathbf{p} := \mathbf{M}\tilde{\mathbf{u}} = \rho\begin{bmatrix} A & 0 & -I_0 \\ 0 & A & 0 \\ -I_0 & 0 & I \end{bmatrix}\begin{bmatrix} \dot{u}_0 \\ \dot{w} \\ \dot{\phi} \end{bmatrix}$$

with $I = \int_A z^2 dA$ and $I_0 = \int_A zdA$. Note that $I \neq 0$ and $I_0 = 0$ $\forall x \in [0,L]$, if we choose centroidal coordinates ($a = -b$).

The potential energy stored in the beam can be described as

$$P = \frac{1}{2}\int_0^L \int_A \begin{bmatrix} \varepsilon \\ E \end{bmatrix}^T \begin{bmatrix} C^E & e \\ -e & \varepsilon^e \end{bmatrix}^T \begin{bmatrix} \varepsilon \\ E \end{bmatrix} dAdx$$

Similarly, to define the potential energy as a line integral, we now compute the integral over the cross sectional area of the potential energy.

If we use the definition of the strain (2) and the definition of the stress in the beam (1), the potential energy can be determined as follows,

$$\begin{aligned}\frac{1}{2}\int_V \sigma\varepsilon dV &= \frac{1}{2}\int_V \left(C^E\varepsilon - eE\right)\varepsilon + (e\varepsilon + \varepsilon^e E)EdV \\ &= \frac{1}{2}\int_0^L \int_A C^E\left(u_0' - z\phi' + \frac{1}{2}\left(u_0' - z\phi'\right)^2 + \frac{1}{2}\phi^2\right)^2 \\ &\quad + eE^2 dAdx.\end{aligned}$$

So the Hamiltonian of a 1D piezoelectric beam is given as

$$H = \frac{1}{2}\int_0^L \mathbf{p}^T\mathbf{M}^{-1}\mathbf{p} + eAE^2 \quad (6)$$

$$+ \int_A C^E\left(u_0' - zw'' + \frac{1}{2}\left(u_0' - zw''\right)^2 + \frac{1}{2}w'^2\right)^2 dx \quad (7)$$

The here defined Hamiltonian will be used in

C. Equations of motion of a piezoelectric beam

Now we derive the equations of motion for the piezoelectric beam. From Section II we know that the following equations of motion in integral form must hold

$$\int_V -\rho\ddot{\mathbf{u}}\delta\mathbf{u} - \sigma\delta\varepsilon dV + \oint_B \mathbf{f}_B^T\delta\mathbf{u}dB = 0.$$

where

$$\begin{aligned}\delta\varepsilon &= \left(1 + u_0' - zw''\right)\delta u_0' - z\left(1 + u_0' - zw''\right)\delta w'' + w'\delta w' \\ \delta\mathbf{u} &= \left(\delta u_0 - z\delta\phi, 0, \delta w\right).^T\end{aligned}$$

Now we reformulate this expression to achieve the equations of motion in differential form. We do this part by part. We start with the kinetic energy. We can rewrite the expression above in the following form

$$\begin{aligned}\delta K &= -\rho\int_0^L \int_A \left(\ddot{u}_0 - z\ddot{w}'\right)\left(\delta u_0 - z\delta w\right) + \ddot{w}\delta wdAdx \\ &= \int_0^L \dot{\mathbf{p}}_1\delta u_0 + \dot{\mathbf{p}}_2\delta w + \dot{\mathbf{p}}_3\delta w'dx,\end{aligned}$$

Then we check the variation of potential mechanical energy P_m of the system.

$$\begin{aligned}\delta P_m &= \int_V -\sigma\delta\varepsilon dV \\ &= \int_A \sigma\left(1 + u_0' - zw''\right)dA\bigg|_0^L \delta u_0 \\ &- \int_0^L \frac{\partial}{\partial x}\int_A \sigma\left(1 + u_0' - zw''\right)dA\,\delta u_0\,dx \\ &+ \int_A -z\sigma\left(1 + u_0' - zw''\right)dA\bigg|_0^L \delta w' \\ &- \int_0^L \frac{\partial}{\partial x}\int_A -z\sigma\left(1 + u_0' - zw''\right)dA\delta w'\,dx \\ &+ \int_A \sigma w'dA\bigg|_0^L \delta w - \int_0^L \frac{\partial}{\partial x}\int_A \sigma w'dA\,\delta w\,dx\end{aligned}$$

For the external force, we assume a pressure applied at the lower part, we get

$$\begin{aligned}\oint_B \mathbf{f}_B^T\delta\mathbf{u}dB &= \int_0^L \int_{-g}^g \mathbf{f}_u\left(\delta u_0 - a\delta\phi\right) + \mathbf{f}_w\delta wdAdx \\ &= 2g\int_0^L \mathbf{f}_u\delta u_0 - a\mathbf{f}_u\delta\phi + \mathbf{f}_w\delta wdx\end{aligned}$$

because these equations must hold for any arbitrary δu_0, δw it follows that the integrand should be zero. Hence, one gets

$$\begin{aligned}\dot{\mathbf{p}}_1 &= \frac{\partial}{\partial x}\int_A C^E\varepsilon\left(1 + u_0' - zw''\right)dA \\ &\quad - \frac{\partial}{\partial x}\left(\left(A + Au_0' - I_0w''\right)eE\right) + 2g\mathbf{f}_u \quad (8) \\ \dot{\mathbf{p}}_2 &= \frac{\partial}{\partial x}\int_A C^E\varepsilon w'dA + -\frac{\partial}{\partial x}\left(w'eAE\right) + 2g\mathbf{f}_w \\ \dot{\mathbf{p}}_3 &= \frac{\partial}{\partial x}\int_A -zC^E\varepsilon\left(1 + u_0' - zw''\right)dA \quad (9) \\ &\quad + \frac{\partial}{\partial x}\left(\left(I_0 + I_0u_0' - Iw''\right)eE\right) - 2ga\mathbf{f}_u \quad (10)\end{aligned}$$

The boundary conditions are fulfilled by the assumptions that the beam is clamped

$$w = \frac{\partial w}{\partial x} = 0, u_0 = \frac{\partial u_0}{\partial x} = 0, \text{ for } x = 0 \text{ or } x = L$$

These equations of motion only hold for the mechanical part, but we also need a equation of motion for the electrical field E. To achieve this equation we use the constitutive equations (1),

$$D = e\varepsilon + \varepsilon^e E \Rightarrow E = \frac{1}{\varepsilon^e} D - \frac{e}{\varepsilon^e}\varepsilon.$$

This equation must hold for every point in the piezoelectric beam $\forall(x,y,z) \in V$. But we are treating a beam where we only want a spatial dependency of x. So we integrate this equation over the cross-sectional area to lose the dependency on y and z.

$$\begin{aligned} AE &= \frac{A}{\varepsilon^e} D - \frac{eA}{\varepsilon^e}\left(u_0' + \frac{1}{2}\left(u_0'\right)^2 + \frac{1}{2}w'^2\right) \\ &\quad - \frac{eI_0}{\varepsilon^e}\left(w'' + u_0' w''\right) + \frac{1}{2}\frac{eI}{\varepsilon^e} w''^2 \end{aligned}$$

If we now use the fact that the electrical displacement in a plate capacitor can be described as $D = \frac{Q}{A_Q}$ we get,

$$\begin{aligned} E &= \frac{1}{\varepsilon^e A_q} Q - \frac{e}{\varepsilon^e}\left(u_0' + \frac{1}{2}\left(u_0'\right)^2 + \frac{1}{2}w'^2\right) \\ &+ \frac{eI_0}{\varepsilon^e A}\left(w'' + u_0' w''\right) + \frac{1}{2}\frac{eI}{\varepsilon^e A} w''^2 \end{aligned}$$

It is obvious that this equation is a constraint to the system. But if we choose the right initial values for the states such that this equation is fulfilled for $t = 0$. We can reformulate the static constraint to a dynamical equation,

$$\begin{aligned} \dot{E} &= \frac{1}{\varepsilon^e}\frac{I_e}{A_q} - \frac{e}{\varepsilon^e A}(A + Au_0' - I_0 w'')\dot{u}_0' \quad (11) \\ &\quad - \frac{e}{\varepsilon^e} w'\dot{w}' + \frac{e}{\varepsilon^e A}\left(I_0 + I_0 u_0' - Iw''\right)\dot{w}'', \quad (12) \end{aligned}$$

where I_e is the current applied to the electrodes.

The equations of motion derived in this section are used in the next section to define the final pH model of a piezoelectric beam.

D. Interconnection structure and final pH model

Now we use the results of the last subsections to derive a interconnection structure which is able to represent the equations of motion of the system in pH form.

The state variables of the pH system are $(\mathbf{p}, \tilde{\varepsilon}, E)^T$. The gradient of the Hamiltonian (6) with respect to these state variables is

$$\nabla H = \begin{bmatrix} \nabla_{p_1} H \\ \nabla_{p_2} H \\ \nabla_{p_3} H \\ \nabla_{u_0'} H \\ \nabla_{\phi} H \\ \nabla_{\phi'} H \\ \nabla_E H \end{bmatrix} = \begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{w}' \\ \int_A C^E \varepsilon\left(1 + u_0' - zw''\right) dA \\ \int_A C^E \varepsilon w' dA \\ \int_A C^E \varepsilon\left(-z - zu_0' + z^2 w''\right) dA \\ \varepsilon^e AE \end{bmatrix}.$$

So using the state variables and the gradient of the Hamiltonian we can write (9) and (11) as

$$\begin{aligned} \dot{p}_1 &= \frac{\partial}{\partial x}\nabla_{u'} H + g_1\left(\nabla_E H\right) + 2g\mathbf{f}_u \\ \dot{p}_2 &= \frac{\partial}{\partial x}\nabla_{\phi} H + g_2\left(\nabla_E H\right) + 2g\mathbf{f}_w \\ \dot{p}_3 &= \frac{\partial}{\partial x}\nabla_{\phi'} H + g_3\left(\nabla_E H\right) - 2ga\mathbf{f}_u \\ \dot{E} &= \frac{1}{\varepsilon^e}\frac{I_e}{A_q} - g_1^*(\nabla_{p_1} H) - g_2^*(\nabla_{p_2} H) - g_3^*(\nabla_{p_3} H), \end{aligned}$$

where

$$\begin{aligned} g_1(\circ) &= -\frac{e}{\varepsilon^e A}\frac{\partial}{\partial x}\left(\left(A + Au_0' - I_0 w''\right)\cdot\circ\right) \\ g_2(\circ) &= -\frac{e}{\varepsilon^e}\frac{\partial}{\partial x}\left(w'\cdot\circ\right) \\ g_3(\circ) &= \frac{e}{\varepsilon^e A}\frac{\partial}{\partial x}\left(\left(I_0 + I_0 u_0' - Iw''\right)\cdot\circ\right) \end{aligned}$$

The formal adjoints of this operators are given

$$\begin{aligned} g_1^*(\circ) &= \frac{e}{\varepsilon^e A}\left(A + Au_0' - I_0 w''\right)\frac{\partial}{\partial x}\circ \\ g_2^*(\circ) &= \frac{e}{\varepsilon^e} w' \frac{\partial}{\partial x}\circ \\ g_3^*(\circ) &= -\frac{e}{\varepsilon^e A}\left(I_0 + I_0 u_0' - Iw''\right)\frac{\partial}{\partial x}\circ \end{aligned}$$

Because we need equations of motion for all state variables we also have to define $\frac{\partial}{\partial t}[u_0', \phi, \phi']^T$. These equations can be stated as

$$\begin{aligned} \frac{\partial}{\partial t} u_0' &= \frac{\partial}{\partial x}\nabla_{p_1} H, \; \frac{\partial}{\partial t}\phi = \frac{\partial}{\partial x}\nabla_{p_2} H \\ \frac{\partial}{\partial t}\phi' &= \frac{\partial}{\partial x}\nabla_{p_3} H \end{aligned}$$

Hence, we can state the following pH model

$$\begin{bmatrix} \dot{\mathbf{p}} \\ \dot{\tilde{\varepsilon}} \\ \dot{E} \end{bmatrix} = \mathbf{J}\nabla H + \begin{bmatrix} 2g & 0 & 0 \\ 0 & 2g & 0 \\ -2ga & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\varepsilon^e A_q} \end{bmatrix} \begin{bmatrix} \mathbf{f}_u \\ \mathbf{f}_w \\ I_e \end{bmatrix} \quad (13)$$

$$\mathbf{y} = \begin{bmatrix} 2g & 0 & 0 \\ 0 & 2g & 0 \\ -2ga & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\varepsilon^e A_q} \end{bmatrix}^T \nabla H,$$

where

$$\mathbf{J} = \begin{bmatrix} 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 & 0 & g_1 \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 & g_2 \\ 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} & g_3 \\ \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 \\ -g_1^* & -g_2^* & -g_3^* & 0 & 0 & 0 & 0 \end{bmatrix}$$

Remark 1: The pH model of a beam without a piezoelectric property can be easily derived from this model if we set D and E to zero. So we get

$$\begin{aligned} \begin{bmatrix} \dot{\mathbf{p}} \\ \dot{\tilde{\varepsilon}} \end{bmatrix} &= \mathbf{J}\nabla H + \begin{bmatrix} 2g & 0 \\ 0 & 2g \\ -2ga & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{f}_u \\ \mathbf{f}_w \end{bmatrix} \\ \mathbf{y} &= \begin{bmatrix} 2g & 0 \\ 0 & 2g \\ -2ga & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^T \nabla H, \end{aligned}$$

where

$$\mathbf{J} = \begin{bmatrix} 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 \end{bmatrix}.$$

With the Hamiltonian defined as

$$\begin{aligned} H(\mathbf{p},\tilde{\varepsilon}) &= \frac{1}{2}\int_0^L \mathbf{p}^T\mathbf{M}^{-1}\mathbf{p} \\ &+ \int_A C^E \left(u_0' - zw'' + \frac{1}{2}\left(u_0' - zw''\right)^2 + \frac{1}{2}w'^2\right)^2 dAdx. \end{aligned}$$

This model is similar as in used continuum mechanics except that we have defined it in the pH framework and we have not neglected any terms.

IV. Modeling of a symmetric piezoelectric composite

In this section we define a system that describes the dynamics of a piezoelectric composite. The composite consists of a base layer to which a piezoelectric layer is bonded. To define the dynamics, we follow here an approach where we first consider the bonding of the composite and then model the composite material as one beam. The result will be one pH model of a beam which represents the physics of the composite. Another way is to first define a pH model for every layer and then do the interconnection of the layers to achieve a pH model of the composite, see [4]. This approach has the disadvantage that the interconnection induces constraints to the system which cause unnecessary difficulties when we want to spatially discretize the system, see [11].

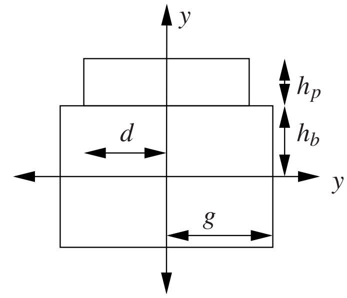

Fig. 2. Cross sectional area of the composite

Therefore, we first define the connection between the two layers. Since in the final system the piezo-electric layer is bonded to the base layer, the strains in all layers have to be the same. These constraints assure the perfect bonding so

$$\varepsilon_b = \varepsilon_p$$

In the sequel we will use the subscript b to identify the base layer, p the piezo-electric layer. From the continuity of strain it also automatically follows that the $\mathbf{u}_b = \mathbf{u}_p \Rightarrow \dot{\mathbf{u}}_b = \dot{\mathbf{u}}_p$.

Before we try to express the total stored energy as a line integral we have to define the geometry of the system, see Figure 2. We assume that the base layer has a constant thickness ($2g$) and a constant height $2h_b$ while the length is L. We also define that the origin of the yz-plane is in the center of mass of the base layer. So the cross sectional area of the base layer A_b is $[-g,g]\times[-h_b,h_b]$. With this it follows that $I_{b,0} = \int_{A_b} z dA_b = 0$. On top of the base layer the piezo-electric layer is bonded. The height of the layer is h_p and the width is given as $2d$. We also assume that the width is symmetric with the x-axis, hence the cross sectional area of the piezo layer is A_p is $[-d,d]\times[h_b,h_b+h_p]$. To simplify notation in the following paragraphs we define $A_{tot} = A_b + A_p$.

The energy stored in the composite will be the sum of the energies stored in the different layers,

$$H_{tot} = H_b + H_p.$$

In Section III we already defined the model for a flexible and piezo-electric beam as a line integral. And thus we can now combine these models to derive a model which describes the dynamics of the piezoelectric composite.

First we find a global expression for the total kinetic energy as a line integral. The total kinetic energy is given as

$$K_{tot} = \frac{1}{2}\int_0^L \mathbf{p}_b^T\mathbf{M}_b^{-1}\mathbf{p}_b + \mathbf{p}_p^T\mathbf{M}_p^{-1}\mathbf{p}_p dx.$$

Now we can combine the kinetic energy in the following way

$$K_{tot} = \frac{1}{2}\int_0^L \mathbf{p}_{tot}^T\mathbf{M}_{tot}^{-1}\mathbf{p}_{tot} dx,$$

with

$$\mathbf{p}_{tot} = \mathbf{M}_{tot}\dot{\mathbf{u}}, \mathbf{M}_{tot} = \mathbf{M}_b + \mathbf{M}_p.$$

Next we do the same for the mechanical potential energy. It is the sum of the mechanical potential energies of the layers, so

$$P_{tot} = P_b + P_p.$$

From Section III we already have an expression as a line integral for each potential energy. So we have to combine these expression to get the total potential energy. So

$$\begin{aligned} P_{tot} &= \int_0^L \int_{A_{tot}} C_{tot}^E \left(u_0' - zw'' + \frac{1}{2}\left(u_0' - zw''\right)^2 + \frac{1}{2} w'^2 \right)^2 dA_{tot} \\ &+ \quad eA_p E^2 dx, \end{aligned}$$

where

$$C_{tot}^E(z) = \begin{cases} C_b^E \text{ for all } z \in [-h_b, h_b] \\ C_p^E \text{ for all } z \in (h_b, h_b + h_p] \end{cases}.$$

With this definition we are able to rewrite the energy function as

$$\begin{aligned} H_{tot} &= \frac{1}{2} \int_0^L \mathbf{p}_{tot}^T \mathbf{M}_{tot}^{-1} \mathbf{p}_{tot} + eA_p E^2 \\ &+ \int_{A_{tot}} C_{tot}^E \left(u_0' - zw'' + \frac{1}{2}\left(u_0' - zw''\right)^2 + \frac{1}{2} w'^2 \right)^2 dA_{tot} dx. \end{aligned}$$

The equations of motion for the system can be calculated in the same way as in Section III. For the state variables of the pH system we choose $(\mathbf{p}, \tilde{\varepsilon}, E)^T$. So the gradient of the Hamiltonian with respect to this state variables is

$$\begin{bmatrix} \nabla_{p_1} H \\ \nabla_{p_2} H \\ \nabla_{p_3} H \\ \nabla_{u_0'} H \\ \nabla_{\phi} H \\ \nabla_{\phi'} H \\ \nabla_E H \end{bmatrix} = \begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{w}' \\ \int_{A_{tot}} C_{tot}^E \varepsilon \left(1 + u_0' - zw''\right) dA_{tot} \\ \int_{A_{tot}} C_{tot}^E \varepsilon w' dA_{tot} \\ \int_{A_{tot}} C_{tot}^E \varepsilon \left(-z - zu_0' + z^2 w''\right) dA_{tot} \\ \varepsilon^e A_p E \end{bmatrix}$$

Due to the fact that the only difference between a model for a piezo composite and a single piezo layer is the stored energy and not the interconnection structure, the equation of motions in pH form are the same as (13) except that we use the Hamiltonian derived in this section. Of course we can also extend this process to derive models with more then one piezo layers to increase the actuation force or to use some layers as sensors while the others are used as actuators.

V. Numerical Results

In this section we show the simulation results of an Euler-Bernoulli beam consisting of an piezo electric composite. The lumped model was derived by using a finite element method which is adapted to the distributed port-Hamiltonian modeling framework [10]. As material for the base layer we have chosen Kapton, and as piezo electric material we choose PVDF. The base layer has a length of $1m$ the thickness and width of the beam are $2cm$. The piezo electric material covers the whole beam and has a thickness of $0.25cm$. For the first simulation we apply a pressure of $10 \cdot t \frac{N}{m^2}$ until we reach a pressure of $5 \frac{N}{m^2}$, see Figure 3. The second simulation shows the response of the beam if we apply a Voltage of $0.1 \cdot t\,V$, see Figure 3. For both simulations we show Snapshots at time $t \in \{0.1, 0.25, 0.5\}$.

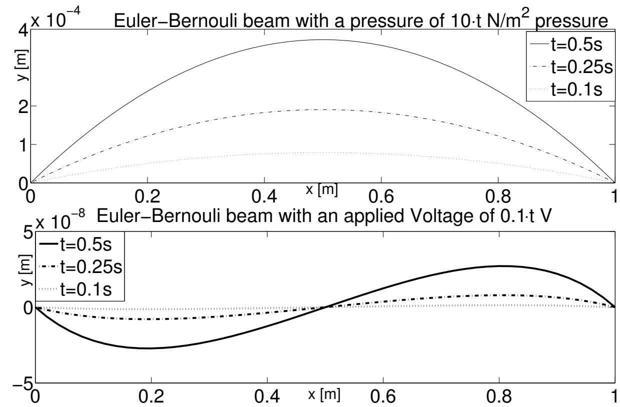

Fig. 3. Simulation of a piezo electric beam

VI. Concluding Remarks

In this paper we have determined a model for an inflatable structure in a pH framework. The modeling was done in a pH formulation in such a way that it can be used for an energy based control method. The achieved model is a nonlinear distributed pH model which can easily be used to represent the dynamics of a piezo electric composite beam with large deformations.

For the future we want to derive a 2D-model.

VII. Acknowledgments

We would like to thank the MicroNed programme for the funding of the research. We also thank Prof. Arjan van der Schaft from the University of Groningen for his support in deriving the model.

References

[1] A.J. van der Schaft & B.M. Maschke, The Hamiltonian formulation of energy conserving physical systems with external ports, *Archiv für Elektronik und Übertragungstechnik*, 49, pp. 362-371, 1995

[2] S.P. Timoschenko & J.N. Goodier, Theory of Elasticity, Third Edition, *McGraw-Hill International Editions*, 1970

[3] C.H.M. Jenkins, Gossamer Spacecraft: Membrane and inflatable structures technology for space applications, *Progress in Astronautics and Aeronautics*, Volume 191, 2001

[4] A. Macchelli, A.J. van der Schaft, C. Melchiorri, Multi-variable Port Hamiltonian Model of Piezoelectric Material, *IEEE/RSJ International Conference on Intelligent Robots and Systems,* IROS 2004

[5] A. Macchelli, Port Hamiltonian systems. A unified approach for modeling and control finite and infinite dimensional physical systems, University of Bologna, *DEIS*, 2003

[6] The Institute of Electrical and Electronics Engineers, Inc, An American National Standard IEEE Standard on Piezo electricity, *IEEE Standards Board* 1987

[7] A. Vinogradov, S. Ji, C. Jenkins, Y. Bar-Cohen, State-of-the-art developments in the field of electro active polymers, *Materials Research Society Fall Meeting*, Boston, MA, November 28, 2005

[8] A. Heckmann, N. Arnold, O. Vaculín, A model multifield approach for an extended flexible body description in multibody dynamics, *Multibody System Dynamics* (2005) 13: 299-322

[9] T. Voß, J.M.A. Scherpen, A.J. van der Schaft, Modeling for control of an inflatable space reflector, the linear 1-D case, MTNS 2008

[10] T. Voß, J.M.A. Scherpen, A.J. van der Schaft, Structure preserving port-Hamiltonian discretization of a 1-D inflatable space reflector, *submitted*

[11] G. Golo, V. Talasila, A.J. van der Schaft, B.M. Maschke, *Hamiltonian discretization of boundary control systems*, Automatica, 40, pp. 757-771, 2004