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A necessary and sufficient condition for input-output realization of switched affine state space models

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Abstract— **This paper presents a necessary and sufficient condition under which a discrete-time switched affine (SWA) state space model admits equivalent representations in the class of SWA input-output models. In particular, it is shown that observability is not a necessary requirement for input-output realization of SWA models. When an equivalent input-output representation exists, a constructive procedure is presented to derive both its parameters and the switching constraints. Numerical examples illustrate and motivate the presented equivalence result.**

I. Introduction

Among models characterized by switches between different linear/affine dynamics, piecewise affine (PWA) and switched affine (SWA) models represent two classes widely used in practice. In PWA models, switching between different modes is state- and input-dependent, being determined by a partition of the state-input domain into a finite number of polyhedral regions [1]. PWA models are suitable to describe, e.g., hybrid phenomena due to physical limits, deadzones and thresholds. In addition, thanks to the universal approximation properties of PWA maps, PWA models can also be used to approximate nonlinear systems. In SWA models, mode switching is determined by an exogenous signal, that is either deterministic or stochastic (e.g., it is generated by a finite-state Markov chain). SWA models arise naturally in multi-modal control systems, systems subject to component/subsystem failures and possibly to repairs, motion segmentation in computer vision, problems in communications, signal processing, econometrics and biometrics (see, e.g., [2] and references therein).

Equivalence results between different classes of switched and hybrid models are important in order to establish the capabilities of representation of each class, and possibly transfer analysis tools between classes. For instance, formal equivalence between discrete-time PWA state space models and other classes of hybrid models such as mixed logical dynamical and linear complementarity models, is addressed in [3], [4]. State space realization of continuous-time SWA input-output models is addressed in [5] using the theory of formal power series. In discrete-time, the realization problem for autonomous SWA and PWA models is investigated in [6], while a stochastic realization theory for jump-Markov linear systems is presented in [7]. Input-output realization of PWA and SWA state space models is another important issue, as it has a number of theoretical and practical implications, e.g., in hybrid system identification [8], [9]. A necessary and sufficient condition for input-output realization of PWA state space models is presented in [10], and it is shown that PWA systems admitting input-output representations are strictly contained in the class of all PWA state space systems. Input-output realization of SWA state space models was firstly investigated in [11], where it was shown that any observable SWA state space model admits a representation as a switched affine autoregressive model with exogenous inputs (SARX).

In the present work, a necessary and sufficient condition for input-output realization of SWA state space models is presented. It takes the form of a condition regarding the span of the observability matrices of all possible switching sequences of finite length. It turns out that observability (i.e., all observability matrices over a sufficiently long finite horizon having full-column rank) is only sufficient to guarantee the existence of equivalent SWA input-output realizations. Examples of non-observable SWA state space models satisfying the proposed necessary and sufficient condition, and thus admitting a SWA input-output realization, are given. SWA models that do not admit an input-output realization are also presented. This proves that the class of SWA systems admitting input-output representations is strictly contained in the class of all SWA state space systems. When an equivalent SARX model exists, a constructive procedure is presented to derive not only its parameters, but also the constraints on the switching and on the initial condition needed to ensure that the input-output model has no extra input-output behaviors with respect to the SWA state space model.

The paper is organized as follows. After recalling some basic notions and posing the considered equivalence problem in Section II, Section III introduces two motivating examples, and presents the equivalence result in detail. Additional numerical examples are reported in Section IV to highlight the role of the proposed necessary and sufficient condition. Finally, conclusions are drawn in Section V.

II. Preliminaries and problem formulation

A. Notation and definitions

The m-ary Cartesian product of a set $\mathbb{X}$ is denoted by $\mathbb{X}^m$. The sets of real, integer, and positive integer numbers are denoted by $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{Z}^+$, respectively. The set of real matrices with m rows and n columns is denoted by $\mathbb{R}^{m\times n}$. An $m \times n$ matrix with 0 everywhere is denoted by $\mathbf{0}_{m\times n}$. The set of all linear combinations of the row vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_p$ is denoted by $\mathrm{span}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_p)$. For a matrix $A \in \mathbb{R}^{n\times m}$, $\mathrm{span}(A)$ denotes the span of the rows of A. The

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vector collecting the values of the n-dimensional discrete-time signal $\boldsymbol{z}(k)$ from time k_1 to time $k_2 \geq k_1$ is denoted by $\boldsymbol{z}_{k_1}^{k_2}$, i.e. $\boldsymbol{z}_{k_1}^{k_2} = [\ \boldsymbol{z}(k_2)^\top\ \boldsymbol{z}(k_2-1)^\top\ \ldots\ \boldsymbol{z}(k_1)^\top\ ]^\top$.

Let $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}^n$. The set $\mathcal{X}$ is *dense* in the set $\mathcal{Y}$ if the closure of $\mathcal{X}$ is $\mathcal{Y}$, while it is *nowhere dense* if the interior of the closure of $\mathcal{X}$ is empty. The set $\mathcal{X}$ is *affine* if it can be expressed as $\mathcal{X} = \{\boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{v}\ :\ \boldsymbol{v} \in \mathcal{L}\}$, where $\boldsymbol{x}_0 \in \mathbb{R}^n$ and $\mathcal{L} \subseteq \mathbb{R}^n$ is a linear subspace. The *dimension* of an affine set $\mathcal{X}$ is the dimension of the corresponding linear subspace $\mathcal{L}$. The closed ball with center $x \in \mathbb{R}^n$ and radius $\varepsilon > 0$ in the Euclidean norm is denoted by $B(x, \varepsilon)$.

B. SWA state space models

A discrete-time SWA model in *state space* form is described by the equations

$$\begin{aligned} \boldsymbol{x}(k+1) &= A_{\sigma(k)}\,\boldsymbol{x}(k) + B_{\sigma(k)}\,\boldsymbol{u}(k) + f_{\sigma(k)} \\ \boldsymbol{y}(k) &= C_{\sigma(k)}\,\boldsymbol{x}(k) + D_{\sigma(k)}\,\boldsymbol{u}(k) + g_{\sigma(k)}, \end{aligned} \tag{1}$$

where $\boldsymbol{x}(k) \in \mathbb{R}^n$, $\boldsymbol{u}(k) \in \mathbb{R}^p$ and $\boldsymbol{y}(k) \in \mathbb{R}^q$ are, respectively, the state, the input and the output of the system at time $k \in \mathbb{Z}$. The discrete-valued signal $\sigma(k)$, specifying which affine dynamics of the system is active at time k, takes values in the finite set $\mathbb{S} = \{1, \ldots, s\}$, where $s \in \mathbb{Z}^+$ is the number of different modes. The real matrices/vectors A_i, B_i, f_i, C_i, D_i and g_i, $i \in \mathbb{S}$, having appropriate dimensions, describe each affine dynamics.

In this paper, the switching sequence $\{\sigma(k)\}_{k=0}^{\infty}$ is assumed to be an arbitrary external signal that does not depend on $\boldsymbol{x}(k)$ and $\boldsymbol{u}(k)$. Moreover, the initial state $\boldsymbol{x}(0)$ and the input sequence $\{\boldsymbol{u}(k)\}_{k=0}^{\infty}$ are assumed to be unconstrained. The output trajectory of model (1) given the initial state $\boldsymbol{x}(0) = \boldsymbol{x}_0$, the input sequence $\{\boldsymbol{u}(k)\}_{k=0}^{\infty}$, and the switching sequence $\{\sigma(k)\}_{k=0}^{\infty}$ is denoted by $\boldsymbol{y}(\cdot\,; \boldsymbol{x}_0, \boldsymbol{u}(\cdot), \sigma(\cdot))$. Matrices and vectors useful to describe the input, state and output trajectories of model (1) are introduced in Table I, where $m \in \mathbb{Z}^+$, and $(j_1, \ldots, j_m) \in \mathbb{S}^m$. In analogy with the linear case, the matrix $\mathcal{O}_{j_1,\ldots,j_m}$ is called the *observability matrix* of the mode sequence $(j_1, \ldots, j_m)$ [12]. The following lemma (see [10]) provides useful expressions for the state and output trajectories of model (1).

Lemma 1: Let the initial state $\boldsymbol{x}(0) = \boldsymbol{x}_0$, the input sequence $\{\boldsymbol{u}(k)\}_{k=0}^{\infty}$, and the switching sequence $\{\sigma(k)\}_{k=0}^{\infty}$ be given for model (1). For fixed $\bar{n} \in \mathbb{Z}^+$ and $k \geq \bar{n}$, let $i_h = \sigma(k-h)$, $h = 0, 1, \ldots, \bar{n}$. Then, for $h = 0, 1, \ldots, \bar{n}-1$, the state and output trajectories of model (1) satisfy

$$\begin{aligned} \boldsymbol{x}(k-h) = A_{i_{h+1},\ldots,i_{\bar{n}}}\,&\boldsymbol{x}(k-\bar{n}) \\ &+ B_{i_{h+1},\ldots,i_{\bar{n}}}\boldsymbol{u}_{k-\bar{n}}^{k-h-1} + f_{i_{h+1},\ldots,i_{\bar{n}}} \end{aligned} \tag{2}$$

$$\begin{aligned} y(k-h) = C_{i_h} A_{i_{h+1},\ldots,i_{\bar{n}}}\,&\boldsymbol{x}(k-\bar{n}) \\ &+ [\ D_{i_h}\ \ C_{i_h} B_{i_{h+1},\ldots,i_{\bar{n}}}\ ]\,\boldsymbol{u}_{k-\bar{n}}^{k-h} \\ &+ \big(g_{i_h} + C_{i_h} f_{i_{h+1},\ldots,i_{\bar{n}}}\big). \end{aligned} \tag{3}$$

By exploiting (3) and

$$\boldsymbol{y}(k-\bar{n}) = C_{i_{\bar{n}}}\boldsymbol{x}(k-\bar{n}) + D_{i_{\bar{n}}}\boldsymbol{u}(k-\bar{n}) + g_{i_{\bar{n}}}, \tag{4}$$

the output sequence $\boldsymbol{y}_{k-\bar{n}}^{k-1}$ can be expressed as

$$\boldsymbol{y}_{k-\bar{n}}^{k-1} = \mathcal{O}_{i_1,\ldots,i_{\bar{n}}}\,\boldsymbol{x}(k-\bar{n}) + \mathcal{D}_{i_1,\ldots,i_{\bar{n}}}\,\boldsymbol{u}_{k-\bar{n}}^{k} + \mathcal{G}_{i_1,\ldots,i_{\bar{n}}}, \tag{5}$$

TABLE I

$$A_{j_1,\ldots,j_m} = A_{j_1} A_{j_2} \ldots A_{j_m}$$

$$B_{j_1,\ldots,j_m} = [\ B_{j_1}\ \ A_{j_1} B_{j_2}\ \ A_{j_1,j_2} B_{j_3}\ \ldots\ A_{j_1,\ldots,j_{m-1}} B_{j_m}\ ]$$

$$f_{j_1,\ldots,j_m} = f_{j_1} + A_{j_1} f_{j_2} + A_{j_1,j_2} f_{j_3} + \ldots + A_{j_1,\ldots,j_{m-1}} f_{j_m}$$

$$\mathcal{O}_{j_1,\ldots,j_m} = \begin{bmatrix} C_{j_1} A_{j_2,\ldots,j_m} \\ C_{j_2} A_{j_3,\ldots,j_m} \\ \vdots \\ C_{j_{m-1}} A_{j_m} \\ C_{j_m} \end{bmatrix}$$

$$\mathcal{D}_{j_1,\ldots,j_m} = \begin{bmatrix} 0\ \ D_{j_1}\ \ C_{j_1} B_{j_2,\ldots,j_m} \\ \mathbf{0}_{q\times 2}\ \ D_{j_2}\ \ C_{j_2} B_{j_3,\ldots,j_m} \\ \vdots \\ \mathbf{0}_{q\times(m-1)}\ \ D_{j_{m-1}}\ \ C_{j_{m-1}} B_{j_m} \\ \mathbf{0}_{q\times m}\ \ D_{j_m} \end{bmatrix}$$

$$\mathcal{G}_{j_1,\ldots,j_m} = \begin{bmatrix} g_{j_1} + C_{j_1} f_{j_2,\ldots,j_m} \\ g_{j_2} + C_{j_2} f_{j_3,\ldots,j_m} \\ \vdots \\ g_{j_{m-1}} + C_{j_{m-1}} f_{j_m} \\ g_{j_m} \end{bmatrix}$$

$$\Gamma_{j_1,\ldots,j_m} = \begin{bmatrix} \mathcal{O}_{j_1,\ldots,j_m} & \mathcal{D}_{j_1,\ldots,j_m} \\ \mathbf{0}_{p(m+1)\times n} & I_{p(m+1)} \end{bmatrix}$$

$$\gamma_{j_1,\ldots,j_m} = \begin{bmatrix} \mathcal{G}_{j_1,\ldots,j_m} \\ \mathbf{0}_{p(m+1)\times 1} \end{bmatrix}$$

$$\nu_{j_1,\ldots,j_m} = [\ C_{j_1} A_{j_2,\ldots,j_m}\ \ D_{j_1}\ \ C_{j_1} B_{j_2,\ldots,j_m}]$$

where $\mathcal{O}_{i_1,\ldots,i_{\bar{n}}}$, $\mathcal{D}_{i_1,\ldots,i_{\bar{n}}}$ and $\mathcal{G}_{i_1,\ldots,i_{\bar{n}}}$ are defined in Table I. Moreover, by introducing the matrix $\Gamma_{i_1,\ldots,i_{\bar{n}}}$ and the vector $\gamma_{i_1,\ldots,i_{\bar{n}}}$ in Table I, the following relation holds:

$$\begin{bmatrix} \boldsymbol{y}_{k-\bar{n}}^{k-1} \\ \boldsymbol{u}_{k-\bar{n}}^{k} \end{bmatrix} = \Gamma_{i_1,\ldots,i_{\bar{n}}} \begin{bmatrix} \boldsymbol{x}(k-\bar{n}) \\ \boldsymbol{u}_{k-\bar{n}}^{k} \end{bmatrix} + \gamma_{i_1,\ldots,i_{\bar{n}}}. \tag{6}$$

C. SARX models

For fixed orders n_a and n_b, an SARX model is defined by introducing the regression vector

$$\begin{aligned} \boldsymbol{r}(k) = [\ &\boldsymbol{y}(k-1)^\top\ \ldots\ \boldsymbol{y}(k-n_a)^\top \\ &\boldsymbol{u}(k)^\top\ \boldsymbol{u}(k-1)^\top\ \ldots\ \boldsymbol{u}(k-n_b)^\top\ ]^\top, \end{aligned} \tag{7}$$

and then by expressing the output $\boldsymbol{y}(k)$ as

$$\boldsymbol{y}(k) = \Theta_{\ell(k)} \begin{bmatrix} \boldsymbol{r}(k) \\ 1 \end{bmatrix}, \tag{8}$$

where $\ell(k)$ specifies the ARX dynamics active at time k, and takes values in the finite set $\bar{\mathbb{S}} = \{1, \ldots, \bar{s}\}$, with $\bar{s} \in \mathbb{Z}^+$ being the number of different ARX dynamics. The real matrices Θ_j, $j \in \bar{\mathbb{S}}$, having appropriate dimensions, describe each ARX submodel. In the following, $\max\{n_a, n_b\}$ will be referred to as the *order* of the SARX model.

Let $\bar{n} = \max\{n_a, n_b\}$. Constraints on the initial regression vector $\boldsymbol{r}(\bar{n})$ and the switching signal $\{\ell(k)\}_{k=\bar{n}}^{\infty}$ can be included in the definition of the SARX model. In particular, an affine set $\mathcal{R}_j \subseteq \mathbb{R}^{n_a+n_b+1}$ and a set $\Gamma(j) \subseteq \bar{\mathbb{S}}$ are associated to each mode j of the SARX model, and it is required that $\boldsymbol{r}(\bar{n})$ and $\{\ell(k)\}_{k=\bar{n}}^{\infty}$ satisfy:

$$\boldsymbol{r}(\bar{n}) \in \mathcal{R}_{\ell(\bar{n})} \tag{9}$$

$$\ell(k+1) \in \Gamma(\ell(k)), \quad k \geq \bar{n}. \tag{10}$$

According to (9), the initial regression vector $\boldsymbol{r}(\bar{n})$ cannot be arbitrarily chosen. Rather, it must belong to an affine set that is consistent with mode $\ell(\bar{n})$. Note that it is not required that two sets $\mathcal{R}_j$ and $\mathcal{R}_{j'}$, with $j \neq j'$, must be disjoint. The constraints in (10) imply that the switching sequence $\{\ell(k)\}_{k=\bar{n}}^{\infty}$ cannot be arbitrarily chosen either. If the mode at time k is $\ell(k)$, the mode at next time $k+1$ must be chosen from the set $\Gamma(\ell(k))$. A switching sequence $\{\ell(k)\}_{k=\bar{n}}^{\infty}$ satisfying (10) is said to be *valid*.

D. Input-output equivalence of switched affine models

The definitions of *input-output trajectory* for models (1) and (8)-(10) are now introduced.

Definition 2.1: The pair $\{\boldsymbol{u}(k), \boldsymbol{y}(k)\}_{k=0}^{\infty}$ is an input-output trajectory of model (1) if there exist an initial state $\boldsymbol{x}(0) \in \mathbb{R}^n$ and a switching sequence $\{\sigma(k)\}_{k=0}^{\infty}$ such that equations (1) are satisfied for all $k \geq 0$.

Definition 2.2: The pair $\{\boldsymbol{u}(k), \boldsymbol{y}(k)\}_{k=0}^{\infty}$ is an input-output trajectory of model (8)-(10) if there exists a switching sequence $\{\ell(k)\}_{k=\bar{n}}^{\infty}$ such that (8), (9) and (10) are satisfied for all $k \geq \bar{n}$, with $\bar{n} = \max\{n_a, n_b\}$.

Based on the above definitions, the following notion of equivalence for models (1) and (8)-(10) will be considered.

Definition 2.3 (Input-output equivalence): Models (1) and (8)-(10) are said to be (input-output) *equivalent* if the sets of input-output trajectories of (1) and (8)-(10) coincide.

III. Main result

The equivalence problem posed in Section II-D is addressed in this section. First, two motivating examples are presented. Then, a necessary and sufficient condition is given for the SWA state space model (1) to admit equivalent SARX representations (8)-(10).

A. Motivating examples

In [11] an observability-based sufficient condition for input-output equivalence of SWA models is proposed. It is stated that, if there exists $\bar{n} \in \mathbb{Z}^+$ such that the observability matrices $\mathcal{O}_{i_1,\ldots,i_{\bar{n}}}$ have full-column rank for all $(i_1, \ldots, i_{\bar{n}}) \in \mathbb{S}^{\bar{n}}$, then model (1) admits equivalent SARX representations. One may wonder what happens when such an observability condition is not satisfied.

Example 3.1: Consider a 2-mode SWA model (1) with $A_1 = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, $C_1 = [\,1\;1\,]$, $A_2 = \left[\begin{smallmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{smallmatrix}\right]$ and $C_2 = [\,1\;0\,]$. All other model parameters are zero for the sake of simplicity. The model thus defined is not observable, since in particular the pair (C_1, A_1) is not observable. Hence, the sufficient condition in [11] is violated, and it cannot be established whether equivalent SARX representations of the given SWA state space model exist. Nevertheless, it can be observed what follows.

Consider the mode sequence $\sigma(k) = i_{2-k}$, $k = 0, 1, 2$, where $(i_0, i_1, i_2) = (1, 2, 1)$. The observability matrix $\mathcal{O}_{2,1} = \left[\begin{smallmatrix} C_2 A_1 \\ C_1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ has not full-column rank. By applying (3), it turns out that

$$y(0) = C_1 \boldsymbol{x}_0,\; y(1) = C_2 A_1 \boldsymbol{x}_0,\; y(2) = C_1 A_2 A_1 \boldsymbol{x}_0, \tag{11}$$

where $\boldsymbol{x}(0) = \boldsymbol{x}_0$ is an arbitrary initial state. The question whether it is possible to find coefficients θ_1, θ_2 and θ_3 such that $y(0)$, $y(1)$ and $y(2)$ given by (11) satisfy an affine relation of the type

$$y(2) = \theta_1\, y(1) + \theta_2\, y(0) + \theta_3 \tag{12}$$

for all $\boldsymbol{x}_0 \in \mathbb{R}^n$, has a positive answer in this case. Indeed, by substituting the relations (11) into (12), one obtains:

$$(C_1 A_2 A_1 - \theta_1 C_2 A_1 - \theta_2 C_1)\, \boldsymbol{x}_0 = \theta_3. \tag{13}$$

Since $C_1 A_2 A_1 = [\,\frac{1}{2}\;\frac{1}{2}\,] \in \mathrm{span}(\mathcal{O}_{2,1}) = \mathrm{span}([\,1\;1\,])$, (13) holds for all $\boldsymbol{x}_0 \in \mathbb{R}^n$ if, e.g., $\theta_1 = \frac{1}{2}$, $\theta_2 = \theta_3 = 0$. Since $C_{i_0} A_{i_1} A_{i_2} \in \mathrm{span}(\mathcal{O}_{i_1,i_2})$ for all the $\bar{s} = 8$ triples $(i_0, i_1, i_2) \in \{1, 2\}^3$, one can repeat the same reasoning and conclude that, even though the SWA model is not observable, there exists a set of coefficients $\{(\theta_{j,1}, \theta_{j,2}, \theta_{j,3})\}_{j=1}^{\bar{s}}$ such that, at any time $k \geq 2$,

$$y(k) = \theta_{j,1}\, y(k-1) + \theta_{j,2}\, y(k-2) + \theta_{j,3} \tag{14}$$

for some $j \in \{1, \ldots, 8\}$.

Example 3.2: Consider a 2-mode SWA model (1) with $A_1 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $C_1 = [\,1\;0\,]$, $C_2 = [\,0\;1\,]$. The state matrix A_2 is unspecified, while all other model parameters are zero for the sake of simplicity. For fixed $\bar{n} \in \mathbb{Z}^+$, choose the mode sequence as $\sigma(k) = 1$ for $k = 0, 1, \ldots, \bar{n}-1$ and $\sigma(\bar{n}) = 2$. It is easy to verify that

$$\begin{aligned} y(0) &= C_1 \boldsymbol{x}_0 = x_{0,1} \\ y(k) &= C_1 A_1^k \boldsymbol{x}_0 = 0, \quad k = 1, \ldots, \bar{n}-1 \\ y(\bar{n}) &= C_2 A_1^{\bar{n}} \boldsymbol{x}(0) = x_{0,2}, \end{aligned} \tag{15}$$

where $\boldsymbol{x}(0) = \boldsymbol{x}_0 = [\,x_{0,1}\; x_{0,2}\,]^\top$ is the initial state. Assume that the model defined above admits an equivalent switched autoregressive representation (8)-(10) of order $n_a = \bar{n}$. Since the switched autoregressive model is equivalent to the SWA model, for any $\boldsymbol{x}_0 \in \mathbb{R}^2$ there exists $j \in \bar{\mathbb{S}}$ such that

$$y(\bar{n}) = \Theta_j \left[\begin{smallmatrix} \boldsymbol{r}(\bar{n}) \\ 1 \end{smallmatrix}\right] = \sum_{h=1}^{\bar{n}} \theta_{j,h}\, y(\bar{n}-h) + \theta_{j,\bar{n}+1}, \tag{16}$$

and substituting the relation in (15):

$$x_{0,2} = \theta_{j,\bar{n}}\, x_{0,1} + \theta_{j,\bar{n}+1}. \tag{17}$$

Note that, for each $j \in \bar{\mathbb{S}}$, (17) defines a line, i.e. a 1-dimensional affine subspace, in $\mathbb{R}^2$. This clearly leads to a contradiction, because the union of a finite number of lines cannot cover the whole 2-dimensional space where $\boldsymbol{x}_0$ can vary. Given the arbitrariness of $\bar{n} \in \mathbb{Z}^+$, the above reasoning shows that the considered SWA state space model does not admit equivalent switched affine autoregressive representations of any order. This definitely depends on the fact that, for any $\bar{n} \in \mathbb{Z}^+$ and for the mode sequence $(i_0, i_1, \ldots, i_{\bar{n}}) = (2, 1, \ldots, 1)$, it holds that $C_{i_0} A_{i_1,\ldots,i_{\bar{n}}} = C_2 A_1^{\bar{n}} = [\ 0\ 1\ ] \notin \operatorname{span}(\mathcal{O}_{i_1,\ldots,i_{\bar{n}}}) = \operatorname{span}(C_1) = \operatorname{span}([\ 1\ 0\ ])$.

B. General equivalence result

Motivated by the examples in Section III-A, the following condition is now introduced.

Condition C1: Let $\bar{n} \in \mathbb{Z}^+$. For every $(\bar{n}+1)$-tuple $(i_0, i_1, \ldots, i_{\bar{n}}) \in \mathbb{S}^{\bar{n}+1}$ there exists $\Xi \in \mathbb{R}^{q \times \bar{n} q}$ such that

$$\Xi \mathcal{O}_{i_1,\ldots,i_{\bar{n}}} = C_{i_0} A_{i_1,\ldots,i_{\bar{n}}}. \tag{18}$$

It is straightforward to see that, if the observability condition holds for $\bar{n} \in \mathbb{Z}^+$, then Condition C1 also holds. The main result of the paper can now be stated.

Theorem 1: The SWA state space model (1) admits an equivalent SARX representation (8)-(10) if and only if there exists $\bar{n} \in \mathbb{Z}^+$ such that Condition C1 is satisfied.

Proof. (Sufficiency) Let $\bar{s} = s^{\bar{n}+1}$, and define a bijective mapping ϕ that associates a positive integer $j \in \bar{\mathbb{S}}$ to each $(\bar{n}+1)$-tuple $(i_0, i_1, \ldots, i_{\bar{n}}) \in \mathbb{S}^{\bar{n}+1}$. A SARX model (8)-(10) with $\bar{s}$ modes and model orders $n_a = n_b = \bar{n}$ is constructed as follows. For the jth mode of the SARX model, $j \in \bar{\mathbb{S}}$, associated to the mode sequence $(i_0, i_1, \ldots, i_{\bar{n}})$ through ϕ, let $\Xi_j^{(1)}$ be a solution of (18) by virtue of Condition C1. Then, define

$$\Xi_j^{(2)} = [\ D_{i_0} \quad C_{i_0} B_{i_1,\ldots,i_{\bar{n}}}\ ] - \Xi_j^{(1)} \mathcal{D}_{i_1,\ldots,i_{\bar{n}}} \tag{19}$$

$$\xi_j = g_{i_0} + C_{i_0} f_{i_1,\ldots,i_{\bar{n}}} - \Xi_j^{(1)} \mathcal{G}_{i_1,\ldots,i_{\bar{n}}}, \tag{20}$$

and let the ARX parameters Θ_j be given by

$$\Theta_j = [\ \Xi_j^{(1)} \quad \Xi_j^{(2)} \quad \xi_j\ ]. \tag{21}$$

The affine set $\mathcal{R}_j$ of initial regression vectors consistent with mode j is defined as the range of the affine transformation

$$\begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{u} \end{bmatrix} = \Gamma_{i_1,\ldots,i_{\bar{n}}} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{u} \end{bmatrix} + \gamma_{i_1,\ldots,i_{\bar{n}}} \tag{22}$$

for all possible $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{u} \in \mathbb{R}^{(\bar{n}+1)p}$. The set $\Gamma(j)$ of feasible transitions from mode j is defined as

$$\Gamma(j) = \left\{ j' \in \bar{\mathbb{S}} \ :\ j' = \phi(i, i_0, \ldots, i_{\bar{n}-1}), \quad i \in \mathbb{S} \right\}. \tag{23}$$

Let $\{\boldsymbol{y}(k)\}_{k=0}^{\infty}$ be the output sequence of model (1) with initial state $\boldsymbol{x}(0) = \boldsymbol{x}_0$, input sequence $\{\boldsymbol{u}(k)\}_{k=0}^{\infty}$ and switching sequence $\{\sigma(k)\}_{k=0}^{\infty}$. Moreover, for fixed $k \geq \bar{n}$, let $i_h = \sigma(k-h)$, $h = 0, 1, \ldots, \bar{n}$, and $j = \phi(i_0, i_1, \ldots, i_{\bar{n}})$. By defining $\boldsymbol{r}(k) = \begin{bmatrix} \boldsymbol{y}_{k-\bar{n}}^{k-1} \\ \boldsymbol{u}_{k-\bar{n}}^{k} \end{bmatrix}$, the following fundamental relationship can be derived through simple substitutions:

$$\begin{aligned} \Theta_j \begin{bmatrix} \boldsymbol{r}(k) \\ 1 \end{bmatrix} &= \Xi_j^{(1)} \boldsymbol{y}_{k-\bar{n}}^{k-1} + \Xi_j^{(2)} \boldsymbol{u}_{k-\bar{n}}^{k} + \xi_j \\ &= C_{i_0} \boldsymbol{x}(k) + D_{i_0} \boldsymbol{u}(k) + g_{i_0} = \boldsymbol{y}(k). \end{aligned} \tag{24}$$

Next, it is proven that every input-output trajectory of model (1) is an input-output trajectory of the constructed SARX model, and vice versa.

Let $\{\boldsymbol{u}(k), \boldsymbol{y}(k)\}_{k=0}^{\infty}$ be any input-output trajectory of model (1). An input-output trajectory $\{\boldsymbol{u}(k), \tilde{\boldsymbol{y}}(k)\}_{k=0}^{\infty}$ of the constructed SARX model is obtained by letting $\tilde{\boldsymbol{y}}(k) = \boldsymbol{y}(k)$ for $k = 0, 1, \ldots, \bar{n}$, and $\ell(k) = \phi(\sigma(k), \sigma(k-1), \ldots, \sigma(k-\bar{n}))$ for all $k \geq \bar{n}$. Thanks to (22)-(23), the input-output trajectory thus generated satisfies (9) and (10) as required. Indeed, since the initial regression vector $\boldsymbol{r}(\bar{n}) = \begin{bmatrix} \boldsymbol{y}_0^{\bar{n}-1} \\ \boldsymbol{u}_0^{\bar{n}} \end{bmatrix}$ satisfies (6) with $k = \bar{n}$ and $i_h = \sigma(\bar{n}-h)$, $h = 0, 1, \ldots, \bar{n}$, it turns out that $\boldsymbol{r}(\bar{n}) \in \mathcal{R}_{\ell(\bar{n})}$. Moreover, the switching sequence $\{\ell(k)\}_{k=\bar{n}}^{\infty}$ is valid for the constructed SARX model since $\ell(k+1) \in \Gamma(\ell(k))$ for $k \geq \bar{n}$ by construction. In order to show that $\tilde{\boldsymbol{y}}(k) = \boldsymbol{y}(k)$ for $k \geq \bar{n}$ (i.e., the two input-output trajectories coincide), it is just pointed out that $\tilde{\boldsymbol{y}}(k) = \Theta_{\ell(k)} \begin{bmatrix} \boldsymbol{r}(k) \\ 1 \end{bmatrix} = \boldsymbol{y}(k)$, where the latter equality follows from (24), provided that the regression vector $\boldsymbol{r}(k)$ of the SARX model equals $\begin{bmatrix} \boldsymbol{y}_{k-\bar{n}}^{k-1} \\ \boldsymbol{u}_{k-\bar{n}}^{k} \end{bmatrix}$. This holds for $k = \bar{n}$ by definition, and follows by induction for $k > \bar{n}$. Given the arbitrariness of $\{\boldsymbol{u}(k), \boldsymbol{y}(k)\}_{k=0}^{\infty}$, it can be concluded that any input-output trajectory of model (1) is also an input-output trajectory of the constructed SARX model.

Vice versa, let $\{\boldsymbol{u}(k), \tilde{\boldsymbol{y}}(k)\}_{k=0}^{\infty}$ be any input-output trajectory of the constructed SARX model with valid switching sequence $\{\ell(k)\}_{k=\bar{n}}^{\infty}$. A switching sequence $\{\sigma(k)\}_{k=0}^{\infty}$ for model (1) is reconstructed from the relations

$$(\sigma(k), \sigma(k-1), \ldots, \sigma(k-\bar{n})) = \phi^{-1}(\ell(k)), \quad k \geq \bar{n}. \tag{25}$$

Note that this is always possible by virtue of (10) and (23). Since $\boldsymbol{r}(\bar{n}) = \begin{bmatrix} \tilde{\boldsymbol{y}}_0^{\bar{n}-1} \\ \boldsymbol{u}_0^{\bar{n}} \end{bmatrix} \in \mathcal{R}_{\ell(\bar{n})}$, there exists $\boldsymbol{x}_0 \in \mathbb{R}^n$ such that (22) is satisfied with $\boldsymbol{y} = \tilde{\boldsymbol{y}}_0^{\bar{n}-1}$, $\boldsymbol{u} = \boldsymbol{u}_0^{\bar{n}}$, $\boldsymbol{x} = \boldsymbol{x}_0$, and $(i_0, i_1, \ldots, i_{\bar{n}}) = \phi(\ell(\bar{n}))$. If the output $\boldsymbol{y}(\cdot) = \boldsymbol{y}(\cdot; \boldsymbol{x}_0, \boldsymbol{u}(\cdot), \sigma(\cdot))$ of model (1) is now considered, it follows from (6) and the choice of $\boldsymbol{x}_0$ made above, that $\boldsymbol{y}(k) = \tilde{\boldsymbol{y}}(k)$ for $k = 0, 1, \ldots, \bar{n}-1$. Identity of $\boldsymbol{y}(k)$ and $\tilde{\boldsymbol{y}}(k)$ for $k \geq \bar{n}$ is proven by showing that $\boldsymbol{y}(k) = \tilde{\boldsymbol{y}}(k)$ if $\boldsymbol{y}(t) = \tilde{\boldsymbol{y}}(t)$ for $t < k$. This follows by recalling that $\tilde{\boldsymbol{y}}(k) = \Theta_{\ell(k)} \begin{bmatrix} \boldsymbol{r}(k) \\ 1 \end{bmatrix}$ with $\boldsymbol{r}(k) = \begin{bmatrix} \tilde{\boldsymbol{y}}_{k-\bar{n}}^{k-1} \\ \boldsymbol{u}_{k-\bar{n}}^{k} \end{bmatrix}$. Since $\boldsymbol{y}_{k-\bar{n}}^{k-1} = \tilde{\boldsymbol{y}}_{k-\bar{n}}^{k-1}$ by assumption, (24) implies that $\Theta_{\ell(k)} \begin{bmatrix} \boldsymbol{r}(k) \\ 1 \end{bmatrix} = \boldsymbol{y}(k)$, and hence $\boldsymbol{y}(k) = \tilde{\boldsymbol{y}}(k)$. Given the arbitrariness of $\{\boldsymbol{u}(k), \tilde{\boldsymbol{y}}(k)\}_{k=0}^{\infty}$, it can be concluded that any input-output trajectory of the constructed SARX model is also an input-output trajectory of model (1).

(Necessity) Let $\bar{n} = \max\{n_a, n_b\}$. Without loss of generality, it can be assumed that the SWA state space model (1) admits an equivalent SARX representation (8)-(10) with model orders $n_a = n_b = \bar{n}$. If either $n_a < \bar{n}$ or $n_b < \bar{n}$, it suffices to pad with zeros the parameter matrices Θ_j in appropriate positions, and to lift the affine sets $\mathcal{R}_j$ to a higher dimensional space. Consider the $(\bar{n}+1)$-tuple $(i_0, i_1, \ldots, i_{\bar{n}}) \in \mathbb{S}^{\bar{n}+1}$, and for all $j \in \bar{\mathbb{S}}$ define the set Ω_j formed by all pairs $\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{u} \end{bmatrix}$, $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{u} \in \mathbb{R}^{(\bar{n}+1)p}$, such that the pair $\begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{u} \end{bmatrix}$ obtained from (22) satisfies $\begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{u} \end{bmatrix} \in \mathcal{R}_j$ and

$$\begin{aligned} \Theta_j \begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{u} \\ 1 \end{bmatrix} &= C_{i_0} A_{i_1,\ldots,i_{\bar{n}}}\, \boldsymbol{x} \\ &\quad + [\ D_{i_0}\ C_{i_0} B_{i_1,\ldots,i_{\bar{n}}}\ ]\, \boldsymbol{u} \\ &\quad + \left(g_{i_0} + C_{i_1} f_{i_{i_1},\ldots,i_{\bar{n}}} \right). \end{aligned} \tag{26}$$

The reason for such a definition is that, if model (1) evolves with $\boldsymbol{x}(0) = \boldsymbol{x}$, $\boldsymbol{u}_0^{\bar{n}} = \boldsymbol{u}$, and $\sigma(k) = i_{\bar{n}-k}$, $k = 0, 1, \ldots, \bar{n}$, then $\boldsymbol{y} = \boldsymbol{y}_0^{\bar{n}-1}$, while the right-hand side of (26) equals $\boldsymbol{y}(\bar{n})$. Since the SARX model is equivalent to model (1), it holds that $\bigcup_{j=1}^{\bar{s}} \Omega_j = \mathbb{R}^{n+(\bar{n}+1)p}$. Then, according to Baire's Theorem [13], there must exist an index j such that the corresponding set Ω_j is not a nowhere dense set. This in turn implies that there exist $\left[\begin{smallmatrix}\bar{\boldsymbol{x}}\\ \bar{\boldsymbol{u}}\end{smallmatrix}\right] \in \Omega_j$ and $\varepsilon > 0$ such that $B(\left[\begin{smallmatrix}\bar{\boldsymbol{x}}\\ \bar{\boldsymbol{u}}\end{smallmatrix}\right], \varepsilon) \bigcap \Omega_j$ is dense in $B(\left[\begin{smallmatrix}\bar{\boldsymbol{x}}\\ \bar{\boldsymbol{u}}\end{smallmatrix}\right], \varepsilon)$. If Θ_j is decomposed as in (21) with $\Xi_j^{(1)} \in \mathbb{R}^{q \times \bar{n}q}$, $\Xi_j^{(2)} \in \mathbb{R}^{q \times (\bar{n}+1)p}$, $\xi_j \in \mathbb{R}^q$, and (22) is substituted into the left-hand side of (26) with $\left[\begin{smallmatrix}\boldsymbol{x}\\ \boldsymbol{u}\end{smallmatrix}\right] \in \Omega_j$, one obtains

$$\Theta_j \begin{bmatrix}\boldsymbol{y}\\ \boldsymbol{u}\\ 1\end{bmatrix} = \Xi_j^{(1)} \mathcal{O}_{i_1,\ldots,i_{\bar{n}}} \boldsymbol{x} + \left(\Xi_j^{(1)} \mathcal{D}_{i_1,\ldots,i_{\bar{n}}} + \Xi_j^{(2)}\right) \boldsymbol{u} + \left(\Xi_j^{(1)} \mathcal{G}_{i_1,\ldots,i_{\bar{n}}} + \xi_j\right). \tag{27}$$

Equality of the right-hand sides of (26) and (27) leads to the set of q linear equations:

$$\begin{aligned} &\left(C_{i_0} A_{i_1,\ldots,i_{\bar{n}}} - \Xi_j^{(1)} \mathcal{O}_{i_1,\ldots,i_{\bar{n}}}\right) \boldsymbol{x} \\ &+ \left(\left[\, D_{i_0} \ C_{i_0} B_{i_1,\ldots,i_{\bar{n}}} \,\right] - \Xi_j^{(1)} \mathcal{D}_{i_1,\ldots,i_{\bar{n}}} - \Xi_j^{(2)}\right) \boldsymbol{u} \\ &+ \left(g_{i_0} + C_{i_1} f_{i_{i_1},\ldots,i_{\bar{n}}} - \Xi_j^{(1)} \mathcal{G}_{i_1,\ldots,i_{\bar{n}}} - \xi_j\right) = \mathbf{0}_{q \times 1}, \end{aligned} \tag{28}$$

that is satisfied by all points $\left[\begin{smallmatrix}\boldsymbol{x}\\ \boldsymbol{u}\end{smallmatrix}\right] \in B(\left[\begin{smallmatrix}\bar{\boldsymbol{x}}\\ \bar{\boldsymbol{u}}\end{smallmatrix}\right], \varepsilon) \bigcap \Omega_j$. Since these points are dense in the full-dimensional set $B(\left[\begin{smallmatrix}\bar{\boldsymbol{x}}\\ \bar{\boldsymbol{u}}\end{smallmatrix}\right], \varepsilon)$, the only possibility is that all coefficients in (28) are zero. This implies in particular $\Xi_j^{(1)} \mathcal{O}_{i_1,\ldots,i_{\bar{n}}} = C_{i_0} A_{i_1,\ldots,i_{\bar{n}}}$, and hence (18) holds for the $(\bar{n}+1)$-tuple $(i_0, i_1, \ldots, i_{\bar{n}})$. Given the arbitrariness of $(i_0, i_1, \ldots, i_{\bar{n}})$, this means that Condition C1 holds. □

The sufficient part of the proof of Theorem 1 is constructive. It provides a systematic procedure to compute an equivalent SARX representation (8)-(10) of a given SWA state space model (1) satisfying Condition C1 for a certain $\bar{n} \in \mathbb{Z}^+$. As in [11], the idea behind the proof establishes a one-to-one correspondence between the mode sequences of length $\bar{n}+1$ of the original SWA state space model and the modes of the constructed equivalent SARX model. Hence, following the proposed construction, the equivalent SARX model has $\bar{s} = s^{\bar{n}+1}$ modes, though this number can be possibly reduced (see Example 4.1 in Section IV). It is stressed that including the constraints (9)-(10) in the definition of the SARX model is paramount to obtain from the constructed SARX model only the same input-output behaviors as the SWA state space model (see again Example 4.1 in Section IV). The following corollary is a straightforward consequence of Theorem 1.

Corollary 1: Let $\bar{n} \in \mathbb{Z}^+$. The SWA state space model (1) admits an equivalent SARX representation (8)-(10) of order $\max\{n_a, n_b\} \leq \bar{n}$ if and only if Condition C1 is satisfied for such $\bar{n}$.

Corollary 1 requires to check Condition C1 only for the candidate model order $\bar{n}$. Note that, if Condition C1 is satisfied for a certain positive integer $\bar{n}$, then it is satisfied for all positive integers $\bar{\bar{n}} > \bar{n}$. Thus, equivalent SARX models of arbitrary order $\bar{\bar{n}}$ greater than $\bar{n}$ can be obtained by applying the construction described in the proof of Theorem 1. This motivates the following definition.

Definition 3.1: Let the SWA state space model (1) admit equivalent SARX representations (8)-(10). An equivalent SARX model is said to have *minimum order* if it has the smallest order among all equivalent SARX models.

Theorem 1 suggests the following remarks.

i) If $s > 1$, a given SWA state space model may not satisfy Condition C1 for any $\bar{n} \in \mathbb{Z}^+$ (see, e.g., Example 3.2 in Section III-A). Since any SARX model admits a realization in state space form, it can be concluded that the class of SARX systems is strictly contained in the class of SWA systems.

ii) If $s > 1$, a given SWA state space model of order n may admit a minimum-order equivalent input-output representation of order $\bar{n}$ greater than n (see Example 4.2 in Section IV).

iii) If $s = 1$, Condition C1 is satisfied for $\bar{n} = n$ in view of the Cayley-Hamilton theorem. This is in agreement with the well-known result that every linear state space model of order n admits a minimum-order equivalent ARX representation of order at most n.

IV. ILLUSTRATIVE EXAMPLES

In this section, two examples are presented to illustrate the equivalence result of Section III.

Example 4.1: Consider a SWA model (1) with $s = 2$ and

$$\begin{aligned} &\text{mode \#1} \begin{cases} A_1 = \begin{bmatrix} -\frac{1}{4} & \frac{1}{8} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix},\ B_1 = \begin{bmatrix} 0\\ 1 \end{bmatrix},\ f_1 = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} \\ C_1 = \begin{bmatrix} 2 & -1 \end{bmatrix},\ D_1 = 0,\ g_1 = \frac{1}{5}, \end{cases} \\ &\text{mode \#2} \begin{cases} A_2 = \begin{bmatrix} 0 & 1 \\ \frac{1}{8} & \frac{1}{4} \end{bmatrix},\ B_2 = \begin{bmatrix} \frac{5}{2} \\ -1 \end{bmatrix},\ f_2 = \begin{bmatrix} 0\\ 0 \end{bmatrix} \\ C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix},\ D_2 = -1,\ g_2 = 0. \end{cases} \end{aligned} \tag{29}$$

The model is not observable, since in particular the pair (C_1, A_1) is not observable. Nevertheless, Condition C1 is satisfied for $\bar{n} = 2$, and an equivalent SARX representation (8)-(10) with $\bar{s} = s^{\bar{n}+1} = 8$ modes and orders $n_a = n_b = \bar{n}$ is obtained by applying the constructive method described in the sufficient part of the proof of Theorem 1. The parameters Θ_j, the sets of feasible transitions $\Gamma(j)$, and the sets of initial regression vectors $\mathcal{R}_j$, $j \in \{1, \ldots, 8\}$, of the equivalent SARX model are reported in Table II, where

$$\begin{aligned} \mathcal{R}_{(1,1)} &= \left\{ \left[\begin{smallmatrix}\boldsymbol{y}\\ \boldsymbol{u}\end{smallmatrix}\right] :\ \boldsymbol{y}{\in}\mathbb{R}^2, \boldsymbol{u}{\in}\mathbb{R}^3, \left[\, 2\ 1\ 0\ 0\ 2 \,\right] \left[\begin{smallmatrix}\boldsymbol{y}\\ \boldsymbol{u}\end{smallmatrix}\right] = \tfrac{13}{5} \right\} \\ \mathcal{R}_{(2,1)} &= \left\{ \left[\begin{smallmatrix}\boldsymbol{y}\\ \boldsymbol{u}\end{smallmatrix}\right] :\ \boldsymbol{y}{\in}\mathbb{R}^2, \boldsymbol{u}{\in}\mathbb{R}^3, \left[\, 1\ \tfrac{1}{8}\ 0\ 1\ 0 \,\right] \left[\begin{smallmatrix}\boldsymbol{y}\\ \boldsymbol{u}\end{smallmatrix}\right] = \tfrac{21}{40} \right\}. \end{aligned} \tag{30}$$

TABLE II

$$\Theta_1 = \Theta_{(1,1,1)} = \left[\, -\tfrac{1}{2}\ 0\ 0\ -1\ 0\ \tfrac{13}{10} \,\right],\ \Gamma(1) = \{1, 5\},\ \mathcal{R}_1 = \mathcal{R}_{(1,1)}$$
$$\Theta_2 = \Theta_{(1,1,2)} = \left[\, -\tfrac{1}{2}\ 0\ 0\ -1\ 0\ \tfrac{13}{10} \,\right],\ \Gamma(2) = \{1, 5\},\ \mathcal{R}_2 = \mathbb{R}^5$$
$$\Theta_3 = \Theta_{(1,2,1)} = \left[\, -\tfrac{29}{8}\ 0\ 0\ \tfrac{19}{8}\ \tfrac{7}{4}\ \tfrac{39}{20} \,\right],\ \Gamma(3) = \{2, 6\},\ \mathcal{R}_3 = \mathcal{R}_{(2,1)}$$
$$\Theta_4 = \Theta_{(1,2,2)} = \left[\, \tfrac{5}{16}\ \tfrac{7}{32}\ 0\ \tfrac{101}{16}\ -\tfrac{21}{8}\ \tfrac{1}{5} \,\right],\ \Gamma(4) = \{2, 6\},\ \mathcal{R}_4 = \mathbb{R}^5$$
$$\Theta_5 = \Theta_{(2,1,1)} = \left[\, -\tfrac{1}{8}\ 0\ -1\ 0\ 0\ \tfrac{21}{40} \,\right],\ \Gamma(5) = \{3, 7\},\ \mathcal{R}_5 = \mathcal{R}_{(1,1)}$$
$$\Theta_6 = \Theta_{(2,1,2)} = \left[\, -\tfrac{1}{8}\ 0\ -1\ 0\ 0\ \tfrac{21}{40} \,\right],\ \Gamma(6) = \{3, 7\},\ \mathcal{R}_6 = \mathbb{R}^5$$
$$\Theta_7 = \Theta_{(2,2,1)} = \left[\, -2\ 0\ -1\ \tfrac{1}{2}\ 1\ 1 \,\right],\ \Gamma(7) = \{4, 8\},\ \mathcal{R}_7 = \mathcal{R}_{(2,1)}$$
$$\Theta_8 = \Theta_{(2,2,2)} = \left[\, \tfrac{1}{4}\ \tfrac{1}{8}\ -1\ \tfrac{11}{4}\ -\tfrac{3}{2}\ 0 \,\right],\ \Gamma(7) = \{4, 8\},\ \mathcal{R}_8 = \mathbb{R}^5$$

Note that the notation $\Theta_j = \Theta_{(i_0,i_1,i_2)}$ in Table II implicitly defines the bijective mapping ϕ between the mode sequences (i_0, i_1, i_2) of the SWA state space model and the modes j of the constructed equivalent SARX model. According to the proposed construction, each region $\mathcal{R}_j$ is the range of the affine transformation (22) with $\bar{n} = 2$ and (i_1, i_2) given by $(i_0, i_1, i_2) = \phi^{-1}(j)$. Since the observability matrices $\mathcal{O}_{i_1,i_2}$ have full rank for $(i_1, i_2) = (1, 2)$, $(2, 2)$, the range of the affine transformation (22) is the whole space $\mathbb{R}^5$ for such (i_1, i_2), and hence $\mathcal{R}_2 = \mathcal{R}_4 = \mathcal{R}_6 = \mathcal{R}_8 = \mathbb{R}^5$. Conversely, $\mathrm{rank}(\mathcal{O}_{i_1,i_2}) = 1$ for $(i_1, i_2) = (1, 1)$, $(2, 1)$, and explicit expressions (30) for the 4-dimensional affine sets $\mathcal{R}_1$, $\mathcal{R}_3$, $\mathcal{R}_5$ and $\mathcal{R}_7$ are obtained by eliminating $\boldsymbol{x}$ in (22).

It is now shown that, if the constraints (9)-(10) are violated, the SARX model has extra input-output behaviors with respect to the SWA model. Let the input be zero, and choose initial output values $y(0) = y(1) = 0$. The initial regression vector $\boldsymbol{r}(2) = \left[\, y(1)\ y(0)\ u(2)\ u(1)\ u(0) \,\right]^\top = \mathbf{0}$ belongs neither to $\mathcal{R}_{(1,1)}$ nor to $\mathcal{R}_{(2,1)}$, and to satisfy (9) one should choose $\ell(2) \in \{2, 4, 6, 8\}$, according to Table II. If the incorrect choice $\ell(2) = 3$ is made, the next output of the SARX model is $y(2) = \Theta_3\, \boldsymbol{r}(2) = \frac{39}{20}$. Then, in view of (5), if one tries to solve the system of linear equations

$$\left[\, y(2)\ y(1)\ y(0) \,\right]^\top = \mathcal{O}_{i_1,i_2,i_3} \boldsymbol{x}_0 + \mathcal{G}_{i_1,i_2,i_3} \tag{31}$$

with $\boldsymbol{x}_0$ the unknown initial state of the SWA model, it turns out that (31) is infeasible for any $(i_1, i_2, i_3) \in \{1, 2\}^3$, and hence the input-output trajectory of the SARX model thus generated is not an input-output trajectory of the SWA model. Similarly, choose $y(0) = \frac{13}{5}$ and $y(1) = 0$, and let the next output be $y(2) = \Theta_1\, \boldsymbol{r}(2) = \frac{13}{10}$. This choice is consistent, since $\boldsymbol{r}(2) \in \mathcal{R}_1$. Then, let $y(3) = \Theta_7\, \boldsymbol{r}(3) = -\frac{8}{5}$, with $\boldsymbol{r}(3) = \left[\, y(2)\ y(1)\ u(3)\ u(2)\ u(1) \,\right]^\top$. Since $7 \notin \Gamma(1)$, (10) is violated and it turns out that the system of linear equations

$$\left[\, y(3)\ y(2)\ y(1)\ y(0) \,\right]^\top = \mathcal{O}_{i_1,i_2,i_3,i_4} \boldsymbol{x}_0 + \mathcal{G}_{i_1,i_2,i_3,i_4} \tag{32}$$

is infeasible for any $(i_1, i_2, i_3, i_4) \in \{1, 2\}^4$. Hence, also the input-output trajectory of the SARX model thus generated is not an input-output trajectory of the SWA model.

Finally, it is observed that in this example the modes 1 and 2 can be merged because $\Theta_1 = \Theta_2$, $\mathcal{R}_1 \subseteq \mathcal{R}_2$ and $\Gamma(1) = \Gamma(2)$. The same holds for the modes 5 and 6, so that the number of modes of the equivalent SARX model can be reduced from 8 to 6.

Example 4.2: This example shows that the minimum $\bar{n}$ for which Condition C1 is satisfied, can be greater than n. Consider a 2-mode SWA model (1) with $A_1 = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$, $C_1 = [\, 1\ 0 \,]$, $A_2 = \left[\begin{smallmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $C_2 = [\, 1\ {-1} \,]$. For $\bar{n} = n = 2$, Condition C1 is not satisfied by $(i_0, i_1, i_2) = (1, 2, 1)$ and $(2, 2, 1)$. On the contrary, Condition C1 is satisfied for $\bar{n} = 3$, since in particular all the observability matrices $\mathcal{O}_{i_1,i_2,i_3}$ have full-column rank. It can be concluded that the SWA state space model admits a minimum-order equivalent SARX representation of order $\bar{n} = 3$.

V. Conclusions

A necessary and sufficient condition for the conversion of SWA models from state space to input-output form has been derived in this paper. It has been shown that observability is not a necessary requirement for input-output realization of SWA models. Moreover, it has been observed that the class of SARX systems is strictly contained in the class of SWA systems, and that the number of modes (and thus the number of parameters) may grow considerably when a SWA state space model is converted into a minimum-order equivalent SARX representation.

Future work will investigate the existence of an index $\bar{n}_{max}(s, n)$ such that, if a given SWA state space model of order n with s modes does not satisfy Condition C1 for $\bar{n} = \bar{n}_{max}(s, n)$, then it is certified that the model cannot be converted into an equivalent SARX representation. Moreover, it would be interesting to figure out to what extent the results presented in this paper and in [10] for discrete-time SWA and PWA models can be extended or adapted to continuous-time models.

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