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Minimum Time Control of Spin Systems via Dynamic Programming

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Abstract— **In this article we show how dynamic programming can be applied to the time optimal control of spin systems. This is done by recasting the system in two ways: (i) As an adjoint system along the lines of [1], (ii) As an impulsive control problem. We illustrate the dynamic programming methodology using numerical examples.**

I. Introduction

In the recent past, there has been a lot of attention to the problem of obtaining time optimal trajectories for open loop control of quantum systems [1], [2], [4], [5]. These problems arise from applications which include NMR spectroscopy (to produce a time optimal trajectory), and the optimal construction of quantum circuits [6], [7] (to minimize the number of logic gates needed to construct a desired unitary transformation).

Spin systems have the mathematical structure of a bilinear right invariant system on a manifold. Hence, in [8], the reachability problem on the spin system $SU(2)$ was solved, using results from geometric control theory. In [1] a Cartan decomposition method was described to solve the control problem on a general spin system and an explicit solution was found for certain special cases.

In this article we demonstrate the application of Dynamic Programming to a spin system having the following interesting property: the available controls can take on arbitrarily large values (this physically corresponds to certain radio frequency pulses). Hence such a spin system can have rapid motion along certain unitaries (elements of the unitary group). The solution to the problem is obtained after reformulating it in the following two ways:

1) As an adjoint system [1] with ordinary (non-impulsive) controls which have an unbounded range.
2) As an impulsive control problem.

The above methods of recasting the problem and the solutions obtained therefrom, yield insights into the nature of the optimal system dynamics.

The outline of the article is as follows: in Section II we describe the original problem and indicate a simple case where the minimum time function lacks differentiability at certain points. Thus we motivate the need for the notion of viscosity solutions on a manifold (which are briefly introduced in the Appendix). This is followed in Section III by a representation of the problem in terms of an adjoint system (as in [1]) and the use of dynamic programming to obtain a solution. An example problem on $SU(2)$ is solved numerically and the results are described. This is followed by recasting the problem into an impulsive control framework in Section IV. Simulation results and sample optimal trajectories are obtained to demonstrate the application of the theory developed to an example problem. The appendix contains the technical definitions for the preceding sections and an outline of the major proofs.

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II. Problem Formulation

In this section we recall the time optimal control problem for spin systems formulated in [1]. Given the compact Lie group $\mathbf{G} = SU(2^n)$ with Lie algebra $\mathfrak{g} = \mathfrak{su}(2^n)$ and right invariant vector fields X_d, $X_1, \ldots. X_m$, let the evolution of the system be given by

$$\dot{U} = [X_d + \sum_{i=1}^{m} v_i X_i]U\,, \qquad U \in \mathbf{G} \tag{1}$$

with initial condition $U(0) = U_0$, where v_i are the piecewise continuous real valued control signals. We denote the Lie subgroup of $\mathbf{G}$ generated by the control vector fields $\{X_1, \ldots, X_m\}$ by $\mathbf{K}$. Assuming that the Lie algebra generated by the set $\{X_d, X_1 \ldots, X_m\}$ is $\mathfrak{g}$, the system is controllable [15]. Hence the minimum time to move between any two points on $\mathbf{G}$ is finite. The objective is to obtain a control strategy to transfer the system from any initial U_0 in $\mathbf{G}$ to the identity element I of $\mathbf{G}$ in minimum time (actually the formulation given in [1] starts at the identity and controls are sought to reach arbitrary elements of the group in minimum time—a simple time reversal shows that this is equivalent to our modification of their formulation).

We define this problem precisely as follows. Let $\mathscr{M}$ denote the class of piecewise continuous functions $v : [0,\infty) \to \mathbb{R}^m$. For $U \in \mathbf{G}$ and $v \in \mathscr{M}$ let

$$t_U(v) = \inf\{t > 0 \,:\, U(0) = U, U(t) = I, \text{ dynamics (1)}\}. \tag{2}$$

Here $U(\cdot)$ denotes the trajectory of (1) with initial condition $U(0) = U$ and control $v \in \mathscr{M}$. The infimium is infinite if the terminal constraint $U(t) = I$ is not attained. The *minimum time function* is defined by

$$T(U) = \inf\{t_U(v) \;:\; v \in \mathscr{M}\}. \tag{3}$$

This formulation differs from the usual minimum time problem formulation (see [3], [17]), in that, here the range of the controls v is unbounded. The availability of unbounded controls along the directions $\{X_1, X_2, \ldots X_m\}$ leads to an interesting property: we can synthesize any element of the subgroup $\mathbf{K}$ instantaneously. Hence we have the property

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that for all elements $U \in \mathbf{G}$, and $k \in \mathbf{K}$, $T(kU) = T(U)$. This intuitively states the fact that once we reach the point U it takes no additional time to travel to any point $\mathbf{K}U$ i.e the coset of U, and vice versa. This feature of the problem formulation is meant to capture the rapid motion caused by pulses in spin systems.

Under the controllability condition mentioned above, we have $0 \leq T(U) < +\infty$. The following example shows that T need not be differentiable everywhere.

Example 2.1: Consider the following system defined on $\mathbf{G} = SU(2)$:

$$\dot{U} = [I_z + vI_x]U \tag{4}$$
$$U(0) = U_0.$$

We now demonstrate that the minimum time function $T : SU(2) \to \mathbb{R}$ defined above is not differentiable at the points $\pm I$.c

Any point P in $SU(2)$ can be represented as: $k_1 \exp(\alpha I_z) k_2$ where k_1, k_2 are elements of the Lie subgroup generated by $\exp(I_x)$ and $\alpha \in [0, \pi]$. From [1], the minimum time function for any such point is given by: $T(P) := \alpha$. For this compact, connected Lie Group the exponential mapping (denoted by ϕ) is a diffeomorphism from an open set around the origin in the Lie algebra $\mathfrak{su}(2)$ to an open set around I in $SU(2)$. Let e_1 correspond to the I_z axis in $\mathfrak{su}(2)$.

From the above, if the function T is differentiable at the identity element then the function $\tilde{T} := T \circ \phi : \mathfrak{su}(2) \to \mathbb{R}$ must be differentiable at the origin in $\mathfrak{su}(2)$. Hence there must must exist a linear function λ s.t

$$\lim_{\|\varepsilon\| \to 0} \left\{ \frac{\tilde{T}(x+\varepsilon) - \tilde{T}(x) - \lambda(\varepsilon)}{\|\varepsilon\|} \right\} = 0, \quad x = I \tag{5}$$

Now, consider a line through the origin in $\mathfrak{su}(2)$ along I_z i.e e_1. Let ε be either $+\delta e_1$ or $-\delta e_1$ (with $\delta > 0$). At the identity element, the value of $\tilde{T}(x)$ is 0 and $\tilde{T}(x+\varepsilon)$ is the same positive value (δ), despite the change in sign of ε. If λ is linear then $\lambda(\delta e_1)$ should change signs with changes in the sign of ε. Thus, there is no linear function which would give the desired limit. Hence the function is not differentiable at I. Similar arguments hold for the element $-I$ of $SU(2)$. □

In view of this example, care is needed when considering dynamic programming equations since derivatives are involved. We will employ the concept of viscosity solutions [17], suitably extended to the manifold setting. Rather than work with the problem directly as formulated in this section, we apply dynamic programming methods to two related formulations, one involving an adjoint system description in Section III, and another involving an impulsive system (Section IV).

III. Dynamic Programming for an Equivalent Problem using an Adjoint System

A. Minimum Time Control of an Adjoint System

In [1] the time optimal control problem was solved for a class of spin systems using an equivalent problem formulated in terms of an adjoint system. To motivate this, note that the system described by (1), can be rewritten as:

$$\dot{Q} = [\sum_{i=1}^{m} v_i X_i] Q \tag{6}$$
$$\dot{P} = (Q^{-1} X_d Q) P, \tag{7}$$

with initial conditions $Q(0) = I$, $P(0) = U_0$.

It can be verified that $Q(t) \in \mathbf{K}$ and $U(t) = Q(t)P(t)$ for all $t \geq 0$. Further, we note that $Q^{-1} X_d Q \in Ad_{\mathbf{K}}(X_d)$, where

$$Ad_{\mathbf{K}}(X_d) = \{k^{-1} X_d k \mid k \in \mathbf{K}\},$$

called the adjoint orbit of X_d, is a compact set (since $\mathbf{K}$ is compact).

The system in (7) can be generalized to yield the following *adjoint system* [1]:

$$\dot{P} = XP, P \in \mathbf{G}, \tag{8}$$

where X is an $Ad_{\mathbf{K}}(X_d)$-valued control signal. The minimum time function for this adjoint system is defined as follows. Let $\mathscr{X}$ denote the class of piecewise continuous functions $X : [0, \infty) \to Ad_{\mathbf{K}}(X_d)$. For $P \in \mathbf{G}$ and $X \in \mathscr{X}$ let

$$t_P^a(X) = \inf\{t > 0 \ : \ P(0) = P, P(t) = I, \text{ dynamics (8)}\}. \tag{9}$$

Then the minimum time function for the adjoint system is defined by

$$T^a(P) = \inf\{t_P^a(X) \ : \ X \in \mathscr{X}\}. \tag{10}$$

It is shown in [1] that the minimum times for the systems (1) and (8) are equivalent. In particular, the adjoint system provides a description of the dynamics on the coset space $\mathbf{G}/\mathbf{K}$. This alternative formulation is technically simpler, because the control value space is compact; importantly, the Riemannian-symmetric structure of the coset space in certain cases was exploited in [1] to obtain explicit expressions for the optimal controls and trajectories. In the next subsection we apply dynamic programming to this equivalent problem. It is a non-standard application of dynamic programming in that the state space is a manifold, rather than flat Euclidean space usually considered in the literature.

B. Dynamic Programming

The dynamic programming principle for the adjoint minimum time problem can be established: for any $t > 0$,

$$T^a(P) = \inf_{X \in \mathscr{X}} \{t \wedge t_P^a(X) + T^a(P(t \wedge t_P^a(X))\} \tag{11}$$

using the dynamics (8). The corresponding Hamilton-Jacobi-Bellman (HJB) equation is

$$\begin{aligned} H(P, DT^a(P)) &= 0, \quad P \in \mathbf{G}/\mathbf{K} \\ T^a(P) &= 0, \quad P \in [\mathbf{K}] \\ T^a(P) &> 0 \quad P \in (\mathbf{G}/\mathbf{K}) \backslash [\mathbf{K}], \end{aligned} \tag{12}$$

where

$$H(P, \lambda) := \sup_{X \in Ad_{\mathbf{K}}} \{-\lambda[XP] - 1\}.$$

Here, $P \in \mathbf{G}$ and $\lambda \in T_P^*(\mathbf{G})$, the cotangent space; thus $\lambda[XP]$ is a number.

In the HJB equation (12), $DT^a(P)[XP]$ is the value of a directional derivative, if the derivative were to exist (recall Example 2.1). An appropriate viscosity definition is given in the Appendix. The Appendix also discusses the continuity of the function T^a and the uniqueness of continuous viscosity solutions of the HJB equation (12).

Assuming regularity conditions, the optimal control policy is generated by the synthesis equations given below [17, Section 1.5]. X^* is optimal for an initial state P_0 if and only if

$$X^*(t) = L(P(t)) \text{ for a.e } t > 0 \tag{13}$$

$$L(P) \in \underset{X \in Ad_{\mathbf{K}}(X_d)}{\text{argmax}} \{-DS(P)[XP] - 1\} \tag{14}$$

Now, in the systems considered, the viscosity solution is continuous but not necessarily differentiable everywhere. We numerically synthesize the optimal controls using techniques such as in [16, Chapter 3] where the solutions to the discretized version tend to the viscosity solution of the original continuous description.

C. Example

In order to verify the dynamic programming approach using a example system, we proceed in this subsection to compute the optimal controls, minimum time functions, and corresponding trajectories by numerically solving the dynamic programming equation (18). The example system is one considered in [1], viz.

$$\dot{U} = (I_z + vI_x)U \ , U \in SU(2) \tag{15}$$

$$U(0) = U_0 \, , U_0 \in SU(2) \tag{16}$$

where

$$I_x = \frac{1}{2} j \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad I_z = \frac{1}{2} j \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Instead of using the value function T^a (as defined in the last section) directly, it is advantageous to use the monotone transformation (Kruskov transform)

$$S(P) = 1 - e^{-T^a(P)}, \tag{17}$$

which leads to the HJB equation

$$\begin{aligned} S(P) + H(P, DS(P)) &= 0, \quad P \in \mathbf{G}/\mathbf{K} \\ S(P) &= 0, \quad P \in [\mathbf{K}] \\ 1 \geq S(P) &> 0 \quad (\mathbf{G}/\mathbf{K}) \backslash [\mathbf{K}]. \end{aligned} \tag{18}$$

The function S can be interpreted as a discounted minimum time function for the adjoint system (8):

$$S(P) = \inf_{X \in \mathscr{X}} \left\{ \int_0^{t \wedge t_P^a(X)} e^{-s} ds + e^{-(t \wedge t_P^a(X))} S(P(t \wedge t_P^a(X))) \right\} \tag{19}$$

This normalization (discounting) is useful for better numerical convergence and is also used in the uniqueness proofs of the dynamic programming equations (Appendix).

To obtain a numerical solution to the dynamic programming problem, we parameterize points in $SU(2)$ by a subset of the Euclidian space using a mapping of the form $\exp(k_1 I_x) \exp(a I_z) \exp(k_2 I_z)$. Note that for the range of k_1, k_2, a which we use, the parametrization is not unique i.e multiple points in the subset of $\mathbb{R}^3$ map to the same point in $SU(2)$.

The minimum time plots are presented as gray-scale images in a three dimensional grid. The axes correspond to the three parameters used for the representation of $SU(2)$ as described above. A lighter shading indicates a larger value of the minimum time function at a point, while a darker shading implies a smaller time to reach the identity element when starting from that point. Note that minimum time function takes on the same value for elements belonging to the same coset (i.e points which have the same values of the parameter a). In other words moving alone the impulsive directions k_1,k_2 does not change the minimum time function.

In Figure 1, we compare the normalized (ranging from 0 to 1) minimum time function obtained by solving the dynamic programming equation, using the Cartan decomposition technique in [1]. The optimal times determined by the dynamic programming technique and by the Cartan decomposition are seen to be numerically identical (to within the round off error).

Time optimal trajectories for these two methods are shown in Figure 2. Due to the way the adjoint system description of the system is set up, the dynamic programming approach drives the state of the system to a point in $\mathbf{K}$ and not necessarily to the identity itself. However, from this point in $\mathbf{K}$ a single pulse can move the state to the identity. Note that the non-uniqueness of the representation leads to the non-over lap of trajectories between the two methods (when represented in flat space).

The discretization of this system for obtaining numerical solution to the HJB equation (18) is carried out using the procedure in [16]. The dynamics of the system can be recast in the form

$$\dot{x} = f(x, v), \tag{20}$$

where x is an element of an n dimensional Euclidian space (whose basis vectors are denoted by e_i) and v is the control signal. For a discretization of this space with a grid spacing h, the value iteration equation (i.e the iteration of the cost function, say S^h) is given by:

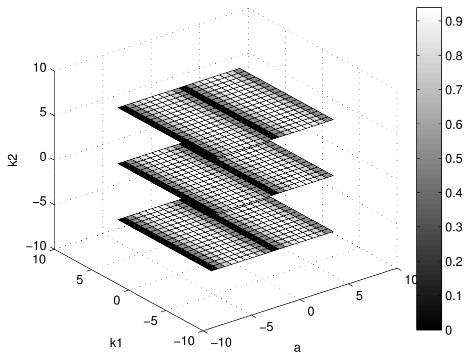
$$S^h(x) = \inf_u \left\{ \frac{h}{h + \|f\|_1} + \frac{\sum_{i=1}^{n} S^{h}{}^{i}_{\pm}(x) f^i_{\pm}(x,u)}{h + \|f\|_1} \right\}$$

where

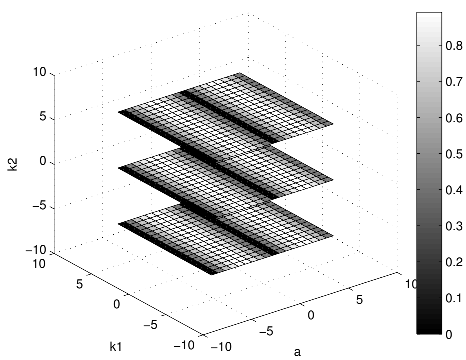
$$\begin{aligned} S^{h}{}^{i}_{\pm} &:= S^h(x \pm h e_i) \\ f^i_+(x,u) &:= \max\left\{ f^i(x + h e_i, u), 0 \right\} \\ f^i_-(x,u) &:= -\min\left\{ f^i(x - h e_i, u), 0 \right\} \\ \|f(x)\|_1 &:= \sum_{i=1}^{n} \left\| f^i(x) \right\| \end{aligned} \tag{21}$$

f^i are the i th components of the vector valued function f.

These equations are used to obtain the simulation results. As can be seen from the resulting figure, at points in $SU(2)$ which have the same value of the a parameter, the minimum time function has the same value. This is due to the fact that it takes arbitrarily small time to flow along each of the cosets.



(a) Minimum time function using the dynamic programming method



(b) Minimum time function using the Cartan Decomposition technique

Fig. 1. Optimal times using the dynamic programming and Cartan Decomposition methods on the adjoint system representation

IV. Dynamic Programming for an Alternative Formulation using Impulsive Controls

A. Minimum Time Control of an Impulsive System

The formulation discussed in Section II featured unbounded controls. An important implication of this for the minimum time problem is that optimal trajectories move rapidly along $\mathbf{K}$-cosets. In practice, this corresponds to a very fast laser or RF pulse. In this section we describe an alternative formulation using impulsive controls.

We replace the system (1) (which employs standard controls) with the following impulsive system:

$$\dot{U}(t) = X_d U(t)\,,\, t \in (\tau^i, \tau^{i+1})\,,\, i = 0, 1 \ldots \qquad (22)$$
$$U(\tau^i{}_+) = k_i U(\tau^i{}_-)\,,\, k_i \in \mathbf{K}$$

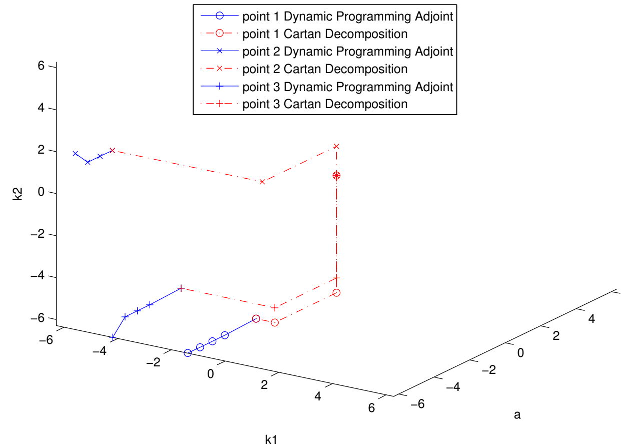


Fig. 2. Optimal trajectories obtained by the dynamic programming and Cartan decomposition methods on the adjoint system representation

with initial condition $U(0) = U$, where

$$U(\tau^i_-) = U(\tau^{i-1}_+) + \int_{\tau^{i-1}}^{\tau^i} X_d U(t)\,dt$$
$$\tau^i \in [0, \infty)\,\forall i$$
$$\tau^i > \tau^j\,, \forall i > j$$

The control of this system involves choosing times τ^i at which an impulse $k_i \in \mathbf{K}$ is applied. The system evolves by drifting between impulses, and rapid impulsive motion along cosets at specified times.

An impulsive control is a sequence

$$\beta = \{(\tau_0, k_0), (\tau_1, k_1), \ldots\}.$$

Let $\mathscr{B}$ denote the set of all such impulsive controls. For $U \in \mathbf{G}$ define

$$t^b_U(\beta) = \inf\{t > 0 \ : \ U(0) = U, U(t) = I, \text{dynamics}(22)\}. \qquad (23)$$

The minimum time function for the impulsive system is defined by

$$T^b(U) = \inf\{t^b_U(\beta) \ : \ \beta \in \mathscr{B}\}. \qquad (24)$$

It can be checked that the minimum time for the impulsive system agrees with that for the original system: $T^b(U)$ is equal to $T(U)$ for all $U \in \mathbf{G}$.

B. Dynamic Programming

The Hamilton-Jacobi-Bellman (HJB) equation for the impulsive minimum time problem is

$$\begin{aligned} \max\{H^b(U, DT^b(U)), T^b(U) - \mathsf{M}[T^b](U)\} &= 0, \quad U \in \mathbf{G}/\mathbf{K} \\ T^b(U) &= 0, \quad U \in [\mathbf{K}] \qquad (25) \\ T^b(U) &> 0 \quad U \in (\mathbf{G}/\mathbf{K}) \backslash [\mathbf{K}], \end{aligned}$$

where

$$H^b(U,\lambda) = -\lambda[X_d] - 1,$$
$$\mathsf{M}[T](U) = \inf_{k \in \mathbf{K}} \{T(kU)\}.$$

This variational inequality will also need to be interpreted in the viscosity sense. This will be investigated in future work. Assuming regularity conditions, the optimal control policy is generated by the synthesis equations given below. K^* is optimal for an initial state U_0 if and only if

$$K^*(t) = L(U(t)) \text{ for a.e } t > 0 \quad (26)$$

$$L(U) \in \underset{k \in \mathbf{K}}{\operatorname{argmax}} \left\{ H^b(U, DT^b(U)), T^b(U) - \mathsf{M}[T^b](U) \right\} \quad (27)$$

where the argument is taken depending on which of H^b or $T^b - \mathsf{M}$ is larger. If the former, then no impulse is applied and the system is allowed to evolve along the drift vector fields.

The optimal controls for the discretized version of the problem are synthesized by methods similar to the one in [16, Chapter 8].

C. Example

We revisit the example described in Subsection III-C and view it in terms of an impulsive control problem.

$$\dot{U} = I_z U, \quad (28)$$
$$U(0) = U_0, U_0 \in SU(2) \quad (29)$$

$$\dot{U}(t) = I_z U(t), t \in (\tau^i, \tau^{i+1}), i = 0, 1 \ldots \quad (30)$$
$$U(\tau^i{}_+) = k_i U(\tau^i{}_-), \qquad U \in SU(2) \quad (31)$$

where k_i an element of the form $\exp(I_x \alpha)$ s.t $\alpha \in \mathbb{R}$.

Figure 3(b) indicates the results obtained using the dynamic programming method for the impulsive system. This simulation result may be compared numerically with the result on the same system obtained by the Cartan decomposition technique (Figure 1(b)). Time optimal trajectories are shown for these two methods in Figure 3(a). The parametrization of points in $SU(2)$ is again taken to be of the form $\exp(k_1 I_x)\exp(aI_z)\exp(k_2 I_z)$, and is not unique; which leads to the non-over lap of the path taken by the two methods. Note that the paths indicated here were obtained using the same starting points as in the example for dynamic programming applied to the adjoint system representation. The initial jump to another quadrant, which occurs in some of the the paths, still allows the path to be optimal since the jump costs no additional time. The optimal time is consistent with the Cartan decomposition in [1], to within the roundoff error.

To obtain numerical solutions to the system in (25), the discretization of this system is carried out in a manner similar to the earlier case of the dynamic programming for the Adjoint system description. As can be seen from the figure, the fact that the impulsive cost is zero leads to points which have the same value of the a parameter having the same value of the minimum time function. Note that the optimal control is not unique at the coset corresponding to the identity element, since it is possible to jump within that coset as many times as desired before ultimately reaching the identity (however all these controls would belong to the same equivalence class consisting of signals that move the state to the identity in the same time starting from the same initial point).

In an n dimensional Euclidian space with a grid spacing h, zero cost of impulse, and basis vectors e_i, the value iteration equation (i.e the iteration of the cost function, say T^h) for the QVI is given by:

$$T^h(x) = \min \left\{ \inf_u \left\{ \frac{h}{h + \|f\|_1} + \frac{\sum_{i=1}^{n} T^{h}{}^{i}_{\pm}(x) f^i_{\pm}(x,u)}{h + \|f\|_1} \right\}, \inf_{k \in \mathbf{K}} [T^h(kx)] \right\} \quad (32)$$

where the notations are similar to the ones used in (21).

V. Discussion and Conclusions

In this article we have described the use of the Dynamic programming method to solve the minimum time control problem on a spin system and have demonstrated a proof of principle of this technique by obtaining a complete solution to an example problem on $SU(2)$.

The numerical procedures outlined herein generalize to higher dimensional cases with the crucial limiting factor being the time taken and storage requirements for these computations (which increases dramatically with the dimension of the system). Owing to the curse of dimensionality, further work is required to develop computational methods of greater efficiency in order to use the Dynamic Programming technique to investigate larger problems of practical interest. An application of the proposed dynamic programming approach to the problem of optimal quantum gate synthesis has been dealt with in [18].

The simulations in this work are based on theoretical results which are quite involved. A rigorous and complete development of the mathematical proofs of the foundations of this article including the convergence of numerical schemes to the solution will be deferred to a future publication.

Appendix

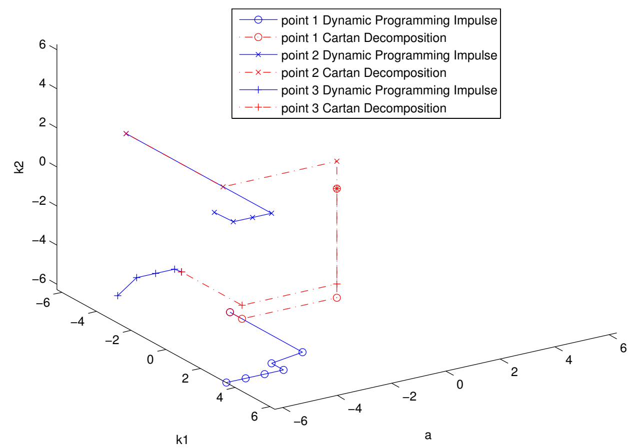
Motivated by [17, Definition 1.1 Chapter 2], we define the notion of a continuous viscosity solution on a manifold as follows:

Definition 1.1: **Continuous Viscosity solution**
Given a manifold $\mathbf{M}$. A function $V \in C(\mathbf{M})$ is a viscosity sub(super) solution of the following PDE in $\mathbf{M}$

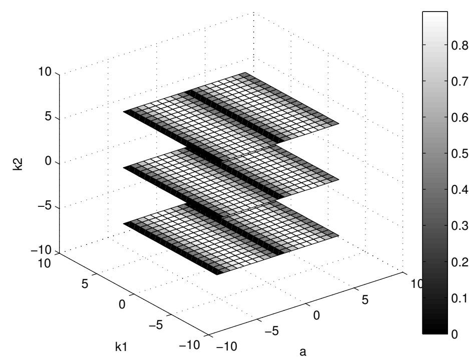
$$\begin{cases} F(x, DV(x)) = 0, x \in \mathbf{M} \\ V(y) = 0, y \in \partial\Omega \text{ (boundary condition)} \end{cases}$$

if $\forall \phi \in C^1(\mathbf{M})$

$$F(x_0, V(x_0), D\phi(x_0)) \leq (\geq) 0$$



(a) Optimal trajectory comparisons for the impulsive control system using dynamic programming and Cartan Decomposition



(b) Optimal time for the dynamic programming with impulsive control

Fig. 3. Path comparison for the dynamic programming method on the impulsive control system and the optimal time using this method

at every point $x_0 \in \mathbf{M}$ where $V - \phi$ has a relative maxima (minima) and such that V satisfies : $V(y) = 0\,, \forall y \in \partial\Omega$. A function is a viscosity solution if and only if it is both a super and sub viscosity solution.

The uniqueness of the viscosity solutions to the HJB equation in the adjoint case arises out of the uniqueness of the viscosity solution to the HJB formed by applying the Kruskov transform to the original HJB equations for the adjoint system. The proof, which requires continuity of the value function, proceeds along the lines of [17, Theorem 2.6 Chapter 4] with appropriate modifications for the manifold setting.

Now, the continuity of the value function required in the uniqueness proof, can be ensured by a small time controllability condition around the target (similar to the one outlined in [17, Theorem 1.23 Chapter 4]). The intuition is that if there is a small region about the boundary of the target in which all points can be reached using the available control signals, then the minimum time function is continuous about the target. Hence using regularity conditions on the dynamics, it follows that the minimum time function is continuous on the open set consisting of the manifold with the target set removed.

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